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Non-separable tree-like Banach spaces and Rosenthal's ℓ_1-theorem

by

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Abstract. We introduce and investigate a class of non-separable tree-like Banach spaces. As a consequence, we prove that we cannot achieve a satisfactory extension of Rosenthal's ℓ_1-theorem to spaces of the type $\ell_1(\kappa)$ for κ an uncountable cardinal.

1. Introduction. Rosenthal's ℓ_1-theorem [9] is one of the most remarkable results in Banach space geometry. It provides a fundamental criterion for the embedding of ℓ_1 into Banach spaces.

THEOREM 1.1 (Rosenthal's ℓ_1-theorem). *Let (x_n) be a bounded sequence in the Banach space X and suppose that (x_n) has no weakly Cauchy subsequence. Then (x_n) contains a subsequence equivalent to the usual ℓ_1-basis.*

A satisfactory extension of Theorem 1.1 to spaces of the type $\ell_1(\kappa)$, for κ an uncountable cardinal, would be desirable, since it would provide a useful criterion for the embedding of $\ell_1(\kappa)$ into Banach spaces. Naturally, therefore, R. G. Haydon [7] posed the following problem: Let κ be an uncountable cardinal. Suppose that X is a Banach space, and A is a bounded subset of X of cardinality κ, which does not contain any weakly Cauchy sequence. Can we deduce that A has a subset equivalent to the usual $\ell_1(\kappa)$-basis?

Before the question was posed, Haydon [6] had already presented a counterexample for the case $\kappa = \omega_1$. A completely different counterexample for the same case had also been obtained by J. Hagler [3]. Finally, a complete solution to the aforementioned problem was given by C. Gryllakis [2] who proved that the answer is always negative with only one exception, namely when both κ and $\mathrm{cf}(\kappa)$ are strong limit cardinals.

In this paper, we first introduce for any infinite cardinal κ a tree-like Banach space X_κ. Our construction is motivated by the well-known James Tree space (JT) [8] and Hagler Tree space (HT) [3]. We also study in detail

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various properties of the space X_κ; we mostly focus on continuous functionals defined on X_κ. As a consequence, we give a very simple answer to Haydon's problem.

Closing this introductory section, we recall some definitions for the sake of completeness. A sequence $(x_n)_{n\in\mathbb{N}}$ in a Banach space X is *weakly Cauchy* if the scalar sequence $(f(x_n))_{n\in\mathbb{N}}$ converges for every f in X^*. A subset $A \subset X$ with cardinality κ is *equivalent to the usual* $\ell_1(\kappa)$*-basis* if there are constants $C_1, C_2 > 0$ such that $C_1 \sum_{i=1}^n |a_i| \le \| \sum_{i=1}^n a_i x_i \| \le C_2 \sum_{i=1}^n |a_i|$, for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in A$ and any scalars $a_1, \dots, a_n$. Given an infinite cardinal κ, we let κ^+ denote the *successor* of κ, i.e. κ^+ is the smallest cardinal greater than κ. We also define the *cofinality* of κ, denoted by $\mathrm{cf}(\kappa)$, to be the smallest cardinal with the following property: there exist cardinals $\{\kappa_i \mid i < \mathrm{cf}(\kappa)\}$ such that $\kappa_i < \kappa$ for every ordinal $i < \mathrm{cf}(\kappa)$, and $\sum_{i<\mathrm{cf}(\kappa)} \kappa_i = \kappa$.

Finally, we should mention that this is not the first time non-separable tree-like Banach spaces have been defined (e.g. see [1], [4] and [5]; our construction is closer to the constructions of [4]).

2. The basic construction. Suppose that κ is an infinite cardinal. Then we set

$$\begin{aligned} \Gamma &= \{0,1\}^\kappa = \{a : \{\xi < \kappa\} \to \{0,1\}\} = \{(a_\xi)_{\xi<\kappa} \mid a_\xi = 0 \text{ or } 1\} \\ \mathcal{D} &= \{0,1\}^{<\kappa} = \bigcup \{\{0,1\}^\eta \mid \mathrm{Ord}(\eta), \eta < \kappa\} \\ &= \{(a_\xi)_{\xi<\eta} \mid \eta \text{ is an ordinal}, \eta < \kappa, a_\xi = 0 \text{ or } 1\}. \end{aligned}$$

The set $\mathcal{D}$ is called the (*standard*) *tree*. The elements $s \in \mathcal{D}$ are called *nodes*. The elements of the set $\Gamma = \{0,1\}^\kappa$ are called *branches*.

If s is a node and $s \in \{0,1\}^\eta$, we say that s is at the η*th level* of $\mathcal{D}$. We denote the level of s by $\mathrm{lev}(s)$. The *initial segment partial ordering* on $\mathcal{D}$, denoted by $\le$, is defined as follows: if $s = (a_\xi)_{\xi<\eta_1}$ and $s' = (b_\xi)_{\xi<\eta_2}$ belong to $\mathcal{D}$ then $s \le s'$ if and only if $\eta_1 \le \eta_2$ and $a_\xi = b_\xi$ for any $\xi < \eta_1$. We also write $s < s'$ if $s \le s'$ and $s \ne s'$. By $s \perp s'$ we indicate that s, s' are *incomparable*, that is, neither $s \le s'$ nor $s' \le s$. If $s \le s'$ we say s' is a *follower* of s. Further, the nodes $s \cup \{0\}$ and $s \cup \{1\}$ are called the *successors* of s, that is, we reserve the word successor for immediate follower. However, we observe that a node does not need to have an *immediate predecessor*.

