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# Dynamics of differentiation and integration operators on weighted spaces of entire functions

by

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**Abstract.** We investigate the dynamical behavior of the operators of differentiation and integration and the Hardy operator on weighted Banach spaces of entire functions defined by integral norms. In particular we analyze when they are hypercyclic, chaotic, power bounded, and (uniformly) mean ergodic. Moreover, we estimate the norms of the operators and study their spectra. Special emphasis is put on exponential weights.

**1. Introduction and notation.** In this article we are concerned with the dynamics of the following three operators on weighted spaces of entire functions: the differentiation operator $Df(z) = f'(z)$, the integration operator $Jf(z) = \int_0^z f(\zeta)\,d\zeta$ and the Hardy operator $Hf(z) = z^{-1}\int_0^z f(\zeta)\,d\zeta$, $z \in \mathbb{C}$.

In [BBF], Bonet, Fernández and the author study these operators on weighted Banach spaces of entire functions defined by means of supremum norms. The continuity of the differentiation and the integration operators on these spaces was considered by Harutyunyan and Lusky [HL], and the spectrum of the differentiation operator was studied by Atzmon and Brive [AB]. Bonet [Bo] investigated when the operator of differentiation is hypercyclic or chaotic on weighted Banach spaces of entire functions.

It is our purpose to extend the work in [BBF] to more general spaces of entire functions such as weighted spaces of entire functions $B_{p,q}(v)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ or $q = 0$, determined by a weight $v$. Bonet and Bonilla [BB] also extend the results of [Bo] to generalized weighted Bergman spaces $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q \in \{0, \infty\}$, giving conditions that ensure that the differentiation operator is chaotic, hypercyclic or frequently hypercyclic. Similar spaces of holomorphic functions on the disc have been considered by Blasco [B] and by Blasco and de Souza [BS].

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Given a Banach space $X$, we denote by $\mathcal{L}(X)$ the space of continuous and linear operators $T : X \to X$. For $x \in X$, we denote by $\mathrm{Orb}(x,T) := \{x, Tx, T^2x, \ldots\}$ its *orbit under* $T$, and we say that a point $x \in X$ is *periodic* if there is some $n \in \mathbb{N}$ such that $T^n x = x$. An operator $T : X \to X$ is called *topologically transitive* if, for any non-empty open subsets $U, V$ of $X$, there exists some $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$, and $T$ is called *topologically mixing* if, for any such $U, V$, there exists some $N \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

The operator $T \in \mathcal{L}(X)$ is called *hypercyclic* if there is $x \in X$ with a dense orbit, and *chaotic* if it is hypercyclic and it has a dense set of periodic points. By *Birkhoff's transitivity criterion* (see [GEP, Theorem 1.16]), if $X$ is separable, then $T$ is hypercyclic if and only if it is topologically transitive. The first simple criterion ensuring that an operator $T$ on a separable completely metrizable topological vector space is hypercyclic (even topologically mixing) was presented by Kitai in her 1982 thesis (see [GEP, Theorem 3.4]). It was discovered independently by Gethner and Shapiro (1987) and was improved by several authors. A weakening of the Kitai–Gethner–Shapiro criterion is the famous Hypercyclicity Criterion (1999) due to Bès and Peris (see [BP] and [BBP, 17]). The assumptions in the weaker form of the criterion given below do not imply that the operator is topologically mixing.

Hypercyclicity Criterion. *Let $T : X \to X$ be an operator on a separable completely metrizable topological vector space $X$. Suppose that there are dense subsets $Y_0, Y_1 \subseteq X$, an increasing sequence $\{n_k\}_k$ of positive integers, and maps $S_{n_k} : Y_1 \to X$, $k \geq 1$, not necessarily linear or continuous, such that:*

(i) $T^{n_k} x \to 0$ *for each* $x \in Y_0$,
(ii) $S_{n_k} y \to 0$ *for each* $y \in Y_1$, *and*
(iii) $T^{n_k} S_{n_k} y \to y$ *for each* $y \in Y_1$.

*Then $T$ is hypercyclic.*

If the Hypercyclicity Criterion is satisfied for the sequence of all positive integers, then the proof shows that $T$ is even topologically mixing. Bès and Peris proved that an operator $T$ satisfies the assumptions of the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic on $X \oplus X$. Only very recently, De La Rosa and Read [DRR] were able to exhibit hypercyclic operators which do not satisfy the Hypercyclicity Criterion, thus solving a long standing problem. Their example was later improved by Bayart and Matheron [BM], who presented examples defined on classical Banach sequence spaces.

A vector $x \in X$ is called *frequently hypercyclic* for $T$ if, for every non-empty open subset $U$ of $X$,

$$\underline{\mathrm{dens}}\{n \in \mathbb{N} : T^n x \in U\} > 0,$$

where

$$\underline{\mathrm{dens}}(A) = \liminf_{N\to\infty} \frac{\sharp\{n \in A : n \le N\}}{N}$$

denotes the lower density of a subset $A$ of $\mathbb{N}$ and $\sharp$ denotes cardinality. The operator $T$ is called *frequently hypercyclic* if it has a frequently hypercylic vector. The orbit of a frequently hypercyclic vector is therefore, in the specified sense, frequently recurrent. Obviously, frequent hypercyclicity is a stronger notion than hypercyclicity.

According to Bayart and Grivaux [BGr], a bounded operator $T$ on a Banach space $X$ is said to have a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure $\sigma$ on the unit circle $\mathbb{T}$ such that for every $\sigma$-measurable subset $A$ of $\mathbb{T}$ of $\sigma$-measure 1, $\mathrm{span}(\bigcup\{\mathrm{Ker}(T-\lambda I) : \lambda \in A\})$ is dense in $X$. Grivaux [Gr, Theorem 1.4] proved that a bounded operator with a perfectly spanning set of eigenvectors associated to unimodular eigenvalues is always frequently hypercyclic. For more background about linear dynamics see the books by Bayart and Matheron [BM] and by Grosse-Erdmann and Peris [GEP].

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\sup_{m\ge 0} \|T^m\| < \infty$, i.e., the orbit $\{x, Tx, T^2x, \ldots\}$ is bounded for every $x \in X$, by the uniform boundedness principle.

Given $T \in \mathcal{L}(X)$, let

$$T_{[m]} := \frac{1}{m}\sum_{j=1}^{m} T^j, \qquad m \in \mathbb{N}, \tag{1.1}$$

denote the Cesàro means of the iterates of $T$. The operator $T$ is said to be *Cesàro power bounded* if the sequence $\{T_{[m]}\}_{m\in\mathbb{N}}$ is bounded, and *mean ergodic* if the limits $Px := \lim_{m\to\infty} T_{[m]}x$, $x \in X$, exist in $X$. A power bounded operator $T$ is mean ergodic precisely when

$$X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Im}(I-T)},$$

where $I$ stands for the identity on $X$ and the bar denotes closure in $X$. In general, $\overline{\mathrm{Im}(I-T)}$ is the set of all $x \in X$ for which the sequence $\{T_{[m]}x\}_{m\in\mathbb{N}}$ converges to 0. If $\{T_{[m]}\}_{m\in\mathbb{N}}$ is convergent in the operator norm, then $T$ is called *uniformly mean ergodic*. Clearly, if $T$ is mean ergodic, then $\|T^m x\|/m \to 0$ for every $x \in X$, and if it is uniformly mean ergodic, then $\|T^m\|/m \to 0$. Conversely, if this last convergence holds, then the operator $T$ is uniformly mean ergodic if and only if $\mathrm{Im}(I-T)$ is closed [L].

Mean ergodic operators in Fréchet spaces and barrelled locally convex spaces have been considered by Albanese, Bonet and Ricker [ABR1], [ABR2].

An operator $T$ is said to be *quasi-compact* if $T^m$ is compact for some $m \in \mathbb{N}$. Quasi-compact operators share some properties of compact operators, in particular their spectrum reduces to eigenvalues and $\{0\}$.

Our notation for functional analysis and operator theory is standard. We refer the reader e.g. to [MV] and [R]. For ergodic theory of operators on Banach spaces, see [K].

**2. Preliminaries.** In what follows, $\mathcal{H}(\mathbb{C})$ and $\mathcal{P}$ denote the spaces of entire functions and of polynomials, respectively. The space $\mathcal{H}(\mathbb{C})$ is endowed with the compact-open topology $\tau_{\mathrm{co}}$. It is easy to see that the three operators, $D$, $J$ and $H$, are continuous on $\mathcal{H}(\mathbb{C})$.

For $r \geq 0$ and $f \in \mathcal{H}(\mathbb{C})$, consider

$$\begin{aligned} M_p(f,r) &:= \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} \quad \text{for } 1 \leq p < \infty, \\ M_\infty(f,r) &:= \sup_{|z|=r} |f(z)|. \end{aligned}$$

By the classical Hardy convexity theorem and the Maximum Modulus Theorem, the mapping $r \mapsto M_p(f,r)$ is increasing and logarithmically convex.

A *weight* $v$ on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$ which is radial, i.e. $v(z) = v(|z|)$, $z \in \mathbb{C}$, $v(r)$ is non-increasing on $[0,\infty[$ and rapidly decreasing, that is, $\lim_{r\to\infty} r^n v(r) = 0$ for each $n \in \mathbb{N}$.

For such a weight, $1 \leq p \leq \infty$, and $q \in \{0,\infty\}$, the *generalized weighted Bergman spaces of entire functions* are defined by

$$\begin{aligned} B_{p,\infty}(v) &:= \Big\{ f \in \mathcal{H}(\mathbb{C}) : |||f|||_{p,v} := \sup_{r>0} v(r) M_p(f,r) < \infty \Big\}, \\ B_{p,0}(v) &:= \Big\{ f \in \mathcal{H}(\mathbb{C}) : \lim_{r\to\infty} v(r) M_p(f,r) = 0 \Big\}. \end{aligned}$$

Both are Banach spaces under the norm $|||\cdot|||_{p,v}$. In case $p = \infty$ they are usually denoted by $H_v(\mathbb{C})$ and $H_v^0(\mathbb{C})$ (see [BBG, BBT, BG, G, Lu3]). The inclusions $B_{p,0} \subseteq B_{p,\infty} \subseteq B_{1,\infty} \subseteq \mathcal{H}(\mathbb{C})$ are continuous for $1 \leq p \leq \infty$. As in [BB], take $r > 0$, select $R_0 > r$, fix $|z| \leq r$ and apply the Cauchy formula, integrating around the circle of center 0 and radius $R_0$, to get

$$\frac{R_0 - r}{R_0} |f(z)| \leq M_1(f,R_0) \leq M_p(f,R_0) \leq M_\infty(f,R_0).$$

This implies

$$
\begin{aligned}
(2.1)\qquad \sup_{|z|\le r}|f(z)| &\le \frac{R_0}{(R_0-r)v(R_0)}v(R_0)M_p(f,R_0)\\
&\le \frac{R_0}{(R_0-r)v(R_0)}|||f|||_{p,v}.
\end{aligned}
$$

Then, for every $1\le p\le\infty$, the closed unit ball of $B_{p,\infty}(v)$, denoted by $C_{p,\infty}$, is bounded on $\mathcal{H}(\mathbb{C})$ and $\tau_{\mathrm{co}}$-closed, since for $r>0$ the mapping $\delta_{p,r}:\mathcal{H}(\mathbb{C})\to\mathbb{C}$, $f\mapsto M_p(f,r)$, is continuous and

$$C_{p,\infty}=\bigcap_{r\ge 0}\delta_{p,r}^{-1}([0,1/v(r)]).$$

As $\mathcal{H}(\mathbb{C})$ is Montel, the set $C_{p,\infty}$ is $\tau_{\mathrm{co}}$-compact.

