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Isometries of the unitary groups in C^*-algebras

by

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To the memory of Professor Junzo Wada

Abstract. We give a complete description of the structure of surjective isometries between the unitary groups of unital C^*-algebras. While any surjective isometry between the unitary groups of von Neumann algebras can be extended to a real-linear Jordan *-isomorphism between the relevant von Neumann algebras, this is not the case for general unital C^*-algebras. We show that the unitary groups of two C^*-algebras are isomorphic as metric groups if and only if the C^*-algebras are isomorphic in the sense that each of them can be decomposed as the direct sum of two C^*-algebras with the first parts being linear *-algebra isomorphic and the second parts being conjugate-linear *-algebra isomorphic. We emphasize that in this paper by an isometry we merely mean a distance preserving transformation; we do not assume that it respects any algebraic operation.

1. Introduction. The study of linear isometries between Banach spaces or Banach algebras has a long tradition dating back to the 1930's. For an excellent comprehensive treatment of related results we refer to the two-volume set [5, 6]. The most prominent results in this area are the Banach–Stone theorem, which describes the structure of linear surjective isometry between the Banach algebras of all continuous functions on compact Hausdorff spaces, and its non-commutative generalization, Kadison's theorem [14], which describes the structure of a linear surjective isometry between general unital C^*-algebras: it is a Jordan *-isomorphism followed by left multiplication by a fixed unitary element. Recall that a Jordan *-isomorphism between C^*-algebras is a complex-linear bijection which preserves the *-operation and the power structure (and hence the Jordan structure). In this paper a real-linear (resp. conjugate-linear) Jordan *-isomorphism is a real-linear (resp. conjugate-linear) bijection which preserves the *-operation and the power structure. We also mention a classical result of similar spirit which also concerns isometries—the celebrated Mazur–Ulam theorem, stating that

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any surjective isometry between normed real-linear spaces is automatically a real-linear isometry followed by a translation.

Recently, several attempts have been made to describe the structure of isometries on non-linear substructures of Banach algebras including C^*-algebras (cf. [18, 19, 8, 13]). Molnár [18] and Molnár and Nagy [20] obtained a result concerning the structure of surjective Thompson isometries between the sets of invertible positive operators on Hilbert spaces. Isometries on quantum states were also considered [22, 21] (cf. [17, Section 2.4]).

The author, Hirasawa, Miura and Molnár [9] have developed a new technique for the study of isometries between substructures of the groups of invertible elements in unital Banach algebras: a Mazur–Ulam theorem for groups. Applying it, the author and Molnár gave a complete description of the structure of a surjective isometry from the unitary group of one von Neumann algebra onto the unitary group of another von Neumann algebra and showed that it can be uniquely extended to an isometry between these von Neumann algebras [11, 12]. The above mentioned result on Thompson isometries was generalized to general unital C^*-algebras by applying the Mazur–Ulam theorem for groups [12]. Another application of the theorem concerns isometries on Lie groups [1].

Our primary aim in this paper is to substantially generalize the above mentioned result on surjective isometries between the unitary groups of von Neumann algebras, namely to generalize them to the setting of general unital C^*-algebras; we give a complete description of the structure of isometries between the unitary groups of general unital C^*-algebras. In particular, we show that a surjective isometry between the principal components of the unitary groups of unital C^*-algebras can be extended to a direct sum of a Jordan *-isomorphism and a conjugate-linear Jordan *-isomorphism. This generalizes the above mentioned result for von Neumann algebras. On the other hand, we also show that an isometry between the unitary groups need not extend to an isometry between the underlying C^*-algebras in general. Such an isometry always exists if the corresponding unitary groups are disconnected.

The problem of equivalence of C^*-algebras with equivalent unitary groups probably dates back at least to the study of isomorphic unitary groups by Dye [4] and Sakai [23] in the 1950's. Al-Rawashdeh, Booth and Giordano [2] proved that within some classes of unital C^*-algebras, two algebras are complex-linear *-algebra isomorphic or conjugate-linear *-algebra isomorphic if and only if their unitary groups are isomorphic as abstract groups or topological groups. A simple example shows that this is not the case for general unital C^*-algebras: there are unital commutative C^*-algebras which are not isomorphic as real algebras while their unitary groups are isomorphic as topological groups (cf. Example 6.1). Applying our main results we shed new light on this problem from a slightly different point of view. Let A_j be a unital C^*-algebra and U_j the unitary group of A_j for $j = 1, 2$. Suppose that $\phi : U_1 \to U_2$ is a group isomorphism and K is a positive constant such that

$$\frac{1}{K}\|u - v\| \leq \|\phi(u) - \phi(v)\| \leq K\|u - v\|, \quad u, v \in U_1.$$

Does it follow that A_1 is real-linear *-algebra isomorphic to A_2? We give a partial answer to this question. As an application of our main result (Theorem 4.1) we show in Corollary 6.3 that if the unitary groups of two C^*-algebras are isomorphic as metric groups, then the C^*-algebras are real-linear *-algebra isomorphic; this is the case for $K = 1$. On the other hand, as is shown in Example 6.1, it is not the case for $K \geq 3$.