A subset T of $\mathcal{D}$ is called a *subtree* if it is order isomorphic to $\{0,1\}^{<\lambda}$ for some cardinal $\lambda \le \kappa$. In this paper, we only use countable subtrees of $\mathcal{D}$, that is, subtrees which are order isomorphic to $\{0,1\}^{<\aleph_0}$. If T is countable, we enumerate its elements as $T = \{t_1, t_2, \dots\}$ where t_1 is the minimum element of T and for each $m \in \mathbb{N}$, t_{2m}, t_{2m+1} are the successors of t_m (in the tree T).

A linearly ordered subset $\mathcal{I}$ of $\mathcal{D}$ is called a *segment* if for every $s < t < s'$, t is contained in $\mathcal{I}$ provided that s, s' belong to $\mathcal{I}$. Consider now a non-empty segment $\mathcal{I}$. Let η_1 be the least ordinal such that there exists a node $s \in \mathcal{I}$ with $\mathrm{lev}(s) = \eta_1$. Suppose further that there are an ordinal η and a node s' at the ηth level such that $s \leq s'$ for every $s \in \mathcal{I}$. Let η_2 be the least ordinal with this property. Then we say that $\mathcal{I}$ is an η_1-η_2 *segment*. A segment is called *initial* if $\eta_1 = 0$, that is, $\emptyset \in \mathcal{I}$.

We next define admissible families of segments in the sense of Hagler [3]. Suppose that $\{\mathcal{I}_j\}_{j=1}^r$ is a finite family of segments. This family is called *admissible* if:

(1) there exist ordinals $\eta_1 < \eta_2$ such that $\mathcal{I}_j$ is an η_1-η_2 segment for each $j = 1, \ldots, r$;
(2) $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ provided that $i \neq j$.

Consider now the vector space $c_{00}(\mathcal{D})$ of finitely supported functions $x : \mathcal{D} \to \mathbb{R}$. For any segment $\mathcal{I}$ of $\mathcal{D}$, we set $\mathcal{I}^* : c_{00}(\mathcal{D}) \to \mathbb{R}$ with $\mathcal{I}^*(x) = \sum_{s \in \mathcal{I}} x(s)$. Then, for any $x \in c_{00}(\mathcal{D})$, we define the norm

$$\|x\| = \sup \Big[\sum_{j=1}^{r} |\mathcal{I}_j^*(x)|^2 \Big]^{1/2}$$

where the supremum is taken over all finite, admissible families $\{\mathcal{I}_j\}_{j=1}^r$ of segments. The space X_κ is the completion of the normed space $(c_{00}(\mathcal{D}), \|\cdot\|)$ just defined.

For every node $s \in \mathcal{D}$, we define $e_s : \mathcal{D} \to \mathbb{R}$ by $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$ otherwise. Clearly, $\|e_s\| = 1$ for any $s \in \mathcal{D}$.

We come now to the final definition. Suppose that $\{s_i \mid i \in I\}$ is a family of nodes of the tree $\mathcal{D}$. This family is called *strongly incomparable* (see [3]) if:

(1) $s_i \perp s_j$ provided that $i \neq j$;
(2) if $\{S_1, \ldots, S_r\}$ is any admissible family of segments, then at most two of the s_i's, $i \in I$, are contained in $S_1 \cup \cdots \cup S_r$.

There is a standard way of constructing strongly incomparable families of nodes. Suppose that $(s_\xi)_{\xi < \eta}$ is a set of nodes, where $\eta < \kappa$, such that $s_0 < s_1 < \cdots$. For any ordinal $\xi < \eta$, let t_ξ be the successor of s_ξ with $t_\xi \perp s_{\xi+1}$. Then the family $\{t_\xi \mid \xi < \eta\}$ is strongly incomparable.

Concerning strongly incomparable sets of nodes, we quote the following proposition whose proof is straightforward.

Proposition 2.1. *Suppose that $\{s_i \mid i \in I\}$ is a strongly incomparable set of nodes in the tree $\mathcal{D}$. Then the family $\{e_{s_i} \mid i \in I\}$ is equivalent to the usual basis of $c_0(I)$. More precisely, for any $n \in \mathbb{N}$, any $i_1, \ldots, i_n \in I$ and* *any scalars* $a_1, \dots, a_n$*, we have*

$$\max_{1\le k\le n}|a_k| \le \Big\|\sum_{k=1}^{n} a_k e_{s_{i_k}}\Big\| \le \sqrt{2}\max_{1\le k\le n}|a_k|.$$

3. The main results. Suppose that $B = (a_\xi)_{\xi<\kappa} \in \Gamma$ is any branch. Then B can be naturally identified with a maximal segment of $\mathcal{D}$, namely $B = \{s_0 < s_1 < \cdots < s_\eta < \cdots\}$ where $s_0 = \emptyset$ and $s_\eta = (a_\xi)_{\xi<\eta}$ for any ordinal $\eta < \kappa$. In Section 2, we defined the linear functional $B^* : c_{00}(\mathcal{D}) \to \mathbb{R}$ by setting $B^*(x) = \sum_{s\in B} x(s)$. Clearly, $\|B^*\| = 1$. This functional can be extended to a bounded functional on X_κ, having the same norm and denoted again by B^*. Let also Γ^* denote the set of all functionals B^* defined above. Then Γ^* is a bounded subset of X_κ^* of cardinality 2^κ.

This section is devoted to the study of the family Γ^*. We first prove the following.