For $1\le p\le\infty$ and $1\le q<\infty$, we consider the space

$$B_{p,q}(v):=\Big\{f\in\mathcal{H}(\mathbb{C}):\|f\|_{p,q,v}:=\Big(2\pi\int_0^\infty rv(r)^qM_p(f,r)^q\,dr\Big)^{1/q}<\infty\Big\}.$$

Given a compact set $K\subseteq\mathbb{C}$ and $z\in K$, by the mean value formula we get

$$|f(z)|\le\frac{1}{\pi}\int_{D(z,1)}|f(\lambda)|\,d\lambda\le\frac{1}{\pi}\int_{D(0,R)}|f(\lambda)|\,d\lambda$$

for every $f\in\mathcal{H}(\mathbb{C})$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}^2$, $D(z,r)$ denotes thee open disc centered at $z$ and of radius $r$, and $R>0$ is such that $z\in K\subseteq\bigcup_{z\in K}D(z,1)\subseteq D(0,R)$. For the unit disc centered at zero we simply write $\mathbb{D}:=D(0,1)$. Thus

$$|f(z)|\le\frac{1}{\pi}\int_0^R r\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\,dr=2\int_0^R rM_1(f,r)\,dr\le 2\int_0^R rM_p(f,r)\,dr,$$

so, applying Hölder's inequality, we see that for every $z\in K$,

$$
\begin{aligned}
(2.2)\qquad |f(z)| &\le 2R^{1-1/q}\Big(\int_0^R r^qM_p(f,r)^q\,dr\Big)^{1/q}\\
&\le\frac{2R^{2-2/q}}{v(R)}\Big(\int_0^\infty rv(r)^qM_p(f,r)^q\,dr\Big)^{1/q}.
\end{aligned}
$$

Therefore, convergence in $B_{p,q}(v)$ implies uniform convergence on compact subsets of $\mathbb{C}$. Thus, $B_{p,q}(v)$ is a closed subset of the Banach space

$$\Big\{f:\mathbb{C}\to\mathbb{C}\text{ measurable}:\int_0^\infty rv(r)^q\Big(\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\Big)^{q/p}dr<\infty\Big\},$$

and therefore a Banach space. The spaces $B_{p,q}(v)$ are called *weighted spaces of entire functions.* Observe that for $p=q$ the last displayed space is usually denoted by $L_v^p(\mathbb{C})$, the Banach space of all complex functions $f$ on $\mathbb{C}$ such that $fv \in L^p(\mathbb{C}, d\lambda)$. When $p = 2$ it is a Hilbert space. For these spaces, we simply write $B_v^p := B_{p,p}(v)$ and denote the norm by $\| \|_{p,v}$. Spaces of this type appear in the study of growth of analytic functions and have been investigated in various articles (see e.g. [BBG, BBT, BG, **?**, Lu2] and the references therein).

By (2.2), the closed unit ball of $B_{p,q}(v)$, denoted by $C_{p,q}$, is bounded on $\mathcal{H}(\mathbb{C})$. It is $\tau_{\mathrm{co}}$-closed since for $r_0 > 0$ the mapping $\delta_{p,q,r_0} : \mathcal{H}(\mathbb{C}) \to \mathbb{C}$, $f \mapsto \int_0^{r_0} rv(r)^q M_p(f, r)^q \, dr$, is continuous and

$$C_{p,q} = \bigcap_{r_0 \geq 0} \delta_{p,q,r_0}^{-1}([0, 1/(2\pi)]).$$

So, also $C_{p,q}$ is $\tau_{\mathrm{co}}$-compact.

Since the weights are rapidly decreasing and

$$\int_{r_0}^{\infty} r^j v(r) dr = \int_{r_0}^{\infty} r^{j+2} v(r) \frac{1}{r^2} \, dr < \infty$$

for every $r_0 > 0$, the polynomials are in $B_{p,q}(v)$ for all $1 \leq p \leq \infty$ and $q = 0$ or $1 \leq q \leq \infty$. By [Lu2, Theorem 2.1] (see also [**?**, Proposition 2.1]), the polynomials are even dense whenever $q \neq \infty$. In particular, $B_{p,q}(v)$ is separable. For $1 < p < \infty$ and $1 \leq q < \infty$ or $q = 0$, the monomials are a Schauder basis of $B_{p,q}(v)$, but this is not the case in general for $p \in \{1, \infty\}$ [Lu2, Theorem 2.3].

Throughout the paper, $B_{p,q}(a, \alpha)$ denotes the space associated to the following weight: $v_{a,\alpha}(r) = e^{-\alpha}$, $r \in [0, 1[$, $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, $r \geq 1$, if $a < 0$, and $v_{a,\alpha}(r) = (a/\alpha)^a e^{-a}$, $r \in [0, a/\alpha[$, $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, $r \geq a/\alpha$, if $a > 0$. Clearly, changing the value of $v$ on a compact interval does not change the spaces and gives an equivalent norm. Moreover, we can assume without loss of generality that the weight is differentiable. In case $a = 0$, we simply write $B_{p,q}(\alpha)$. The norms will be denoted by $\| \|_{p,q,a,\alpha}$ and $\| \|_{p,q,\alpha}$, respectively. In case $q = \infty$, we write $||| \; |||_{p,a,\alpha}$ and $||| \; |||_{p,\alpha}$. If, in addition, $p = \infty$, then the spaces are denoted by $H_{a,\alpha}(\mathbb{C})$, $H^0_{a,\alpha}(\mathbb{C})$, $H_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$, and the norms by $\| \|_{a,\alpha}$ and $\| \|_\alpha$, respectively. For $p = \infty$ we adopt the convention $1/p := 0$ and for $q \in \{0, \infty\}$, $1/q := 0$.

For the estimates of the norms of the operators on these spaces, we use the Stirling formulas

$$n! \sim \sqrt{2\pi n}(n/e)^n \quad \text{and} \quad \Gamma(x+1) \sim \sqrt{2\pi x}\,(x/e)^x,\ x > 0,$$

where $\Gamma$ denotes the Gamma function. Recall that $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$, $z \in \mathbb{C}$, and $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{N}$. Given two real functions $h(x)$ and $g(x)$, the expression $h(x) \sim g(x)$ means $\lim_{x\to\infty} h(x)/g(x) = 1$ and $h(x) \lesssim g(x)$ means that there exists some constant $D > 0$ such that

$h(x) \le Dg(x)$ for every $x \in \mathbb{R}$. When $h(x) \lesssim g(x) \lesssim h(x)$, we simply write $h(x) \approx g(x)$.

Whereas the behavior of the iterates of the differentiation and the integration operators depends heavily on the weights, the Hardy operator is power bounded and uniformly mean ergodic in all cases.

The next lemma is an extension of [BB, Lemma 2.2]:

LEMMA 2.1. *Given a weight $v$, $a > 0$, $1 \le p \le \infty$, and $q = 0$ or $1 \le q \le \infty$, the following are equivalent:*

(i) $\{e^{a\theta z} : |\theta| = 1\} \subseteq B_{p,q}(v)$,
(ii) *there is* $\theta \in \mathbb{C}$, $|\theta| = 1$, *such that* $e^{a\theta z} \in B_{p,q}(v)$,
(iii) $\lim_{r\to\infty} v(r)\frac{e^{ar}}{r^{1/(2p)}} = 0$ *if* $q = 0$, *and* $r^{1/q-1/(2p)}e^{ar} \in L_v^q([r_0,\infty[)$ *for some* $r_0 > 0$ *if* $q \ne 0$.

*Proof.* (i)⇔(ii)⇔(iii) is proved for $q = 0$ in [BB, Lemma 2.2], where it is shown that for each $1 \le p < \infty$ there are $d_p, D_p, r_0 > 0$ such that, for each $|\theta| = 1$ and each $r > r_0$,

$$d_p\frac{e^{ar}}{r^{1/(2p)}} \le M_p(e^{a\theta z}, r) \le D_p\frac{e^{ar}}{r^{1/(2p)}}. \tag{2.3}$$

This equivalence is also satisfied for $1 \le q < \infty$, since for every $s > r_0$,

$$\begin{aligned} d_p^q \int_s^\infty r^{1-q/(2p)} v(r)^q e^{arq}\,dr &\le \int_s^\infty r v(r)^q M_p(e^{a\theta z}, r)^q\,dr \\ &\le D_p^q \int_s^\infty r^{1-q/(2p)} v(r)^q e^{arq}\,dr. \ \blacksquare \end{aligned} \tag{2.4}$$

COROLLARY 2.2. *For the exponential weight $v(r) = e^{-\alpha r}$, $\alpha > 0$, we have $e^{\lambda z} \in B_{p,\infty}(\alpha)$ if and only if $e^{\lambda z} \in B_{p,0}(\alpha)$, for every $1 \le p < \infty$ and $\lambda \in \mathbb{C}$. This is not satisfied for $p = \infty$, since $e^{\alpha z} \in H_v(\mathbb{C}) \setminus H_v^0(\mathbb{C})$.*

LEMMA 2.3. *For every $1 \le p \le \infty$, the unit ball $C_{p,0}$ is $\tau_{\mathrm{co}}$-dense in $C_{p,\infty}$.*

*Proof.* Given $f \in C_{p,\infty}$, let $f_r(z) := f(rz)$, $r \in (0,1)$. Fix $r$ and $\varepsilon > 0$ and pick $n \in \mathbb{N}$ such that $r^{n+1} < \varepsilon/4$. If $f(z) = \sum_{k=0}^\infty a_k z^k$ is the Taylor series representation of $f$ at 0, then the Taylor polynomial $P_n(z) := \sum_{k=0}^n a_k z^k$ is in $B_{p,0}(v)$, so there exists $R > 0$ such that $v(s)M_p(P_n, s) < \min(\varepsilon/2, 1)$ for all $s > R$. Since $r \mapsto M_p(f, r)$ is increasing for every $f \in \mathcal{H}(\mathbb{C})$, for all $s > R$ we have

$$v(s)M_p((P_n)_r, s) = v(s)M_p(P_n, rs) \le v(s)M_p(P_n, s) < \varepsilon/2. \tag{2.5}$$