2. Preliminaries. To make the presentation complete, in this section we recall the necessary definitions and briefly summarize the results of [9] that we shall need in the proofs in Section 3. In Definitions 2.1, 2.2, and Proposition 2.3, (X_j, d_j) denotes a metric space, and X_j is a twisted subgroup of a group G_j in the sense that

$$yx^{-1}y \in X_j \quad \text{for all } x, y \in X_j.$$

Definition 2.1 (Condition $\mathrm{B}(\cdot,\cdot)$). Let $a, b \in X_j$. We say that $\mathrm{B}(a, b)$ holds for (X_j, d_j) if:

(B1) For all $x, y \in X_j$ we have

$$d_j(bx^{-1}b, by^{-1}b) = d_j(x, y).$$

(B2) There exists a constant $K > 1$ such that

$$d_j(bx^{-1}b, x) \geq K d_j(x, b)$$

for all $x \in L_{a,b} = \{x \in X_j : d_j(a, x) = d_j(ba^{-1}b, x) = d_j(a, b)\}$.

Definition 2.2 (Condition $\mathrm{C}_1(\cdot,\cdot)$). Let $a, b \in X_j$. We say that $\mathrm{C}_1(a, b)$ holds for (X_j, d_j) if:

(C1) For every $x \in X_j$ we have $ax^{-1}b$, $bx^{-1}a \in X_j$.
(C2) $d_j(ax^{-1}b, ay^{-1}b) = d_j(x, y)$ for all $x, y \in X_j$.

Proposition 2.3. *Let $\phi : X_1 \to X_2$ be a surjective isometry. Pick $a, b \in X_1$. Suppose that the condition $\mathrm{B}(a, b)$ holds for (X_1, d_1), and $\mathrm{C}_1(\phi(a), \phi(ba^{-1}b))$ holds for (X_2, d_2). Then*

$$\phi(ba^{-1}b) = \phi(b)(\phi(a))^{-1}\phi(b).$$

Theorem 6 in [11] is an analogue of the Mazur–Ulam theorem for groups of isometries and is stated only for self-maps of subgroups of full unitary groups, but a similar statement for surjective isometries between any two such subgroups acting on any two Banach spaces is possible. For a complex

Banach space $\mathcal{B}$ we denote by $\mathrm{Iso}(\mathcal{B})$ the group of complex-linear isometries from $\mathcal{B}$ onto itself. Proposition 2.4 below plays an important role in the proof of Lemma 3.3, describing the algebraic structure of isometries between the principal components of unitary groups.

PROPOSITION 2.4. *Let $\mathcal{B}_j$ be a complex Banach space and $\mathcal{G}_j$ be a subgroup of $\mathrm{Iso}(\mathcal{B}_j)$ equipped with the metric d_j coming from the operator norm for $j = 1, 2$. Suppose that $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a surjective isometry, that is, ϕ is just a distance preserving surjection. Then*

$$\phi(VU^{-1}V) = \phi(V)(\phi(U))^{-1}\phi(V) \tag{2.1}$$

for all $U, V \in \mathcal{G}_1$ that satisfy $d_1(U, V) < 1/2$.

The proof is an application of Proposition 2.3, which is very similar to the one of Theorem 6 given in [11], and is omitted.

Note that the assumption $d_1(U, V) < 1/2$ in Proposition 2.4 is essential: see the example just after the proof of Theorem 6 in [11].

3. Isometries between the principal components. Let A_j be a unital C^*-algebra. The real-linear space of all self-adjoint elements in A_j is denoted by A_{jS}. The principal component of the unitary group U_j of A_j is denoted by U_j^0. The principal component is a normal subgroup of U_j. The quotient group U_j/U_j^0 is denoted by Λ_j. We exhibit the form of isometries (distance preserving maps without additional assumptions about respecting any algebraic operations) between the principal components of the unitary groups of two unital C^*-algebras.

THEOREM 3.1. *Let A_j be a unital C^*-algebra for $j = 1, 2$. Suppose that ϕ is a map from U_1^0 into U_2^0. The map ϕ is a surjective isometry if and only if there exists a central projection p in A_2 and a Jordan *-isomorphism J from A_1 onto A_2 such that*

$$\phi(a) = \phi(1)(pJ(a) + (1 - p)J(a)^*), \qquad a \in U_1^0. \tag{3.1}$$

The unitary group of a von Neumann algebra is connected, hence the principal component of the unitary group is the unitary group itself. Theorem 3.1 thus generalizes Corollary 3 in [12], which describes the structure of surjective isometries between the unitary groups of von Neumann algebras. The proof of Corollary 3 in [12] depends on the fact that the unitary group of a von Neumann algebra consists precisely of the elements of the form $\exp(ix)$ for self-adjoint elements x in the algebra. This is not the case for the principal component of a general unital C^*-algebra (cf. [16, 4.6.9]), hence the proof of [12, Corollary 3] does not work for Theorem 3.1.