Theorem 3.1. *Suppose that* $(B_n)_{n\in\mathbb{N}}$ *is a sequence of branches such that* $B_n \neq B_m$ *for* $n \neq m$*. Then* $(B_n^*)_{n\in\mathbb{N}}$ *contains a subsequence equivalent to the usual* ℓ_1*-basis.*

Proof. Consider the set $\mathcal{A}$ of all ordinals $\eta < \kappa$ which satisfy the following: there are nodes $\varphi \neq t$ with $\mathrm{lev}(\varphi) = \mathrm{lev}(t) = \eta$ and there are positive integers $m_1 \neq m_2$ such that $\varphi \in B_{m_1}$ and $t \in B_{m_2}$. Clearly $\mathcal{A}$ is a non-empty set, so we can consider its least element, say η. Then η cannot be a limit ordinal. Indeed, let $\varphi = (a_\xi)_{\xi<\eta}$ and $t = (b_\xi)_{\xi<\eta}$ be as above. Since $\varphi \neq t$, there exists $\eta_1 < \eta$ with $a_{\eta_1} \neq b_{\eta_1}$. We set $\tilde\varphi = (a_\xi)_{\xi<\eta_1+1}$ and $\tilde t = (b_\xi)_{\xi<\eta_1+1}$. Then $\tilde\varphi \neq \tilde t$, these nodes are at the same level and $\tilde\varphi \le \varphi$, $\tilde t \le t$. Hence, $\tilde\varphi \in B_{m_1}$ and $\tilde t \in B_{m_2}$. By the minimality of η, we conclude that $\eta = \eta_1 + 1$.

Furthermore, the minimality of η also implies that there exists a node s_1 at level η_1 so that $s_1 \in B_m$ for every $m \in \mathbb{N}$, and the nodes φ, t at level $\eta = \eta_1 + 1$ are precisely the successors of s_1. Now, we set $\varphi_1 = \varphi$ and $t_1 = t$. We may assume that there are infinitely many terms of the sequence $(B_m)_{m\in\mathbb{N}}$ which pass through the node φ_1. Then we choose a branch B_{l_1} passing through the node t_1 (clearly such a branch does exist). B_{l_1} is just the first term of the desired subsequence.

We next set $N_1 = \{m \in \mathbb{N} \mid m > l_1 \text{ and } \varphi_1 \in B_m\}$. Then N_1 is an infinite subset of $\mathbb{N}$. Repeating the previous argument for the branches $(B_m)_{m\in N_1}$, we find an ordinal $\eta_2 > \eta_1 + 1$ and a node s_2 at the η_2th level, with successors φ_2 and t_2, such that

- all branches B_m, $m \in N_1$, pass through s_2;
- infinitely many branches of the sequence $(B_m)_{m\in N_1}$ pass through φ_2 and the set $\{m \in N_1 \mid t_2 \in B_m\}$ is non-empty.

We also choose a branch B_{l_2} so that $t_2 \in B_{l_2}$.

Continuing in the obvious manner, we inductively construct a sequence $s_1 < s_2 < \cdots$ of nodes of $\mathcal{D}$, with the successors of s_i denoted by φ_i and t_i, and a sequence $l_1 < l_2 < \cdots$ of positive integers such that:

(1) $s_1 < \varphi_1 \leq s_2 < \varphi_2 \leq \cdots$;
(2) $s_i \in B_{l_j}$ for any $j \geq i$, but the branches B_{l_j}, $j > i$, pass through φ_i while the branch B_{l_i} passes through t_i.

We prove now that $(B^*_{l_m})_{m\in\mathbb{N}}$ is equivalent to the usual ℓ_1-basis. Let $M \in \mathbb{N}$ and $a_1, \ldots, a_M \in \mathbb{R}$ be given. We set $x = \sum_{i=1}^M \operatorname{sgn}(a_i) e_{t_i}$. Condition (1) of the above construction implies that the sequence (t_i) is strongly incomparable. Hence, $\|x\| = \sqrt{2}$ by Proposition 2.1. Furthermore, condition (2) implies that $t_i \in B_{l_i} \setminus \bigcup\{B_{l_j} \mid j \neq i\}$, thus $B_{l_j}(e_{t_i}) = \delta_{ij}$. Therefore

$$\Big\|\sum_{i=1}^M a_i B^*_{l_i}\Big\| \geq \frac{1}{\|x\|}\Big|\sum_{i=1}^M a_i B^*_{l_i}(x)\Big| = \frac{1}{\sqrt{2}}\Big|\sum_{i=1}^M a_i \operatorname{sgn}(a_i)\Big| = \frac{1}{\sqrt{2}}\sum_{i=1}^M |a_i|.$$

Hence $\frac{1}{\sqrt{2}}\sum_{i=1}^M |a_i| \leq \|\sum_{i=1}^M a_i B^*_{l_i}\|$. Since clearly $\|\sum_{i=1}^M a_i B^*_{l_i}\| \leq \sum_{i=1}^M |a_i|$, the proof is complete. ∎

Corollary 3.2. *The set Γ^* contains no weakly Cauchy sequence.*

We pass now to the second result concerning the set of functionals $\{B^* \mid B \in \Gamma\}$.

Theorem 3.3. *No subset of Γ^* is equivalent to the usual $\ell_1(\kappa^+)$-basis.*

For the proof we need to establish some lemmas. Before proceeding, let us introduce some notation. First of all, if A is any set, then $|A|$ denotes the cardinality of A. Suppose now that $\Delta \subseteq \Gamma$ is a set of branches. For any node $s \in \mathcal{D}$, we denote by Δ_s the set of all branches $B \in \Delta$ passing through s, that is, $\Delta_s = \{B \in \Delta \mid s \in B\}$. We also set $\Delta^c_s = \Delta \setminus \Delta_s = \{B \in \Delta \mid s \notin B\}$.