Moreover, if we consider $g := f - P_n$ and $h(z) := \sum_{k=n+1}^\infty a_k z^{k-(n+1)}$, $z \in \mathbb{C}$, then $g(z) = z^{n+1}h(z)$. For $s > R$, by Minkowski's inequality, we get

$$\begin{aligned}(2.6)\quad v(s)M_p(g, rs) &= v(s)M_p(z^{n+1}h, rs) = r^{n+1}s^{n+1}v(s)M_p(h, rs)\\ &\le r^{n+1}s^{n+1}v(s)M_p(h, s) = r^{n+1}v(s)M_p(g, s)\\ &= r^{n+1}v(s)M_p(f - P_n, s)\\ &\le r^{n+1}v(s)M_p(f, s) + r^{n+1}v(s)M_p(P_n, s)\\ &\le r^{n+1}(|||f|||_{p,v} + \min(\varepsilon/2, 1)) \le 2r^{n+1} \le \varepsilon/2.\end{aligned}$$

By (2.5) and (2.6) we obtain

$$\begin{aligned}v(s)M_p(f_r, s) &= v(s)M_p(g_r + (P_n)_r, s)\\ &\le v(s)M_p(g_r, s) + v(s)M_p((P_n)_r, s) \le \varepsilon\end{aligned}$$

for all $s > R$, which implies that $f_r \in B_{p,0}(v)$. Moreover, $f_r \in C_{p,0}$ since

$$|||f_r|||_{p,v} = \sup_{s\ge 0} v(s)M_p(f, rs) \le |||f|||_{p,v} \le 1,$$

and it is easy to see that $f_r$ converges to $f$ in $\tau_{\mathrm{co}}$ as $r \to 1$, since $f$ is uniformly continuous on compact subsets of $\mathbb{C}$. ∎

The next lemma is inspired by [Bo, Proposition 1.1].

LEMMA 2.4. *Let $T : (\mathcal{H}(\mathbb{C}), \tau_{\mathrm{co}}) \to (\mathcal{H}(\mathbb{C}), \tau_{\mathrm{co}})$ be a continuous linear operator such that $T(\mathcal{P}) \subseteq \mathcal{P}$, let $v$ be a weight and $1 \le p \le \infty$. The following conditions are equivalent:*

(i) $T(B_{p,\infty}(v)) \subseteq B_{p,\infty}(v)$,
(ii) $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ *is continuous,*
(iii) $T(B_{p,0}(v)) \subseteq B_{p,0}(v)$,
(iv) $T : B_{p,0}(v) \to B_{p,0}(v)$ *is continuous.*

*If* (i)–(iv) *hold, then* $\|T\|_{\mathcal{L}(B_{p,\infty}(v))} = \|T\|_{\mathcal{L}(B_{p,0}(v))}$.

*Proof.* The equivalences (i)⇔(ii) and (iii)⇔(iv) follow from the closed graph theorem, since $B_{p,\infty}(v) \hookrightarrow \mathcal{H}(\mathbb{C})$ continuously and $T$ is continuous on $(\mathcal{H}(\mathbb{C}), \tau_{\mathrm{co}})$. Moreover, (ii)⇒(iii) comes easily from the fact that the polynomials are dense in $B_{p,0}(v)$, $T(\mathcal{P}) \subseteq \mathcal{P}$ and $B_{p,0}(v)$ is closed in $B_{p,\infty}(v)$. Clearly $\|T\|_{\mathcal{L}(B_{p,0}(v))} \le \|T\|_{\mathcal{L}(B_{p,\infty}(v))}$.

(iv)⇒(i). By Lemma 2.3, the unit ball of $B_{p,0}(v)$ is $\tau_{\mathrm{co}}$-dense in the unit ball of $B_{p,\infty}(v)$, so given $f$ in $C_{p,\infty}$ there exists $\{f_\alpha\}_\alpha$ in $C_{p,0}$ such that $\{f_\alpha\}_\alpha$ converges to $f$ in $\tau_{\mathrm{co}}$ and $|||Tf_\alpha|||_{p,v} \le \|T\|_{\mathcal{L}(B_{p,0}(v))}$. Since $T$ is $\tau_{\mathrm{co}}$-continuous, $Tf_\alpha$ converges to $Tf$ in $\tau_{\mathrm{co}}$, and since the unit ball of $B_{p,\infty}(v)$ is $\tau_{\mathrm{co}}$-closed, we have $|||Tf|||_{p,v} \le \|T\|_{\mathcal{L}(B_{p,0}(v))}$. Since this happens for every $f$ in the unit ball of $B_{p,\infty}(v)$, the operator $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ is continuous with $\|T\|_{\mathcal{L}(B_{p,\infty}(v))} \le \|T\|_{\mathcal{L}(B_{p,0}(v))}$, and thus the norms coincide. ∎

In what follows we write $|||T|||_{p,v}$ instead of $\|T\|_{\mathcal{L}(B_{p,\infty}(v))} = \|T\|_{\mathcal{L}(B_{p,0}(v))}$ for $1 \le p \le \infty$, and $\|T\|_{p,v}$ instead of $\|T\|_{\mathcal{L}(B^p_v)}$. For $1 \le q < \infty$ we use the notation $\|T\|_{p,q,v}$. Moreover, $|||T|||_{p,a,\alpha}$, $\|T\|_{p,a,\alpha}$ and $\|T\|_{p,q,a,\alpha}$ refer to the norm of the operator acting on the respective spaces associated to the weight $v_{a,\alpha}$. For $v(r) = e^{-\alpha r}$, $r \geq 0$, we omit the $a$.

Using Lemma 2.4 and arguing as in the proof of [BBF, Proposition 2.3] we get the proposition below. In fact, inspecting the proof we even find that if $J$ is mean ergodic on $B_{p,\infty}(v)$ or on $B_{p,0}(v)$, then $\lim_m (J+\cdots+J^m)(f)/m = 0$ for every $f$ in the corresponding space. Observe also that as the polynomials are dense in $B_{p,0}(v)$, the operator $D$ is mean ergodic on $B_{p,0}(v)$ if and only if it is Cesàro power bounded. In this case, $P(f) = 0$ for every $f \in B_{p,0}(v)$.

Proposition 2.5. *Let $T = D$ or $T = J$ and assume that $T$ is continuous on $B_{p,\infty}(v)$, and equivalently on $B_{p,0}(v)$. The following conditions are equivalent:*

(i) $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ *is uniformly mean ergodic,*
(ii) $T : B_{p,0}(v) \to B_{p,0}(v)$ *is uniformly mean ergodic,*
(iii) $\lim_{m\to\infty} |||T + \cdots + T^m|||_{p,v}/m = 0$.

Also from the proof of [BBF, Proposition 2.3] we obtain the next general lemma:

Lemma 2.6. *If $T \in \mathcal{L}(X)$ is a uniformly mean ergodic operator such that $\lim_{m\to\infty} \|T + \cdots + T^m\|/m = 0$, then $1 \notin \sigma(T)$.*

For every $1 \leq p \leq \infty$ and $n \in \mathbb{N}$ we have $M_p(z^n, r) = r^n$, and thus $\|z^n\|_{p,q,v} = \|z^n\|_{\infty,q,v}$. In what follows, we denote it simply by $\|z^n\|_{q,v}$. As in [Bo], it is important to estimate the norms of monomials. In fact, from the inequalities $\|1\|_{q,v} n! \leq \|D^n\|_{p,q,v}\|z^n\|_{q,v}$ and $\|z^n\|_{q,v}/n! = \|J^n(1)\|_{q,v} \leq \|J^n\|_{p,q,v}\|1\|_{q,v}$ we get:

Lemma 2.7. *Let $v$ be a weight such that the differentiation operator $D$ and the integration operator $J$ are continuous on $B_{p,q}(v)$, $1 \leq p \leq \infty$, $q = 0$ or $1 \leq q \leq \infty$.*

(i) *If $D$ is power bounded (resp. uniformly mean ergodic), then $\inf_n \frac{\|z^n\|_{q,v}}{n!} > 0$ (resp. $\|z^n\|_{q,v}/(n-1)! \to \infty$).*
(ii) *If $J$ is power bounded (resp. mean ergodic), then $\{\|z^n\|_{q,v}/n!\}_n$ is bounded (resp. $\|z^n\|_{q,v}/(n!n) \to 0$).*

In [BBF] it is shown that for the weights $v(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r \geq r_0$ for some $r_0 \geq 0$,

$$\|z^n\|_v \approx \left(\frac{n+a}{e\alpha}\right)^{n+a}, \tag{2.7}$$

with equality for $v(r) = e^{-\alpha r}$, $r \geq 0$. In what follows the symbol $\approx$ will appear as a consequence of the fact that the value of a given weight $v$ can be changed on a compact interval in order to satisfy some required conditions without changing the spaces and giving an equivalent norm.

For $1 \le q < \infty$, the Stirling formula yields

$$\|z^n\|_{q,a,\alpha} \approx \Big(2\pi \int_0^\infty r^{(a+n)q+1} e^{-\alpha r q}\, dr\Big)^{1/q} = \Big(2\pi \frac{\Gamma((a+n)q+2)}{(\alpha q)^{(a+n)q+2}}\Big)^{1/q} \approx \Big(\frac{(a+n)q+1}{e\alpha q}\Big)^{a+n+3/(2q)}. \tag{2.8}$$

Observe that (2.8) tends to (2.7) as $q \to \infty$. Applying again the Stirling formula yields

$$\frac{\|z^n\|_{q,a,\alpha}}{n!} \approx \frac{n^{a+3/(2q)-1/2}}{\alpha^n}. \tag{2.9}$$

## 3. The integration operator

Proposition 3.1. *The operator $J$ is not hypercyclic on $\mathcal{H}(\mathbb{C})$, nor on $B_{p,q}(v)$ for $1 \le p \le \infty$, and $q = 0$ or $1 \le q < \infty$, whenever it is continuous. Moreover, $J - \lambda I$ is injective on $\mathcal{H}(\mathbb{C})$ for all $\lambda \in \mathbb{C}$, and $J$ has no periodic points different from $0$ on $\mathcal{H}(\mathbb{C})$.*

*Proof.* $Jf(0) = 0$ for every $f \in \mathcal{H}(\mathbb{C})$, thus $\mathrm{Im}(J)$, and the orbit of an element, cannot be dense. What is more, $J^m f$ tends to zero in the compact-open topology for every $f \in \mathcal{H}(\mathbb{C})$. Indeed, given $f(z) = \sum_{k=0}^\infty a_k z^k \in \mathcal{H}(\mathbb{C})$, we have $J^m f(z) = \sum_{k=0}^\infty a_k z^{k+m} \frac{k!}{(k+m)!}$, so

$$|J^m f(z)| \le R^m \sum_{k=0}^\infty |a_k| R^k \frac{k!}{(k+m)!} \le \frac{R^m}{m!} \sum_{k=0}^\infty |a_k| R^k$$

for every $z \in \mathbb{C}$, $|z| \le R$. Thus, $J^m f$ tends to zero in the compact-open topology.