Lemma 3.4. *Let $\Delta \subseteq \Gamma$ be a set of branches with $|\Delta| = \kappa^+$. Then there exists a node $s \in \mathcal{D}$ such that $|\Delta_{s\cup\{0\}}| = |\Delta_{s\cup\{1\}}| = \kappa^+$.*

Proof. Assume that the assertion is not true. Then for every node $s \in \mathcal{D}$ there is a successor $s \cup \{\epsilon\}$ of s, where $\epsilon = 0$ or 1, such that $|\Delta_{s\cup\{\epsilon\}}| < \kappa^+$. With this assumption and using transfinite induction we construct a branch $B = \{s_\eta\}_{\eta<\kappa} = \{s_0 < s_1 < \cdots\}$ with $|\Delta_{s_\eta}| = \kappa^+$ for any $\eta < \kappa$.

We start with $s_0 = \emptyset$. Clearly, $|\Delta_\emptyset| = |\Delta| = \kappa^+$. Suppose now that η is an ordinal, $\eta < \kappa$, and we have defined the nodes $\{s_\xi\}_{\xi<\eta}$ with $\operatorname{lev}(s_\xi) = \xi$ and $|\Delta_{s_\xi}| = \kappa^+$ for any $\xi < \eta$.

If $\eta = \eta_0 + 1$, then by the inductive hypothesis we have $|\Delta_{s_{\eta_0}}| = \kappa^+$. Clearly, $\Delta_{s_{\eta_0}} = \Delta_{s_{\eta_0}\cup\{0\}} \cup \Delta_{s_{\eta_0}\cup\{1\}}$. Therefore, there exists a successor $s_{\eta_0} \cup \{\epsilon\}$ (where $\epsilon = 0$ or 1) of s_{η_0} such that $|\Delta_{s_{\eta_0}\cup\{\epsilon\}}| = \kappa^+$. Let $s_\eta = s_{\eta_0} \cup \{\epsilon\}$.

If η is a limit ordinal, we set $s_\eta = \bigcup_{\xi<\eta} s_\xi$. Then s_η is a node at the ηth level of $\mathcal{D}$. It remains to show that $|\Delta_{s_\eta}| = \kappa^+$. Since $\Delta = \Delta_{s_\eta} \cup \Delta^c_{s_\eta}$, it suffices to prove that $|\Delta^c_{s_\eta}| \le \kappa$.

Let us consider a branch B belonging to $\Delta^c_{s_\eta}$, that is, $s_\eta \notin B$. We also denote by S the initial segment $\{s_\xi\}_{\xi\le\eta}$. We consider now the set $\mathcal{A}$ containing all ordinals $\xi \le \eta$ such that at the ξth level of $\mathcal{D}$, the segments B and S do not pass through the same node. The set $\mathcal{A}$ is non-empty as $\eta \in \mathcal{A}$. Therefore $\mathcal{A}$ has a minimum element, say ξ_0. The minimality implies that ξ_0 cannot be a limit ordinal. Hence $\xi_0 = \xi + 1$. Further, by the minimality of ξ_0, at level ξ we have $s_\xi \in B$ and $s_\xi \in S$, while at level $\xi+1$, $s_{\xi+1} \in S$ and $s_{\xi+1} \notin B$. Consequently,

$$\Delta^c_{s_\eta} = \bigcup_{\xi<\eta} \{B \in \Delta \mid s_\xi \in B \text{ and } s_{\xi+1} \notin B\} = \bigcup_{\xi<\eta} (\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}).$$

Observe that $s_{\xi+1}$ is a successor of s_ξ, $|\Delta_{s_\xi}| = |\Delta_{s_{\xi+1}}| = \kappa^+$ and $\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}$ consists of all branches $B \in \Delta$ which pass through the other successor of s_ξ. By our assumption at the beginning of the proof, we have $|\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}| \le \kappa$ and therefore $|\Delta^c_{s_\eta}| \le \sum_{\xi<\eta} \kappa = \kappa$.

Thus a branch $B = \{s_\eta\}_{\eta<\kappa}$ has been constructed with $|\Delta_{s_\eta}| = \kappa^+$ for any $\eta < \kappa$. To complete the proof, we only need to repeat our last argument. Consider a branch $\tilde{B} \in \Delta$ with $\tilde{B} \ne B$. Let ξ_0 be the minimum ordinal such that at the ξ_0th level the branches $\tilde{B}, B$ do not pass through the same node. The minimality of ξ_0 implies that $\xi_0 = \xi + 1$, $s_\xi \in \tilde{B}$ and $s_{\xi+1} \notin \tilde{B}$. Therefore

$$\Delta \subseteq \{B\} \cup \bigcup_{\xi<\kappa} (\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}).$$

Since $|\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}| \le \kappa$, it follows that $|\Delta| \le \kappa$ and we have reached a contradiction. ∎

Lemma 3.5. *Let $\Delta \subset \Gamma$ be a set of branches with $|\Delta| = \kappa^+$. Then there exists a countable subtree T of $\mathcal{D}$, $T = \{t_1, t_2, \ldots\}$, such that:*

(1) *$|\Delta_{t_m}| = \kappa^+$ for any node $t_m \in T$;*
(2) *for any $t_m \in T$ there exists a node $s_m \in \mathcal{D}$ such that $t_m \le s_m$ and t_{2m}, t_{2m+1} are the successors of s_m (that is, when we look at the tree $\mathcal{D}$, the successors of t_m remain the successors of some $s_m \in \mathcal{D}$).*

Proof. Let $t_1 = \emptyset$. By Lemma 3.4, there exists a node $s_1 \in \mathcal{D}$ with $t_1 \le s_1$ such that $|\Delta_{s_1\cup\{0\}}| = |\Delta_{s_1\cup\{1\}}| = \kappa^+$. We set $t_2 = s_1 \cup \{0\}$ and $t_3 = s_1 \cup \{1\}$. Then t_2, t_3 are the successors of t_1 in T, and they are also the successors of s_1 when we look at the tree $\mathcal{D}$.