If $\lambda = 0$, then $J$ is injective, since $Jf = 0$ implies $f = DJf = 0$. If $\lambda \neq 0$ and $Jf - \lambda f = 0$, then $f - \lambda Df = 0$, so $f(z) = Ce^{\frac{1}{\lambda} z}$ for some $C \in \mathbb{C}$. But $f(0) = \frac{1}{\lambda} Jf(0) = 0$, which implies $0 = f(0) = C$, and thus $f = 0$.

Now suppose that $J^m f = f$ for some $f \neq 0$ and some $m \in \mathbb{N}$. Using the trivial decomposition $J^m - I = (J - \theta_1 I) \dots (J - \theta_m I)$, $\theta_j^m = 1$, $j = 1, \dots, m$, we conclude that there is $g \in \mathcal{H}(\mathbb{C})$, $g \neq 0$, and $\theta \in \mathbb{C}$, $\theta^m = 1$, such that $(J - \theta I)g = 0$. But $J - \theta I$ is injective, so we get a contradiction. ∎

Proposition 3.2. *Let $v$ be a weight such that $J$ is continuous on $B_{p,q}(v)$, $1 \le p \le \infty$, $1 \le q \le \infty$ or $q = 0$, and assume that $v(r)e^{\alpha r}$ is non-decreasing for some $\alpha > 0$. Then $\sigma(J) \supseteq (1/\alpha)\mathbb{D}$.*

*Proof.* To see that $(1/\alpha)\mathbb{D} \subseteq \sigma(J)$ we show that $J - \lambda I$ is not surjective on $B_{p,q}(v)$ for $|\lambda| < 1/\alpha$. For $\lambda = 0$, $J$ is not surjective on any $B_{p,q}(v)$

(without any additional assumption) since $Jf(0) = 0$ for each $f$, hence the equation $Jf = 1$ has no solution. Now assume that $\lambda \neq 0$ and that there is $f \in B_{p,q}(v)$ such that $Jf - \lambda f = 1$. Then $f - \lambda f' = 0$ and, as by Lemma 2.1, $e^{z/\lambda} \notin B_{p,q}(v)$, we have $f \equiv 0$, and thus $Jf - \lambda f \neq 1$. ■

Following [AB] we define, for every $\lambda \in \mathbb{C}$, an integral operator $K_\lambda$ on $\mathcal{H}(\mathbb{C})$ by

$$K_\lambda f(z) = e^{\lambda z} \int_0^z e^{-\lambda\zeta} f(\zeta)\, d\zeta, \qquad f \in \mathcal{H}(\mathbb{C}),\ z \in \mathbb{C}.$$

It maps $\mathcal{H}(\mathbb{C})$ into itself continuously and it is a right inverse of $D - \lambda I$. Integrating along the segment that joins 0 to $z$, we obtain, for $f \in \mathcal{H}(\mathbb{C})$,

$$K_\lambda f(z) = z \int_0^1 e^{\lambda z(1-t)} f(zt)\, dt, \qquad z \in \mathbb{C}. \tag{3.1}$$

Observe that for $\lambda = 0$, it is just the integration operator $J$.

Proposition 3.3. *Let $v$ be a weight such that $v(r)e^{\alpha r}$ is non-increasing for some $\alpha > 0$ and let $1 \leq p \leq \infty$. If $|\lambda| < \alpha$, then the operator $K_\lambda$ is continuous on $B_{p,\infty}(v)$ and on $B_{p,0}(v)$ with $|||K_\lambda|||_{p,v} \leq 1/(\alpha - |\lambda|)$. As a consequence, $J$ is continuous on $B_{p,\infty}(v)$ with $|||J|||_{p,v} \leq 1/\alpha$. In particular, $\sigma(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$. Moreover, $|||J^m|||_{p,a,\alpha} \approx 1/\alpha^m$ for all $m \in \mathbb{N}_0$ and $a \leq 0$, with equality for $a = 0$.*

*Proof.* Given $f \in B_{p,\infty}(v)$, we have

$$\begin{aligned} M_p(K_\lambda f, r) &= \left( \frac{1}{2\pi} \int_0^{2\pi} |K_\lambda f(re^{i\theta})|^p\, d\theta \right)^{1/p} \\ &= r \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 e^{\lambda re^{i\theta}(1-t)} f(tre^{i\theta})\, dt \right|^p d\theta \right)^{1/p}. \end{aligned}$$

So, applying the Minkowski integral inequality we obtain

$$\begin{aligned} M_p(K_\lambda f, r) &\leq r \int_0^1 \left( \frac{1}{2\pi} \int_0^{2\pi} e^{|\lambda| r(1-t)p} |f(tre^{i\theta})|^p\, d\theta \right)^{1/p} dt \\ &= r \int_0^1 e^{|\lambda| r(1-t)} M_p(f, rt)\, dt. \end{aligned}$$

Thus, by hypothesis, for $|\lambda| < \alpha$,

$$v(r) M_p(K_\lambda f, r) \leq r \int_0^1 v(tr) M_p(f, rt) e^{r(t-1)(\alpha - |\lambda|)}\, dt \leq \frac{|||f|||_{v,p}}{\alpha - |\lambda|}$$

and $K_\lambda : B_{p,\infty}(v) \to B_{p,\infty}(v)$ is continuous with $|||K_\lambda|||_{p,v} \le 1/(\alpha - |\lambda|)$. Let us see now that $K_\lambda(B_{p,0}(v)) \subseteq B_{p,0}(v)$. Since

$$K_\lambda(1) = -\frac{1}{\lambda} + \frac{1}{\lambda} e^{\lambda z} \in B_{p,0}(v),$$

and integrating by parts yields

$$K_\lambda(z^n) = -\frac{1}{\lambda} z^n + \frac{n}{\lambda} K_\lambda(z^{n-1}), \qquad n \in \mathbb{N},$$

we get

$$K_\lambda(\mathcal{P}) \subseteq \mathcal{P} \oplus \operatorname{span}(e^{\lambda z}) \subseteq B_{p,0}(v).$$

Since the polynomials are dense in $B_{p,0}(v)$, $K_\lambda : B_{p,0}(v) \to B_{p,0}(v)$ is continuous.

If we consider $\lambda = 0$, we get $|||J|||_{p,v} \le 1/\alpha$, and the spectral radius formula yields $\sigma(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$. As a consequence, $|||J|||_{p,a,\alpha} \lesssim 1/\alpha$.

The lower estimate is satisfied for all $1 \le q \le \infty$ whenever $J$ is continuous, using (2.9):

$$(3.2) \qquad \|J^m\|_{p,q,a,\alpha} \ge \sup_{k\in\mathbb{N}} \frac{\|J^m(z^k)\|_{p,q,a,\alpha}}{\|z^k\|_{q,a,\alpha}} = \sup_{k\in\mathbb{N}} \frac{\|z^{k+m}\|_{q,a,\alpha}}{\|z^k\|_{q,a,\alpha}} \frac{k!}{(k+m)!}$$
$$\gtrsim \lim_k \frac{1}{\alpha^m}\left(1 + \frac{m}{k}\right)^{a+3/(2q)-1/2} = \frac{1}{\alpha^m}. \ \blacksquare$$

PROPOSITION 3.4. *Let $v$ be a weight such that $v(r)e^{\alpha r}$ is non-increasing for some $\alpha > 0$ and let $1 \le p < \infty$, $p > 1/\alpha$. Then $K_\lambda$ is continuous on $B_v^p$ if $|\lambda| < \alpha$ and $J$ is continuous on $B_v^p$ with*

$$\|J^m\|_{p,v} \lesssim \left(\frac{p}{\alpha p - 1}\right)^m \qquad \textit{for every } m \in \mathbb{N}.$$

*In particular, $\sigma(J) \subseteq \frac{p}{\alpha p-1}\overline{\mathbb{D}}$. Moreover, $\|J^m\|_{p,a,\alpha} \gtrsim 1/\alpha^m$ for all $m \in \mathbb{N}_0$.*

*Proof.* The continuity is proved in [AB, Theorem 4] for weights of the type $v(z) = \exp(-\varphi(|z|))$, $z \in \mathbb{C}$, where $\varphi$ is a non-negative concave function on $\mathbb{R}_+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t)/\log t = \infty$. For our weights, to get an estimate for the norm we proceed similarly. Since $K_\lambda : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is continuous, it would be enough to show that $K_\lambda$ is bounded on $L_v^p(\mathbb{C})$. But this is not the case for $1 \le p < \infty$. Simple examples show that $K_\lambda$ is not even defined on this space for $1 \le p \le 2$. However, for $1 \le p < \infty$, the measure $v(z)^p d\lambda(z)$ can be replaced by another positive Borel measure $\mu$ on $\mathbb{C}$ such that the space $L^p(\mathbb{C}, d\mu)$ includes $B_v^p$, the restriction of its norm $N_p$ to $B_v^p$ is equivalent to the $L_v^p(\mathbb{C})$ norm, and $K_\lambda$ maps $L^p(\mathbb{C}, d\mu)$ continuously into itself.

Since $v(r)e^{\alpha r}$ is non-increasing, the function $\rho(r) := v(r)$ for $0 \le r \le 1$, and $\rho(r) := v(r)r^{1/p}$ for $r > 1$, satisfies

$$\frac{\rho'(r)}{\rho(r)} = \frac{v'(r)}{v(r)} < -\alpha \quad \text{if } r \le 1$$

and

$$\frac{\rho'(r)}{\rho(r)} = \frac{v'(r)}{v(r)} + \frac{1}{rp} < -\alpha + \frac{1}{p} \quad \text{if } r > 1.$$

So, applying the mean value theorem to the function $\log \rho$, we get

$$\rho(t) \le \rho(x)e^{(1/p-\alpha)(t-x)}, \quad 0 < x < t, \tag{3.3}$$

which yields

$$\int_x^\infty \rho(t)^p\,dt \le \frac{\rho(x)^p}{\alpha p - 1}, \quad x \in \mathbb{R}_+. \tag{3.4}$$

Hence, as $\rho$ satisfies the hypothesis of [AB, Proposition 2(1)], the operator $V_\gamma : L^p_\rho(\mathbb{R}_+) \to L^p_\rho(\mathbb{R}_+)$ given by

$$V_\gamma f(x) = e^{\gamma x}\int_0^x e^{-\gamma t} f(t)\,dt, \quad x \in \mathbb{R}_+,$$

is continuous. Moreover, since (3.7) in the proof of [AB, Proposition 2] is satisfied for a constant $C := 1/(1-\alpha p)$, we even get $\|V_0\| \le Cp = p/(\alpha p - 1)$. By [AB, Theorem 4], $N_p(K_\lambda) \le \|V_\lambda\|$. In particular, $N_p(J) \le \|V_0\| \le p/(\alpha p-1)$. Since the norms $N_p$ and $\|\ \|_{p,v}$ are equivalent on $B^p_v$, we conclude that $\|J^m\|_{p,v} \lesssim \big(\frac{p}{\alpha p-1}\big)^m$ for every $m \in \mathbb{N}$. Therefore, the conclusion about the norm and the spectrum follows. The lower estimate is calculated in (3.2). ∎

Corollary 3.5. *The spectrum of $J : B_{p,q}(a,\alpha) \to B_{p,q}(a,\alpha)$ satisfies*

$$\sigma(J) = (1/\alpha)\overline{\mathbb{D}} \quad \textit{for } 1 \le p \le \infty,\ q \in \{0,\infty\},$$
$$(1/\alpha)\overline{\mathbb{D}} \subseteq \sigma(J) \subseteq \frac{p}{\alpha p - 1}\overline{\mathbb{D}} \quad \textit{for } 1 \le p < \infty,\ p > 1/\alpha,\ p = q.$$

*Proof.* For each $\beta < \alpha$, the function $v_{a,\alpha}(r)e^{\beta r}$ is decreasing in $[r_0,\infty[$ for some $r_0 > 0$. Therefore, by Propositions 3.3 and 3.4, the integration operator $J$ is continuous on $B_{p,q}(a,\alpha)$, and for an equivalent norm, $\|J^m\| \le 1/\beta^m$ on $B_{p,\infty}(v)$ and $\|J^m\| \le \big(\frac{p}{\beta p-1}\big)^m$ on $B^p_v$ for $1 \le p < \infty$. Thus, the spectral radius $r(J)$ satisfies $r(J) \le 1/\beta$ and $r(J) \le p/(\beta p-1)$, respectively. Since $\beta < \alpha$ is arbitrary, we have $\sigma(J) \subseteq (1/\alpha)\overline{\mathbb{D}}$ and $\sigma(J) \subseteq \frac{p}{\alpha p-1}\overline{\mathbb{D}}$. On the other hand, $v_{a,\alpha}(r)e^{\gamma r}$ is non-decreasing for every $\gamma > \alpha$. By Proposition 3.2, $\sigma(J) \supseteq (1/\gamma)\mathbb{D}$ for every $\gamma > \alpha$, and thus $\sigma(J) \supseteq (1/\alpha)\mathbb{D}$. ∎

THEOREM 3.6.

(a) *Let* $1 \le p \le \infty$, *and* $1 \le q \le \infty$ *or* $q = 0$, *and assume* $J : B_{p,q}(v) \to B_{p,q}(v)$ *is continuous. Then:*

  (i) *If* $r^a e^{-\alpha r} = O(v(r))$ *for* $\alpha < 1$, $a \in \mathbb{R}$, *or* $\alpha = 1$, $a > 1/2 - 3/(2q)$, *then* $J$ *is not power bounded on* $B_{p,q}(v)$.

  (ii) $J$ *is not uniformly mean ergodic on* $B_{p,q}(v)$ *if* $v(r)e^{\beta r}$ *is non-decreasing for all* $\beta > 1$. *In particular* $J$ *is not uniformly mean ergodic on* $B_{p,q}(a, 1)$, *for any* $a \in \mathbb{R}$.

  (iii) *If* $r^{3/2-3/(2q)}e^{-r} = O(v(r))$, *then* $J$ *is not mean ergodic on* $B_{p,q}(v)$. *In particular, it is not mean ergodic on* $B_{p,q}(a, \alpha)$ *when* $\alpha < 1$, $a \in \mathbb{R}$.

(b) *For* $1 \le p \le \infty$ *and* $q \in \{0, p, \infty\}$, *we get:*

  (iv) $J$ *is power bounded on* $B_{p,q}(v)$ *for* $q \in \{0, p, \infty\}$ *and mean ergodic for* $q \in \{0, p\}$ *provided that* $v(r)e^{(1+1/q)r}$ *is non-increasing. In particular, this monotonicity condition is satisfied for the weight* $v_{a,1+1/q}$ *for every* $a \le 0$.

  (v) $J$ *is uniformly mean ergodic on* $B_{p,q}(v)$ *if* $v(r)e^{\alpha r}$ *is non-increasing for some* $\alpha > 1 + 1/q$.

*Proof.* (i) We have $\|z^n\|_{q,a,\alpha}/n! = O(\|z^n\|_v/n!)$ and (2.9) implies that the sequence $\{\|z^n\|_{q,a,\alpha}/n!\}_n$ is unbounded if $\alpha < 1$, $a \in \mathbb{R}$, or $\alpha = 1$, $a > 1/2 - 3/(2q)$. So, by Lemma 2.7(ii), $J$ is not power bounded.

(ii) If for all $\beta > 1$, $v(r)e^{\beta r}$ is non-decreasing in some interval $[r_0, \infty[$, then $\sigma(J) \supseteq \overline{\mathbb{D}}$. Since $1 \in \sigma(J)$, Lemma 2.6 yields the conclusion.

(iii) By (2.9), the sequence $\{\|z^n\|_{q,3/2-3/(2q),1}/(n!n)\}_n$ does not tend to zero and $\|z^n\|_{q,3/2-3/(2q),1} = O(\|z^n\|_{p,q,v})$. By Lemma 2.7(ii), $J$ is not mean ergodic on $B_{p,q}(v)$.

(iv) The first statement follows from the estimates of the norm of $J^m$ in Propositions 3.3 and 3.4. Moreover, for each $k \in \mathbb{N}$,

$$\|J^m(z^k)\|_{p,q,v} = \frac{k!}{(m+k)!}\|z^{m+k}\|_{q,v} \lesssim \frac{k!}{(m+k)!}\|z^{m+k}\|_{q,1+1/q}.$$

So, by (2.9), the successive iterates tend to zero on the polynomials. As $J$ is power bounded and the polynomials are a dense subset, $\{J^m f\}_m$ converges to zero for each $f \in B_{p,q}(v)$, and thus $m^{-1}\sum_{j=1}^m J^j f$ also converges to 0.

(v) $\{\|J^n\|_{p,q,v}\}_n$ tends to zero by Propositions 3.3 and 3.4, therefore

$$\Big\|\frac{1}{m}\sum_{j=1}^m J^j\Big\|_{p,q,v} \le \frac{1}{m}\sum_{j=1}^m \|J^j\|_{p,q,v} \to 0. \ \blacksquare$$

From Theorem 3.6 and [BBF, Corollary 3.6], we obtain:

Corollary 3.7. *Let $1 \le p \le \infty$. The integration operator $J$ is uniformly mean ergodic on $B_{p,q}(\alpha)$, $q \in \{0, p, \infty\}$, if $\alpha > 1 + 1/q$, and it is not mean ergodic on these spaces if $1/q < \alpha < 1$. Moreover, $J$ is power bounded and mean ergodic on $B_{p,q}(1 + 1/q)$, $q \in \{0, p\}$, not uniformly mean ergodic on $B_{p,q}(1)$, $q \in \{0, p, \infty\}$, and not mean ergodic on $H_1(\mathbb{C})$.*

**4. The Hardy operator.** The next theorem is an analogue of [BBF, Theorem 3.12] for weighted spaces of entire functions.

Theorem 4.1. *Given a weight $v$, $1 \le p \le \infty$, and $q = 0$ or $1 \le q \le \infty$, the Hardy operator $H : B_{p,q}(v) \to B_{p,q}(v)$, $Hf(z) = z^{-1} \int_0^z f(\zeta)\, d\zeta$, $z \in \mathbb{C}$, is well defined and continuous with norm $\|H\| = 1$. Moreover, $H^2$ is compact and $H^2(B_{p,\infty}(v)) \subseteq B_{p,0}(v)$. If the integration operator $J : B_{p,q}(v) \to B_{p,q}(v)$ is continuous, then $H$ is compact. Moreover, $H(B_{p,\infty}(v)) \subseteq B_{p,0}(v)$.*

*Proof.* For every $f \in \mathcal{H}(\mathbb{C})$ and $r \ge 0$ we have

$$\begin{aligned} M_p(Hf, r)^p &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{re^{i\theta}} \int_0^{re^{i\theta}} f(\omega)\, d\omega \right|^p d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^1 f(tre^{i\theta})\, dt \right|^p d\theta \\ &\le \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} |f(tre^{i\theta})|^p\, d\theta\, dt \le M_p(f, r)^p. \end{aligned}$$

Hence, for every $f \in B_{p,q}(v)$ we have $\|Hf\|_{p,q,v} \le \|f\|_{p,q,v}$ and $\|H\| := \|H\|_{p,q,v} \le 1$. On the other hand, since $H(c) = c$ for every $c \in \mathbb{C}$, taking $g := c/\|c\|_{q,v} \in B_{p,q}(v)$ we obtain $\|H\| = 1$.

Given $f = \sum_{k=0}^{\infty} a_k z^k \in B_{p,q}(v)$, the Cauchy and Jensen inequalities imply

$$|a_k| \le \frac{1}{2\pi} \left| \int_{|\omega|=R} \frac{f(\omega)}{\omega^{k+1}}\, d\omega \right| = \frac{1}{R^k} M_1(f, R) \le \frac{1}{R^k} M_p(f, R) \tag{4.1}$$

for every $R > 0$, so $|a_k|\, \|z^k\|_{q,v} \le \|f\|_{p,q,v}$ for every $k \in \mathbb{N}_0$. As $H^2 f(z) = \sum_{k=0}^{\infty} \frac{a_k}{(k+1)^2} z^k$, one has

$$\begin{aligned} \left\| H^2 f - \sum_{k=0}^{N} \frac{a_k}{(k+1)^2} z^k \right\|_{p,q,v} &\le \sum_{k=N+1}^{\infty} \frac{|a_k|\, \|z^k\|_{q,v}}{(k+1)^2} \\ &\le \|f\|_{p,q,v} \sum_{k=N+1}^{\infty} \frac{1}{(k+1)^2}, \end{aligned} \tag{4.2}$$

which shows that the finite rank operators $H_N^2(\sum_{k=0}^{\infty} a_k z^k) := \sum_{k=0}^{N} \frac{a_k}{(k+1)^2} z^k$ are bounded on $B_{p,q}(v)$ and that

$$\|H^2 - H_N^2\|_{p,q,v} \leq \sum_{k=N+1}^{\infty} \frac{1}{(k+1)^2},$$

proving the compactness of $H^2$. Since $H^2 f$ belongs to the closure of the polynomials, it belongs to $B_{p,0}(v)$ if $f \in B_{p,\infty}(v)$.