Applying Lemma 3.4 to the family $\Delta_{s_1\cup\{0\}} = \Delta_{t_2}$ we find a node $s_2 \in \mathcal{D}$ with $t_2 \le s_2$ such that $|\Delta_{s_2\cup\{0\}}| = |\Delta_{s_2\cup\{1\}}| = \kappa^+$. Then the successors of

t_2 in T are the nodes $t_4 = s_2 \cup \{0\}$ and $t_5 = s_2 \cup \{1\}$. We continue in the obvious manner. ■

Proof of Theorem 3.3. Assume that $\Delta \subseteq \Gamma$ is a set of branches with $|\Delta| = \kappa^+$, and $\Delta^* = \{B^* \mid B \in \Delta\}$ is equivalent to the usual $\ell_1(\kappa^+)$-basis. Then there exists a constant $\delta > 0$ such that for any $n \in \mathbb{N}$, any $B_1, \ldots, B_n \in \Delta$ and any scalars $a_1, \ldots, a_n$,

$$\delta \sum_{i=1}^{n} |a_i| \leq \Big\| \sum_{i=1}^{n} a_i B_i^* \Big\| \leq \sum_{i=1}^{n} |a_i|.$$

Let T be the countable subtree of $\mathcal{D}$ given by Lemma 3.5 and let $n \in \mathbb{N}$ be any positive integer. Then we choose branches $B_1, \ldots, B_n$ and $B_{n+1}, \ldots, B_{2n}$ belonging to Δ as follows. We work at the nth level of T, which consists of the nodes $t_{2^n}, t_{2^n+1}, \ldots, t_{2^{n+1}-1}$. If we consider the pair t_{2^n}, t_{2^n+1}, the construction of T implies that these nodes are the successors of some node of $\mathcal{D}$. Therefore they belong to the same level of $\mathcal{D}$, say level ξ_1. Similarly the nodes t_{2^n+2}, t_{2^n+3} are placed at the same level of $\mathcal{D}$, say ξ_2, and so on. Finally, let $\xi_{2^{n-1}} = \mathrm{lev}(t_{2^{n+1}-2}) = \mathrm{lev}(t_{2^{n+1}-1})$. We may assume, without loss of generality, that $\xi_1 = \max\{\xi_k \mid 1 \leq k \leq 2^{n-1}\}$. Then we choose branches B_1 and B_{n+1} of the family Δ such that B_1 passes through t_{2^n} and B_{n+1} passes through t_{2^n+1} (such branches exist by Lemma 3.5). If ψ_1 denotes the immediate predecessor of the nodes t_{2^n}, t_{2^n+1} (in $\mathcal{D}$), then the branches B_1, B_{n+1} coincide up to the level of ψ_1 and they separate each other at the next level.

The nodes t_{2^n}, t_{2^n+1} are followers of t_2 in the tree T. We now forget the followers of t_2 and we repeat the previous procedure for the nodes belonging to the nth level of T which are followers of t_3. That is, we detect the pair, say t_{2^n+2k}, t_{2^n+2k+1}, which is placed at the highest level of $\mathcal{D}$ (if this is not unique, we simply choose one). Then we choose branches B_2, B_{n+2} belonging to Δ such that B_2 passes through the left-hand node of the pair, i.e. t_{2^n+2k}, and B_{n+2} passes through the right-hand node t_{2^n+2k+1}. Let ψ_2 denote the immediate predecessor of t_{2^n+2k}, t_{2^n+2k+1} in $\mathcal{D}$. Then $\mathrm{lev}(\psi_1) \geq \mathrm{lev}(\psi_2)$. The branches B_2, B_{n+2} coincide up to the level of ψ_2. We also notice that the branches B_1, B_2 separate each other before the level of t_2, t_3 and this happens for the branches B_{n+1}, B_{n+2}. The nodes t_{2^n+2k}, t_{2^n+2k+1} are followers of either t_6 or t_7. If t_6 is a predecessor of t_{2^n+2k}, t_{2^n+2k+1}, then we forget the followers of t_6 and we continue with the nodes belonging to the nth level of T which are followers of t_7.

After $n-1$ iterations of the previous argument, we find branches $B_1, \ldots, B_{n-1}$ and $B_{n+1}, \ldots, B_{2n-1}$ belonging to the family Δ and nodes $\psi_1, \ldots, \psi_{n-1}$ of $\mathcal{D}$. At this stage only one pair of nodes at the nth level of T has been left. Let ψ_n be the immediate predecessor on $\mathcal{D}$ of these nodes. We choose $B_n, B_{2n} \in \Delta$ such that B_n passes through the left-hand node and B_{2n} passes through the right-hand node.

Now we observe that $B_1, \dots, B_n$ are pairwise disjoint below the level of ψ_n and this also holds for $B_{n+1}, \dots, B_{2n}$. Therefore, if $\eta_1 = \mathrm{lev}(\psi_n)$ and $\eta_2 = \mathrm{lev}(\psi_1)$, then:

(1) All segments $B_i \cap \{s \mid \mathrm{lev}(s) \geq \eta_2 + 1\}$, $i = 1, \dots, 2n$, are pairwise disjoint.
(2) The segments $B_i \cap \{s \mid \eta_1 + 1 \leq \mathrm{lev}(s) \leq \eta_2\}$ for $i = 1, \dots, n$ are pairwise disjoint. Hence they are admissible $(\eta_1{+}1)$-$(\eta_2{+}1)$ segments. Similarly, $B_i \cap \{s \mid \eta_1 + 1 \leq \mathrm{lev}(s) \leq \eta_2\}$, $i = n+1, \dots, 2n$, form an admissible family.
(3) $B_i \cap \{s \mid \mathrm{lev}(s) \leq \eta_1\} = B_{n+i} \cap \{s \mid \mathrm{lev}(s) \leq \eta_1\}$ for any $i = 1, \dots, n$. Let us also denote $S_i = B_i \cap \{s \mid \mathrm{lev}(s) \leq \eta_1\}$.