Finally, suppose that $J : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. Since $M_p(Hf,r) = \frac{1}{r} M_p(Jf,r)$ for every $r \geq 0$, the Hardy operator $H : B_{p,\infty}(v) \to B_{p,0}(v)$ is well defined, as for every $r \geq 0$,

$$v(r)M_p(Hf,r) = v(r)\frac{1}{r}M_p(Jf,r) \leq \frac{|||J|||_{p,v}}{r}|||f|||_{p,v}.$$

Take a sequence $\{f_n\}_n$ in the unit ball of $B_{p,q}(v)$. As it is compact with respect to the compact-open topology $\tau_{\text{co}}$, there exists a subsequence $\{n_k\}_k$ such that $f_{n_k}$ tends to some $f$ in the unit ball of $B_{p,q}(v)$ in $\tau_{\text{co}}$. Given $\varepsilon > 0$, take $R > 0$ such that $R > \frac{2}{\varepsilon}\|J\|_{p,q,v}$ in order to get

$$v(r)M_p(Hf_{n_k} - Hf, r) \leq v(r)\frac{1}{r}M_p(Jf_{n_k} - Jf, r) \leq \frac{2}{R}\|J\|_{p,q,v} \leq \varepsilon$$

for $r \geq R$ and

$$\begin{aligned} 2\pi \int_R^{\infty} rv(r)^q M_p(Hf_{n_k} - Hf, r)^q \, dr &\leq 2\pi \int_R^{\infty} r^{1-q} v(r)^q M_p(Jf_{n_k} - Jf, r)^q \, dr \\ &\leq \frac{1}{R^q}\|Jf_{n_k} - Jf\|_{p,q,v}^q \leq \frac{2^q}{R^q}\|J\|_{p,q,v}^q < \varepsilon^q. \end{aligned}$$

Since the Hardy operator $H : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ is continuous, there exists $k_0$ such that $\|Hf_{n_k} - Hf\|_{p,q,v} \leq \varepsilon$ for $k \geq k_0$, and therefore $H$ is compact. ∎

Proceeding as in [BBF, Corollary 3.13] yields:

COROLLARY 4.2. *The Hardy operator $H$ is power bounded and uniformly mean ergodic on $B_{p,q}(v)$ for $1 \leq p \leq \infty$, and $q = 0$ or $1 \leq q \leq \infty$. Moreover, its spectrum is $\sigma(H) = \{1/n\}_{n\in\mathbb{N}} \cup \{0\}$.*

REMARK 4.3. Observe that unlike the operators $J$ and $D$, the Hardy operator $H$ is mean ergodic and 1 belongs to its spectrum on $B_{p,q}(v)$. The Cesàro means of the iterates of $H$ do not converge to zero on the polynomials. Being power bounded, $H$ cannot be hypercyclic on $B_{p,q}(v)$. In fact, since $\delta_0(H^n f) = f(0)$ for each $f \in \mathcal{H}(\mathbb{C})$, $H$ is not hypercyclic on $\mathcal{H}(\mathbb{C})$. Moreover, it is not difficult to show that the spectrum of $H : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ reduces to its eigenvalues $\{1/n\}_{n\in\mathbb{N}}$, since by the Cauchy–Hadamard theorem, $H - \lambda I$ is surjective for $\lambda \notin \{1/n\}_{n\in\mathbb{N}}$.

**5. The differentiation operator.** The results of the first part of this section are inspired by [Bo] and [BB].

PROPOSITION 5.1. *Let $v$ be a weight such that $C := \sup_{r>0} \frac{v(r)}{v(r+1)} < \infty$. Then the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous for every $1 \leq p \leq \infty$, and $q = 0$ or $1 \leq q \leq \infty$.*

*Proof.* The case $q \in \{0, \infty\}$ is proved in [BB, Proposition 2.1], where it is shown that $M_p(f', r) \leq \frac{r+1}{2r+1} M_p(f, r+1)$ for every $f \in \mathcal{H}(\mathbb{C})$, $r > 0$ and $1 \leq p \leq \infty$. Therefore,

$$\begin{aligned}\|Df\|_{p,q,v}^q &= 2\pi \int_0^\infty r v(r)^q M_p(f', r)^q \, dr \\ &\leq C^q 2\pi \int_0^\infty (r+1) v(r+1)^q M_p(f, r+1)^q \, dr \leq C^q \|f\|_{p,q,v}^q,\end{aligned}$$

and so $D$ is continuous. ∎

THEOREM 5.2. *Let $1 \leq p \leq \infty$, and $1 \leq q < \infty$ or $q = 0$. Assume that the differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. The following conditions are equivalent:*

(i) *$D$ satisfies the hypercyclicity criterion.*
(ii) *$D$ is hypercyclic.*
(iii) $\liminf_{n\to\infty} \|z^n\|_{q,v}/n! = 0$.

*Proof.* As in the proof of [BB, Theorem 2.8], we find that if we assume that $D$ is hypercyclic, then there is $f \in B_{p,q}(v)$ such that $\{f^{(n)}(0)\}_n$ is unbounded in $\mathbb{C}$. Fix $n \in \mathbb{N}$. By the Cauchy inequalities, for each $r > 0$,

$$r^n \frac{|f^{(n)}(0)|}{n!} = \frac{r^n}{2\pi} \left| \int_{|w|=r} \frac{|f(w)|}{w^{n+1}} \, dw \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta \leq M_p(f, r),$$

which yields $|f^{(n)}(0)| \, \|z^n\|_{q,v}/n! \leq \|f\|_{p,q,v}$ for every $n \in \mathbb{N}$. Since $\{f^{(n)}(0)\}_n$ is unbounded, there exists an increasing sequence $\{n_k\}_k \subseteq \mathbb{N}$ such that $\lim_{k\to\infty} |f^{(n_k)}(0)| = \infty$. Hence, $\liminf_{n\to\infty} \|z^n\|_{q,v}/n! = 0$. Since the polynomials are dense in $B_{p,q}(v)$, proceeding as in [Bo, Theorem 2.3] we get (iii)⇒(i). ∎

THEOREM 5.3. *Assume that $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. The following conditions are equivalent:*

(i) *$D$ is topologically mixing.*
(ii) $\lim_{n\to\infty} \|z^n\|_{q,v}/n! = 0$.

*Proof.* By the proof of [Bo, Theorem 2.4], if $D$ is topologically mixing, then $\lim_{n\to\infty} \|\delta_0 \circ D^n\| = \infty$. Proceeding as in the proof of Theorem 5.2, for each $f \in B_{p,q}(v)$ with $\|f\|_{p,q,v} \le 1$ and each $n \in \mathbb{N}$ we have

$$|\delta_0 \circ D^n(f)| \frac{\|z^n\|_{q,v}}{n!} = |f^{(n)}(0)| \frac{\|z^n\|_{q,v}}{n!} \le \|f\|_{p,q,v}.$$

So, (ii) holds. Since the polynomials are dense in $B_{p,q}(v)$, (ii) implies that $D$ satisfies the assumptions of the criterion of Kitai–Gethner–Shapiro, and thus it is topologically mixing. ∎

Lemma 5.4. *Let $A \subseteq \overline{\alpha\mathbb{D}}$, $\alpha > 0$, be a subset with at least one accumulation point in $\alpha\mathbb{D}$ or such that $A \cap \delta(\alpha\mathbb{D})$ is dense in $\delta(\alpha\mathbb{D}) := \{z \in \mathbb{C} : |z| = \alpha\}$. If $1 \le p \le \infty$ and $\lim_{r\to\infty} v(r)e^{\alpha r}/r^{1/(2p)} = 0$, then the set $Y := \operatorname{span}(\{e_a : a \in A\})$ is dense in $B_{p,0}(v)$, where $e_\omega(z) := e^{\omega z}$ with $z, \omega \in \mathbb{C}$. If for some $r_0 > 0$, $r^{1/q-1/(2p)}e^{\alpha r} \in L^q_v([r_0,\infty[)$, $1 \le p \le \infty$ and $1 \le q < \infty$, then $Y$ is dense in $B_{p,q}(v)$. Under these assumptions, $z^n e_\zeta(z) \in B_{p,q}(v)$ for every $n \in \mathbb{N}$ and $\zeta \in \mathbb{C}$, $|\zeta| \le \alpha$.*

*Proof.* Let $u$ be a continuous functional on $B_{p,q}(v)$, and assume that $u(f) = 0$ for each $f \in Y$. Consider the function $S : \overline{\alpha\mathbb{D}} \to B_{p,q}(v)$, $\zeta \mapsto e_\zeta$, and define $\widetilde{u} := u \circ S : \overline{\alpha\mathbb{D}} \to \mathbb{C}$, so $\widetilde{u}(\zeta) = u(e_\zeta)$ for $\zeta \in \overline{\alpha\mathbb{D}}$. By Lemma 2.1, $S$ is well defined and bounded. Indeed, by (2.3), for $1 \le p \le \infty$ and $\zeta \in \overline{\alpha\mathbb{D}}$,

$$|||S(\zeta)|||_{p,v} = |||e^{\zeta z}|||_{p,v} = \sup_{r\ge0} v(r)M_p(e^{\zeta z}, r) \le D_p \sup_{r\ge0} v(r)\frac{e^{\alpha r}}{r^{1/(2p)}}$$

in case $q = 0$, whereas for $1 \le q < \infty$ there exists some constant $D_{p,q} > 0$ such that

$$\|S(\zeta)\|_{p,q,v} = \|e^{\zeta z}\|_{p,q,v} \le D_{p,q}\Big(2\pi \int_0^\infty r^{1-q/(2p)} v(r)^q e^{\alpha r q}\, dr\Big)^{1/q} =: M.$$

Since $S$ is locally bounded (even bounded), we proceed analogously to the proof of [BB, Theorem 2.3] to show that $S$ is holomorphic on $\alpha\mathbb{D}$ with $z^n e_\zeta(z) = S^{(n)}(\zeta) \in B_{p,q}(v)$.

Let us see now that $S : \overline{\alpha\mathbb{D}} \to B_{p,q}(v)$ is continuous. The case $q = 0$ can be found in the proof of [BB, Theorem 2.3]. For $1 \le q < \infty$ observe that, given $\zeta_0$ in the boundary of $\alpha\mathbb{D}$ and a sequence $\{\zeta_j\}_j \in \overline{\alpha\mathbb{D}}$ converging to $\zeta_0$, by (2.3) there exist $C, r_0 > 0$ such that

$$\begin{aligned}\|S(\zeta_j) - S(\zeta_0)\|^q_{p,q,v} &= 2\pi \int_0^\infty r v(r)^q M_p(e^{\zeta_j z} - e^{\zeta_0 z}, r)^q\, dr \\ &\le C \int_{r_0}^\infty r^{1-q/(2p)} v(r)^q e^{\alpha r q}\, dr.\end{aligned}$$

Given $\varepsilon > 0$, the hypothesis implies that there exists $r_1 > r_0$ such that

$$\int_{r_1}^\infty r^{1-q/(2p)} v(r)^q e^{\alpha r q}\, dr < \frac{\varepsilon}{2C}.$$

Since the map $\mathbb{C} \to \mathcal{H}(\mathbb{C})$, $\zeta \mapsto e^{\zeta z}$, is continuous, there exists $j_0 \in \mathbb{N}$ such that

$$\int_{r_0}^{r_1} rv(r)^q M_p(e^{\zeta_j z} - e^{\zeta_0 z}, r)^q\, dr \le \int_{r_0}^{r_1} rv(r)^q M_\infty(e^{\zeta_j z} - e^{\zeta_0 z}, r)^q\, dr < \frac{\varepsilon}{2C}.$$

So, $S$ is continuous. Since $u \circ S$ is holomorphic on $\alpha\mathbb{D}$, continuous at the boundary and vanishes in $A$, it is zero in $\alpha\mathbb{D}$. In particular, we get $0 = (u \circ S)^{(n)}(0) = u(S^{(n)}(0)) = u(z^n)$ for each $n \in \mathbb{N}_0$. As the polynomials are dense in $B_{p,q}(v)$, it follows that $u = 0$. By the Hahn–Banach theorem we conclude that $Y$ is dense in $B_{p,q}(v)$. ∎

THEOREM 5.5. *Assume that $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous. If $\lim_{r\to\infty} v(r)e^r/r^{1/(2p)} = 0$ for $q = 0$, and $r^{1/q-1/(2p)}e^r \in L^q_v([r_0,\infty[)$ for some $r_0 > 0$ if $1 \le q < \infty$, then $D$ is frequently hypercyclic, and thus hypercyclic. Moreover, it is topologically mixing on $B_{p,q}(v)$ for $1 \le p \le \infty$ when $q = 0$, and for $1 < p \le \infty$ when $1 \le q < \infty$.*

*Proof.* By [Gr, Theorem 1.4], to prove that $D$ is frequently hypercyclic, it is enough to show that $D$ has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues. As a probability measure we consider the normalized Lebesgue measure on $\mathbb{T}$. If a subset $A$ of $\mathbb{T}$ has Lebesgue measure 1, then $A$ is dense in $\mathbb{T}$. By Lemma 5.4, $\mathrm{span}(\{e_a : a \in A\})$ is dense in $B_{p,q}(v)$, and the condition is satisfied.

Let us see now that $D : B_{p,q}(v) \to B_{p,q}(v)$ is topologically mixing. By Theorem 5.3, it is equivalent to study when $\lim_n \|z^n\|_{q,v}/n! = 0$. For $q = 0$, the hypothesis implies that given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $v(r) \le \varepsilon r^{1/(2p)}e^{-r}$ for every $r \ge r_\varepsilon$. Let $r_n$ be a global maximum point of the function $r \mapsto v(r)r^n$; by [HL, Lemma 1.2], $r_n \to \infty$ as $n \to \infty$, so there exists some $n_\varepsilon$ such that for $n \ge n_\varepsilon$,

$$\frac{\|z^n\|_v}{n!} = \sup_{r\ge r_\varepsilon} v(r)\frac{r^n}{n!} \le \varepsilon \sup_{r\ge r_\varepsilon} r^{1/(2p)}e^{-r}\frac{r^n}{n!} \le \varepsilon \frac{\|z^n\|_{1/(2p),1}}{n!}. \tag{5.1}$$

By (2.9), $\|z^n\|_{1/(2p),1}/n!$ converges to 0 for $1 < p \le \infty$ and to 1 for $p = 1$. Therefore, since (5.1) holds for every $\varepsilon > 0$, we have $\lim_n \|z^n\|_v/n! = 0$. For $1 \le q < \infty$,

$$\begin{aligned}\frac{\|z^n\|^q_{q,v}}{n!^q} &\lesssim \int_{r_0}^{\infty} rv(r)^q \frac{r^{nq}}{n!^q}\, dr = \int_{r_0}^{\infty} \frac{r^{nq+q/(2p)}e^{-rq}}{n!^q} v(r)^q r^{1-q/(2p)}e^{rq}\, dr \\ &\le \frac{\|z^n\|^q_{1/(2p),1}}{n!^q} \int_{r_0}^{\infty} v(r)^q r^{1-q/(2p)} e^{rq}\, dr.\end{aligned}$$

Since $\|z^n\|_{1/(2p),1}/n! \to 0$ for $1 < p \le \infty$, we get $\lim_n \|z^n\|_{q,v}/n! = 0$. ∎

Theorem 5.6. *Assume that $D : B_{p,q}(v) \to B_{p,q}(v)$ is continuous for some $1 \le p \le \infty$, and $q = 0$ or $1 \le q < \infty$. The following conditions are equivalent:*

(i) *$D$ is chaotic.*
(ii) *$D$ has a periodic point different from 0.*
(iii) *$\lim_{r\to\infty} v(r)e^r/r^{1/(2p)} = 0$ if $q = 0$, and $r^{1/q-1/(2p)}e^r \in L^q_v([r_0,\infty[)$ for some $r_0 > 0$ if $1 \le q < \infty$.*

*Proof.* Clearly (i) implies (ii). Let us see (ii)⇒(iii). By hypothesis, there exists a function $0 \ne f \in B_{p,q}(v)$ such that $D^n f = f$ for some $n \in \mathbb{N}$. Using the trivial decomposition $D^n - I = (D - \theta_1 I)\dots(D - \theta_n I)$, $\theta_j^n = 1$, $j = 1,\dots,n$, we conclude that there is $\theta \in \mathbb{C}$, $|\theta| = 1$, and $g \in B_{p,q}(v)$, $g \ne 0$, such that $(D - \theta I)g = 0$. This yields $e^{\theta z} \in B_{p,q}(v)$. Using Lemma 2.1, we obtain (iii).

(iii)⇒(i). Denote by $P$ the linear span of the functions $e^{\theta z}$, $\theta \in \mathbb{C}$, $\theta^n = 1$ for some $n \in \mathbb{N}$. Obviously, $P$ is formed by periodic points and, by Lemma 5.4, it is dense in $B_{p,q}(v)$. On the other hand, since $D$ is hypercyclic by Theorem 5.5, it is chaotic. ■

Observe that Theorems 5.6 and 5.5 show that any chaotic continuous differentiation operator $D : B_{p,q}(v) \to B_{p,q}(v)$ is frequently hypercyclic, even topologically mixing on $B_{p,q}(v)$ for $1 \le p \le \infty$ when $q = 0$ and for $1 < p \le \infty$ when $1 \le q < \infty$.

In [BB, Corollaries 2.6, 2.7 and 2.10], some examples of weights for which the differentiation operator on $B_{p,q}(v)$, $1 \le p \le \infty$, $q = 0$, is topologically mixing, chaotic, or none of them are shown. We present here some examples for the case $1 \le q < \infty$.

Corollary 5.7. *Consider the weight $v_{a,\alpha}$, $a \in \mathbb{R}$, $\alpha > 0$, $1 \le p \le \infty$, and $q = 0$ or $1 \le q < \infty$.*

(a) *If $\alpha < 1$, then $D$ is neither hypercyclic nor chaotic on $B_{p,q}(v)$.*
(b) *If $\alpha > 1$ then $D$ is topologically mixing and chaotic on $B_{p,q}(v)$.*
(c) *If $\alpha = 1$, $D$ is hypercyclic (even topologically mixing) if and only if $a < 1/2 - 3/(2q)$, and $D$ is chaotic if and only if $a < 1/(2p) - 2/q$.*

*Proof.* Note that Theorem 5.6 yields the conclusion about chaos since $v_{a,\alpha}(r)e^r r^{-1/(2p)} = e^{r(1-\alpha)}r^{a-1/(2p)}$ tends to zero as $r \to \infty$ if and only if $\alpha > 1$, or $\alpha = 1$ and $a < 1/(2p)$, and $\int_{r_0}^\infty r^{1+aq-q/(2p)}e^{-rq(\alpha-1)}\,dr < \infty$ if and only if $\alpha > 1$, or $\alpha = 1$ and $a < 1/(2p) - 2/q$. Theorem 5.3 and (2.9) yield the conclusion about hypercyclicity. ■

Corollary 5.8. *Assume that $1 \le p \le \infty$, and $q = 0$ or $1 \le q < \infty$.*

(a) *If $v(r) = r^{1/(2p)-1/q}e^{-r}/\varphi(r)$ for $r$ large enough, where $\varphi(r)$ is a positive increasing continuous function with $\lim_{r\to\infty}\varphi(r) = \infty$ in* *the case* $q = 0$, *or* $1/\varphi(r) \in L^q([r_0,\infty[)$ *for* $1 \le q < \infty$, *and* $\sup_{r>0} \varphi(r+1)/\varphi(r) < \infty$, *then* $D : B_{p,q}(v) \to B_{p,q}(v)$ *is chaotic, and thus a frequently hypercyclic continuous operator.*

(b) *If* $v(r) = r^{1/(2p)-1/q}e^{-r}$ *for* $r$ *large enough, then* $D : B_{p,q}(v) \to B_{p,q}(v)$ *is continuous, but it is hypercyclic* (*even topologically mixing*) *if and only if* $1/p + 1/q < 1$. *For this weight,* $D$ *is never chaotic. Observe that for* $q = 0$ *and* $p > 1$ *it is always topologically mixing, but not chaotic.*

*Proof.* (a) is trivial from Proposition 5.1 and Theorem 5.6; (b) follows from Corollary 5.7 by considering $a = 1/(2p) - 1/q$. ∎

From now on we restrict our attention to the spaces $B_{p,q}(a,\alpha)$, $a \in \mathbb{R}$, $\alpha > 0$.

Proposition 5.9. *Let* $1 \le p \le \infty$ *and* $1 \le q \le \infty$. *If* $a > 0$, *then*

$$\|D^n\|_{p,q,a,\alpha} = O\biggl(n!\Bigl(\frac{e\alpha}{n}\Bigr)^n\biggr).$$

*If* $a \le 0$ *and* $n > |a|$, *then*

$$\|D^n\|_{p,q,a,\alpha} = O\biggl(n!\Bigl(\frac{e\alpha}{n+a}\Bigr)^{n+a}\biggr).$$

*For* $1 \le q < \infty$,

$$n!\biggl(\frac{e\alpha q}{(a+n)q+1}\biggr)^{n+a+3/(2q)} = O(\|D^n\|_{p,q,a,\alpha}),$$

*and for* $q = \infty$,

$$n!\biggl(\frac{e\alpha}{n+a}\biggr)^{n+a} = O(|||D^n|||_{p,a,\alpha}),$$

*with equality for* $a = 0$.