After the choice of $(B_i)_{i=1}^{2n}$ has been completed, our next purpose is to estimate the norm of the functional $\sum_{i=1}^{2n} a_i B_i^*$ for any scalars $a_1, \dots, a_{2n}$ and to contradict the assumption that Δ^* is equivalent to the usual $\ell_1(\kappa^+)$-basis. For this reason, we consider a finitely supported vector $x = \sum_{s \in \mathcal{D}} \lambda_s e_s \in X_\kappa$ with $\|x\| \leq 1$. We can write $x = x_1 + x_2 + x_3$, where $x_1 = \sum_{\mathrm{lev}(s) \leq \eta_1} \lambda_s e_s$, $x_2 = \sum_{\eta_1 + 1 \leq \mathrm{lev}(s) \leq \eta_2} \lambda_s e_s$ and $x_3 = \sum_{\eta_2 + 1 \leq \mathrm{lev}(s)} \lambda_s e_s$. Clearly, $\|x_j\| \leq \|x\| = 1$ for any $j = 1, 2, 3$. Then

$$\Big|\sum_{i=1}^{2n} a_i B_i^*(x)\Big| \leq \Big|\sum_{i=1}^{2n} a_i B_i^*(x_1)\Big| + \Big|\sum_{i=1}^{2n} a_i B_i^*(x_2)\Big| + \Big|\sum_{i=1}^{2n} a_i B_i^*(x_3)\Big|.$$

Now we have

$$\Big|\sum_{i=1}^{2n} a_i B_i^*(x_3)\Big| \leq \Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{2n} |B_i^*(x_3)|^2\Big)^{1/2} \leq \Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2},$$

$$\begin{aligned}\Big|\sum_{i=1}^{2n} a_i B_i^*(x_2)\Big| &\leq \Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2} \Big(\sum_{i=1}^{n} |B_i^*(x_2)|^2 + \sum_{i=n+1}^{2n} |B_i^*(x_2)|^2\Big)^{1/2} \\ &\leq \Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2} (2\|x_2\|^2)^{1/2} \leq \sqrt{2}\Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2},\end{aligned}$$

$$\begin{aligned}\Big|\sum_{i=1}^{2n} a_i B_i^*(x_1)\Big| &= \Big|\sum_{i=1}^{n} (a_i B_i^*(x_1) + a_{n+i} B_{n+i}^*(x_1))\Big| \\ &= \Big|\sum_{i=1}^{n} (a_i + a_{n+i}) S_i^*(x_1)\Big| \leq \sum_{i=1}^{n} |a_i + a_{n+i}|\, |S_i^*(x_1)| \\ &\leq \sum_{i=1}^{n} |a_i + a_{n+i}|.\end{aligned}$$

Summarizing, for any finitely supported $x \in X_\kappa$ with $\|x\| \le 1$ we have

$$\Big|\sum_{i=1}^{2n} a_i B_i^*(x)\Big| \le (\sqrt{2}+1)\Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2} + \sum_{i=1}^{n} |a_i + a_{n+i}|.$$

Therefore, $\|\sum_{i=1}^{2n} a_i B_i^*\| \le (\sqrt{2}+1)(\sum_{i=1}^{2n} a_i^2)^{1/2} + \sum_{i=1}^{n}|a_i + a_{n+i}|$. On the other hand, Δ^* is equivalent to the usual $\ell_1(\kappa^+)$-basis. It follows that

$$\delta \sum_{i=1}^{2n} |a_i| \le (\sqrt{2}+1)\Big(\sum_{i=1}^{2n} a_i^2\Big)^{1/2} + \sum_{i=1}^{n} |a_i + a_{n+i}|.$$

If we choose $a_1 = \cdots = a_n = 1$ and $a_{n+1} = \cdots = a_{2n} = -1$, then we obtain $\delta \le (\sqrt{2}+1)/\sqrt{2n}$ for any $n \in \mathbb{N}$, a contradiction. ∎

4. The non-separable version of Rosenthal's ℓ_1-theorem. In this section, we show that we cannot achieve a satisfactory extension of Rosenthal's ℓ_1-theorem to spaces of the type $\ell_1(\kappa)$ for κ an uncountable cardinal. As mentioned in the introduction, this extension is possible in only one case, namely when both κ and $\mathrm{cf}(\kappa)$ are strong limit cardinals. For the proof of this result we refer to [2]; we shall discuss the other cases.

Suppose first that κ is not a strong limit cardinal. This means that there exists a cardinal $\lambda < \kappa$ with $\kappa \le 2^\lambda$. We now consider the space X_λ and the corresponding family of functionals $\Gamma^* \subset X_\lambda^*$. Then Γ^* is a bounded subset of X_λ^* whose cardinality is equal to $2^\lambda \ge \kappa$. Further, by Corollary 3.2, the set Γ^* contains no weakly Cauchy sequence and, by Theorem 3.3, no subset of Γ^* is equivalent to the usual $\ell_1(\kappa)$-basis.

We next consider the case where κ is a strong limit cardinal but $\mathrm{cf}(\kappa)$ is not. This case is not so simple as the previous one, but it is essentially based on the arguments developed in Section 3.