*Proof.* For the lower estimate we use

$$\|D^n\|_{p,q,a,\alpha} \ge \biggl\|D^n\biggl(\frac{z^n}{\|z^n\|_{q,a,\alpha}}\biggr)\biggr\|_{p,q,a,\alpha} = \frac{n!\|1\|_{q,a,\alpha}}{\|z^n\|_{q,a,\alpha}}$$

and (2.8). Applying Jensen's inequality and Fubini's Theorem as in [BB, Proposition 2.1], we get

$$\begin{aligned} M_p(f^{(n)},r) &= \biggl(\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^p\,d\theta\biggr)^{1/p} \\ &= \biggl(\frac{1}{2\pi}\int_0^{2\pi}\biggl|\frac{n!}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\varphi})iRe^{i\varphi}}{(Re^{i\varphi}-re^{i\theta})^{n+1}}\,d\varphi\biggr|^p\,d\theta\biggr)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&\le n!\bigg(\frac{1}{2\pi}\int_0^{2\pi}\bigg(\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(Re^{i\varphi})|R}{|Re^{i\varphi}-re^{i\theta}|^{n+1}}\,d\varphi\bigg)^p\,d\theta\bigg)^{1/p}\\
&=\frac{n!R}{R^2-r^2}\bigg(\frac{1}{2\pi}\int_0^{2\pi}\bigg(\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(Re^{i\varphi})|}{|Re^{i\varphi}-re^{i\theta}|^{n-1}}P_{r/R}(\theta-\varphi)\,d\varphi\bigg)^p\,d\theta\bigg)^{1/p}\\
&\le\frac{n!R}{(R^2-r^2)(R-r)^{n-1}}\bigg(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\varphi})|^pP_{r/R}(\theta-\varphi)\,d\varphi\,d\theta\bigg)^{1/p}\\
&=\frac{n!R}{(R^2-r^2)(R-r)^{n-1}}\bigg(\frac{1}{2\pi}\int_0^{2\pi}|f(Re^{i\varphi})|^p\frac{1}{2\pi}\int_0^{2\pi}P_{r/R}(\theta-\varphi)\,d\theta\,d\varphi\bigg)^{1/p}\\
&=\frac{n!R}{(R^2-r^2)(R-r)^{n-1}}M_p(f,R)
\end{aligned}
$$

for every $R>r$, where $P_s(t)=\frac{1-s^2}{1-2s\cos t+s^2}$, $0\le s<1$, is the Poisson kernel for the unit disc. Then, if we consider $R=r+\varepsilon$ for some $\varepsilon>0$, we get

$$M_p(f^{(n)},r)\le\frac{n!}{\varepsilon^{n-1}}\,\frac{r+\varepsilon}{\varepsilon^2+2r\varepsilon}M_p(f,r+\varepsilon)\le\frac{n!}{\varepsilon^n}M_p(f,r+\varepsilon).$$

If $a>0$, then

$$\frac{v_{a,\alpha}(r)}{v_{a,\alpha}(r+\varepsilon)}=\frac{r^ae^{-\alpha r}}{(r+\varepsilon)^ae^{-\alpha(r+\varepsilon)}}\le e^{\alpha\varepsilon}$$

for $r$ large enough. Thus,

$$v_{a,\alpha}(r)M_p(f^{(n)},r)\le\frac{n!}{\varepsilon^n}e^{\alpha\varepsilon}v_{a,\alpha}(r+\varepsilon)M_p(f,r+\varepsilon)\tag{5.2}$$

for $r$ large enough. This implies that there exists a constant $A>0$ such that $\|D^n\|_{p,q,a,\alpha}\le A(n!/\varepsilon^n)e^{\alpha\varepsilon}$ for every $\varepsilon>0$. If we take $\varepsilon=n/\alpha$, which minimizes $e^{\alpha\varepsilon}/\varepsilon^n$, we get

$$\|D^n\|_{p,q,a,\alpha}\le An!(e\alpha/n)^n.$$

If $a\le 0$, then there exists a constant $B>0$ such that

$$\frac{v_{a,\alpha}(r)}{v_{a,\alpha}(r+\varepsilon)}=\frac{r^ae^{\alpha\varepsilon}}{(r+\varepsilon)^a}\le B\frac{e^{\alpha\varepsilon}}{\varepsilon^a}$$

for $r$ large enough and $\varepsilon>\varepsilon_0$, for some $\varepsilon_0>0$. Thus,

$$v_{a,\alpha}(r)M_p(f^{(n)},r)\le B\frac{n!e^{\alpha\varepsilon}}{\varepsilon^{n+a}}v_{a,\alpha}(r+\varepsilon)M_p(f,r+\varepsilon)\tag{5.3}$$

for $r,\varepsilon$ large enough. Therefore, if we take $\varepsilon=(n+a)/\alpha\ge\varepsilon_0$, we deduce that there exists some $D_2>0$ such that

$$\|D^n\|_{p,q,a,\alpha}\le D_2n!\bigg(\frac{e\alpha}{n+a}\bigg)^{n+a}$$

for every $n\in\mathbb{N}$. ∎

PROPOSITION 5.10. *The spectrum of* $D : B_{p,q}(a,\alpha) \to B_{p,q}(a,\alpha)$ *satisfies*

$$\sigma(D) = \alpha\overline{\mathbb{D}}$$

*for* $1 \le p \le \infty$, *and* $q = 0$ *or* $1 \le q \le \infty$.

*Proof.* If $|\lambda| < \alpha$, the function $e_\lambda(z) := e^{\lambda z}$ belongs to $B_{p,q}(a,\alpha)$ by Lemma 2.1 and satisfies $De_\lambda = \lambda e_\lambda$. Therefore, $\alpha\mathbb{D} \subseteq \sigma(D)$. On the other hand, the spectral radius of $D$ satisfies $r(D) = \lim_{n\in\mathbb{N}} \|D^n\|_{p,q,a,\alpha}^{1/n}$. Using the Stirling formula and the upper estimates for the norms in Proposition 5.9, we obtain $r(D) \le \alpha$. ∎

By [AB, Proposition 4], $D-\lambda I$ *is not surjective on* $B_{p,q}(a,\alpha)$ *for* $|\lambda| = \alpha$. Furthermore $D - \lambda I$ is injective if and only if $e^{\lambda z} \notin B_{p,q}(a,\alpha)$. So Lemma 2.1 yields:

PROPOSITION 5.11. *For the weight* $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, *r large enough, and* $1 \le p \le \infty$, $D - \lambda I$ *is injective on* $B_{p,q}(a,\alpha)$ *if and only if either* $|\lambda| > \alpha$, *or* $|\lambda| = \alpha$ *and*

- (i) $a \ge 1/(2p)$ *when* $q = 0$,
- (ii) $a > 1/(2p)$ *when* $q = \infty$,
- (iii) $a \ge 1/(2p) - 2/q$ *if* $1 \le q < \infty$.

By Propositions 3.3 and 3.4, we get:

PROPOSITION 5.12. *Let* $v$ *be a weight such that* $D$ *is continuous on* $B_{p,q}(v)$, $1 \le p \le \infty$, $q \in \{0, p, \infty\}$, *and* $v(r)e^{\alpha r}$ *is non-increasing. If* $|\lambda| < \alpha$, *the operator* $D - \lambda I$ *is surjective on* $B_{p,q}(v)$ *and it even has*

$$K_\lambda f(z) = z\int_0^1 e^{\lambda z(1-t)} f(zt)\,dt, \qquad z \in \mathbb{C},$$

*as a continuous linear right inverse. In particular, this holds for the weight* $v_{a,\alpha}$, $a \le 0$, $\alpha > 0$.

THEOREM 5.13. *Given* $1 \le p \le \infty$, *and* $q = 0$ *or* $1 \le q \le \infty$:

- (i) *For* $\alpha > 1$, *or* $\alpha = 1$ *and* $a < 1/2 - 3/(2q)$, $D$ *is not power bounded on* $B_{p,q}(a,\alpha)$.
- (ii) *If* $D$ *is chaotic, then* $D$ *is not mean ergodic on* $B_{p,q}(a,\alpha)$. *Consequently,* $D$ *is not mean ergodic on* $B_{p,q}(a,\alpha)$ *if* $\alpha > 1$, *or* $\alpha = 1$ *and* $a < 1/(2p) - 2/q$.
- (iii) *For* $\alpha < 1$, $D$ *is power bounded and uniformly mean ergodic on* $B_{p,q}(a,\alpha)$.
- (iv) $D$ *is not uniformly mean ergodic on* $B_{p,q}(a,1)$, $a \in \mathbb{R}$.

*Proof.* (i) By (2.9),

$$\|D^n\|_{p,q,a,\alpha} \geq \frac{n!\|1\|_{q,a,\alpha}}{\|z^n\|_{q,a,\alpha}} \gtrsim \frac{\alpha^n}{n^{a+3/(2q)-1/2}}$$

and this tends to infinity for the values of $\alpha$ as in the hypothesis.

(ii) If $D$ is mean ergodic, then for each $f \in B_{p,q}(a,\alpha)$, we have $(f' + f'' + \cdots + f^{(N)})/N \to 0$, which is not the case if $D$ is chaotic, since $e^z \in B_{p,q}(a,\alpha)$.

(iii) Since

$$n!\left(\frac{e\alpha}{n}\right)^n \leq n!\left(\frac{e\alpha}{n-a}\right)^{n-a}$$

for every $a > 0$ and $n$ large enough, Proposition 5.9 yields

$$\|D^n\|_{p,q,a,\alpha} = O\left(n!\left(\frac{e\alpha}{n-|a|}\right)^{n-|a|}\right).$$

Applying the Stirling formula we get

$$\|D^n\|_{p,q,a,\alpha} = O\left(\left(\frac{n}{n-|a|}\right)^{n-|a|} n^{|a|+1/2}\alpha^{n-|a|}\right).$$

Therefore, for $\alpha < 1$, $\lim_{n\to\infty} \|D^n\|_{p,q,a,\alpha} = 0$, and thus

$$\lim_{m\to\infty} \left\|\frac{1}{m}\sum_{j=1}^m D^j\right\|_{p,q,a,\alpha} \leq \lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^m \|D^j\|_{p,q,a,\alpha} = 0.$$

(iv) Since $1 \in \sigma(D)$, the conclusion follows from Lemma 2.6. ∎

By Proposition 5.13 and [BBF, Corollary 3.11] we get:

COROLLARY 5.14. *Given $v_\alpha(r) = e^{-\alpha r}$, $\alpha > 0$, $1 \leq p \leq \infty$, and $q = 0$ or $1 \leq q \leq \infty$, we have:*

(i) *If $\alpha > 1$, then $D$ is not mean ergodic on $B_{p,q}(v)$.*
(ii) *If $\alpha < 1$, then $D$ is uniformly mean ergodic on $B_{p,q}(v)$.*
(iii) *If $\alpha = 1$, then $D$ is not uniformly mean ergodic on $B_{p,q}(v)$. It is not mean ergodic for $p = q = \infty$, and for $1 \leq p < \infty$ and $2/q < 1/(2p)$, $q \neq \infty$.*

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