Since $\mathrm{cf}(\kappa)$ is not strong limit, there exists a cardinal $\lambda < \mathrm{cf}(\kappa)$ with $\mathrm{cf}(\kappa) \le 2^\lambda$. By the definition of $\mathrm{cf}(\kappa)$, there are cardinals $\{\kappa_i \mid i < \mathrm{cf}(\kappa)\}$ such that $\kappa_i < \kappa$ for any ordinal $i < \mathrm{cf}(\kappa)$, and $\kappa = \sum_{i<\mathrm{cf}(\kappa)} \kappa_i$. We next consider the space X_κ and we choose a family of branches $A \subset \Gamma$ as follows. We focus on the level λ of the tree $\mathcal{D}$. This level consists of the nodes $\{0,1\}^\lambda = \{(a_\xi)_{\xi<\lambda} \mid a_\xi = 0 \text{ or } 1\}$. Therefore, there are 2^λ nodes at level λ. Since $\mathrm{cf}(\kappa) \le 2^\lambda$, we can choose nodes $\{t_i \mid i < \mathrm{cf}(\kappa)\}$ at level λ with $t_i \ne t_j$ provided that $i \ne j$. Now we observe that for any $i < \mathrm{cf}(\kappa)$, the set of all branches passing through t_i has cardinality 2^κ. Hence, for any $i < \mathrm{cf}(\kappa)$, we can choose a family of branches $A_i \subset \Gamma$ such that $|A_i| = \kappa_i$ and each branch belonging to A_i passes through t_i. Finally, let $A = \bigcup_{i<\mathrm{cf}(\kappa)} A_i$ and let A^* be the family of the corresponding functionals, that is, $A^* = \{B^* \mid B \in A\}$.

Clearly, the choice of A implies that $|A^*| = |A| = \sum_{i<\mathrm{cf}(\kappa)} \kappa_i = \kappa$. Furthermore, by Corollary 3.2, A^* contains no weakly Cauchy sequence. So, it remains to show that no subset of A^* is equivalent to the usual $\ell_1(\kappa)$-basis. The proof follows the lines of the proof of Theorem 3.3. We describe briefly the part corresponding to Lemma 3.4.

LEMMA 4.1. *Let Δ be a subset of A with $|\Delta| = \kappa$. Then there exists a node $s \in \mathcal{D}$ such that* $\mathrm{lev}(s) < \lambda$ *and* $|\Delta_{s\cup\{0\}}| = |\Delta_{s\cup\{1\}}| = \kappa$. (*Recall that* $\Delta_s = \{B \in \Delta \mid s \in B\}$.)

Proof. Assuming that the assertion is not true, we construct an initial segment $S = \{s_\eta\}_{\eta<\lambda} = \{s_0 < s_1 < \cdots\}$ such that $|\Delta_{s_\eta}| = \kappa$ for any $\eta < \lambda$. We start with $s_0 = \emptyset$. If $\eta = \eta_0 + 1$, then s_η is one of the followers of s_{η_0}. If η is a limit ordinal, then we set $s_\eta = \bigcup_{\xi<\eta} s_\xi$. Clearly, s_η is a node at the ηth level of $\mathcal{D}$. We next show that

$$\Delta^c_{s_\eta} = \bigcup_{\xi<\eta} (\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}).$$

Therefore,

$$|\Delta^c_{s_\eta}| = \sum_{\xi<\eta} |\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}| < \kappa,$$

since $|\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}| < \kappa$ and $\eta < \lambda < \mathrm{cf}(\kappa)$. Hence $|\Delta_{s_\eta}| = \kappa$ and this completes the construction of S.

Finally, we set $s_\lambda = \bigcup_{\xi<\lambda} s_\xi$. Then s_λ belongs to level λ and as previously we show $|\Delta_{s_\lambda}| = \kappa$. However, the choice of A indicates that $|\Delta_s| < \kappa$ for any node s at level λ, and we have reached a contradiction. ■

Using Lemma 4.1, we construct a countable subtree $T = \{t_1, t_2, \ldots\}$ of $\mathcal{D}$ such that:

(1) $|\Delta_{t_m}| = \kappa$ for any $m = 1, 2, \ldots$ (therefore, $\mathrm{lev}(t_m) < \lambda$);
(2) the successors t_{2m}, t_{2m+1} of t_m are the successors of some $s_m \in \mathcal{D}$.

Finally, we repeat the proof of Theorem 3.3 to show that no subset Δ^* of A^* is equivalent to the usual $\ell_1(\kappa)$-basis.

5. The structure of the subspaces of X_κ. The structure of subspaces of the James Tree space (JT) and the Hagler Tree space (HT) has been studied extensively, since it has provided answers to several questions about Banach spaces. By analogy, the structure of subspaces of X_κ seems quite interesting. This section is devoted to some remarks concerning this issue.

First of all, X_κ contains a lot of subspaces isomorphic to $c_0(\kappa)$. Indeed, let $B = \{s_\eta\}_{\eta<\kappa}$ be any branch and, for any $\eta < \kappa$, let t_η be the successor of s_η with $t_\eta \neq s_{\eta+1}$. Then $\{t_\eta \mid \eta < \kappa\}$ is a strongly incomparable family of nodes. By Proposition 2.1, the subspace $\overline{\mathrm{span}}\{e_{t_\eta} \mid \eta < \kappa\}$ is isomorphic to $c_0(\kappa)$. Furthermore, it is easy to verify that for any ordinal $\eta < \kappa$ the subspace $\overline{\mathrm{span}}\{e_s \mid s \in \{0,1\}^\eta\}$ is isometrically isomorphic to $\ell_2(2^\eta)$. The main properties of the spaces JT and HT suggest now the following problem about subspaces of X_κ.

PROBLEM 5.1. *Is it true that no subspace of X_κ is isomorphic to $\ell_1(\kappa)$?*

Concerning the above problem, we prove a partial result. Assume that $B = \{s_\eta\}_{\eta<\kappa}$ is any branch of the tree $\mathcal{D}$. Then we show that the subspace generated by this branch, that is, $\overline{\operatorname{span}}\{e_{s_\eta}\}_{\eta<\kappa}$, does not contain any copy of $\ell_1(\kappa)$.

For convenience, we first define a Banach space isometrically isomorphic to the subspace generated by any branch. Let κ be an infinite cardinal. We consider the vector space $c_{00}(\{\eta \mid \eta < \kappa\})$ consisting of all finitely supported functions $x : \{\eta \mid \eta < \kappa\} \to \mathbb{R}$. For any such x, we set

$$\|x\| = \sup\{|S^*(x)|\}$$

where the supremum is taken over all segments $S \subseteq \{\eta \mid \eta < \kappa\}$. If E_κ denotes the completion of the normed space just defined, then E_κ is isometrically isomorphic to the subspace of X_κ generated by any branch.

As usual, for any ordinal $\eta < \kappa$, we consider the vector $e_\eta \in E_\kappa$ with $e_\eta(\xi) = 1$ if $\xi = \eta$ and $e_\eta(\xi) = 0$ otherwise. We now define some projections on the space E_κ. Let η be any ordinal, $\eta < \kappa$. We define $P_\eta : \operatorname{span}\{e_\xi\}_{\xi<\kappa} \to \operatorname{span}\{e_\xi\}_{\xi<\eta}$ as follows: if $x = \sum_{\xi<\kappa} x(\xi)e_\xi$ is finitely supported, then $P_\eta(x) = \sum_{\xi<\eta} x(\xi)e_\xi$. Clearly, P_η is a linear projection with $\|P_\eta\| = 1$. We can also extend P_η continuously to obtain a projection $P_\eta : E_\kappa \to E_\kappa$ onto $\overline{\operatorname{span}}\{e_\xi\}_{\xi<\eta}$ with $\|P_\eta\| = 1$. We next prove the following.

PROPOSITION 5.2. *The space E_κ contains no isomorphic copy of $\ell_1(\kappa)$.*

Proof. Suppose, on the contrary, that $\ell_1(\kappa)$ embeds isomorphically into E_κ. Then we find a subset $\{x_\xi \mid \xi < \kappa\}$ of E_κ which is equivalent to the usual $\ell_1(\kappa)$-basis. Without loss of generality, we may assume that x_ξ is finitely supported and $\|x_\xi\| = 1$ for any $\xi < \kappa$.

We inductively construct a sequence $(y_m)_{m=0}^\infty$ belonging to $\operatorname{span}\{e_\xi\}_{\xi<\kappa}$ with the following properties:

(1) $\|y_m\| = 1$ for each m;
(2) if $A_m \subset \{\xi < \kappa\}$ is the support of y_m then $\max A_m < \min A_{m+1}$ for any m;
(3) $(y_m)_{m=0}^\infty$ is a block sequence of $(x_\xi)_{\xi<\kappa}$, that is, there are ordinals $\eta_0 < \eta_1 < \cdots$ such that $y_m \in \operatorname{span}\{x_\xi \mid \eta_m \le \xi < \eta_{m+1}\}$.

We start with $y_0 = x_0$, $\eta_0 = 0$ and $\eta_1 = 1$. Let $\xi_1 = \max A_0 + 1$. We claim that there exists $y \in \operatorname{span}\{x_\xi\}_{\xi\ge1}$, $y \ne 0$, such that $P_{\xi_1}(y) = 0$. Indeed, if we assume that $P_{\xi_1}(y) \ne 0$ for all $y \in \operatorname{span}\{x_\xi\}_{\xi\ge1}$, $y \ne 0$, then the linear operator $P_{\xi_1} : \operatorname{span}\{x_\xi\}_{\xi\ge1} \to \operatorname{span}\{e_\xi\}_{\xi<\xi_1}$ is one-to-one. Since $\{x_\xi\}_{\xi\ge1}$ are linearly independent, it follows that the (algebraic) dimension of $\operatorname{span}\{e_\xi\}_{\xi<\xi_1}$ is κ, which is a contradiction. Therefore, there is $y \in \operatorname{span}\{x_\xi\}_{\xi\geq 1}$ such that $y \neq 0$ and $P_{\xi_1}(y) = 0$. We set $y_1 = y/\|y\|$. Since $P_{\xi_1}(y) = 0$, we have $\max A_0 < \min A_1$. Moreover, we can choose an ordinal $\eta_2 > \eta_1$ such that $y \in \operatorname{span}\{x_\xi \mid \eta_1 \leq \xi < \eta_2\}$. Repeatedly applying the previous argument, we construct the desired sequence $(y_m)_{m=0}^{\infty}$.

Since $(x_\xi)_{\xi<\kappa}$ is equivalent to the usual $\ell_1(\kappa)$-basis, it is easy to verify that (y_m) is equivalent to the usual ℓ_1-basis. Furthermore, (y_m) belongs to $\operatorname{span}\{e_\xi \mid \xi \in \bigcup_{m=0}^{\infty} A_m\}$. The latter space is isometrically isomorphic to $E_{\aleph_0}$, which in turn is isomorphic to c_0 (see [3]). That is, in a space isomorphic to c_0 we have found a copy of ℓ_1, which is a contradiction. ∎

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