To prove Theorem 3.1 we employ two lemmas. The first one concerns the structure of the principal components of unitary groups.

LEMMA 3.2. *We have*

$$U_j^0 = \{\exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n) : \\ n \text{ is a positive integer, } x_k \in A_{jS} \text{ for } 0 \le k \le n\}.$$

Proof. Denote by V_j the right hand side above. Then $1 \in V_j$ by taking $n = 1$ and $x_1 = x_0 = 0$. We claim that V_j is open and closed in U_j. Let

$$v = \exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n) \in V_j$$

be arbitrary. Suppose $w \in U_j$ and $\|w - v\| < 2$. As $\exp(ix_k)$ is a unitary for each $x_k \in A_{jS}$, we have

$$\|\exp(-ix_0/2)\exp(-ix_1)\cdots\exp(-ix_n)w\exp(-ix_n)\cdots\exp(-ix_1)\exp(-ix_0/2) - 1\| < 2.$$

We infer

$$\sigma(\exp(-ix_0/2)\exp(-ix_1)\cdots\exp(-ix_n)w\exp(-ix_n)\cdots\exp(-ix_1)\exp(-ix_0/2)) \\ \subset \{z \in \mathbb{C} : |z| = 1,\ z \ne -1\},$$

where $\sigma(\cdot)$ denote the spectrum. Therefore there exists $y \in A_{jS}$ such that

$$\exp(iy) = \exp(-ix_0/2)\exp(-ix_1)\cdots\exp(-ix_n)w\exp(-ix_n)\cdots\exp(-ix_1)\exp(-ix_0/2),$$

whence

$$w = \exp(ix_n)\cdots\exp(ix_1)\exp(ix_0/2)\exp(iy)\exp(ix_0/2)\exp(ix_1)\cdots\exp(ix_n)$$

is in V_j. Now, for a general metric space (S, d), a subset K of S for which there is a positive real number r (independent of the choice of an element s of K) such that

$$\{t \in S : d(t, s) < r\} \subset K$$

for every $s \in K$, is open and closed in S. Thus in our case V_j is open and closed in U_j.

We assert that V_j is connected. Let

$$\exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n) \in V_j.$$

For $0 \le t \le 1$, put

$$a_t = \exp(itx_n)\cdots\exp(tix_1)\exp(itx_0)\exp(itx_1)\cdots\exp(itx_n).$$

Then $a_t \in V_j$, $a_0 = 1$ and

$$a_1 = \exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n).$$

Hence V_j is arcwise connected. Thus we conclude that V_j is a connected component of U_j which contains 1, i.e., $V_j = U_j^0$. ■

Suppose that $\phi : U_1^0 \to U_2^0$ is a surjective isometry. Then ϕ_0 defined by $\phi_0(\cdot) = (\phi(1))^{-1}\phi(\cdot)$ is a surjective isometry from U_1^0 onto U_2^0 with $\phi_0(1) = 1$. The second lemma we employ in the proof of Theorem 3.1 is the following.

LEMMA 3.3. *Let A_j be a unital C^*-algebra for $j = 1, 2$. Suppose that ϕ is a map from U_1^0 into U_2^0. If ϕ is a surjective isometry, then for any positive integer n and any $n + 1$ elements $x_0, x_1, \ldots, x_n \in A_{1S}$,*

$$\begin{aligned}(3.2)\qquad &\phi_0(\exp(ix_n)\ldots\exp(ix_1)\exp(ix_0)\exp(ix_1)\ldots\exp(ix_n)) \\ &= \phi_0(\exp(ix_n))\cdots\phi_0(\exp(ix_1))\phi_0(\exp(ix_0))\phi_0(\exp(ix_1))\cdots\phi_0(\exp(ix_n)).\end{aligned}$$

Proof. We apply the Mazur–Ulam theorem for groups (Proposition 2.4), and then a one-parameter-group argument. Suppose that ϕ is a surjective isometry. As already noted, ϕ_0 is also an isometry from U_1^0 onto U_2^0. We prove (3.2) by induction on n.

Suppose that $n = 1$. Let $x, x_1 \in A_{1S}$. Choose any real numbers t_1 and t_2 and set $a_1 = \exp(ix_1)$, $a = \exp(it_1x)$ and $b = \exp(it_2x)$. We shall prove

$$(3.3)\qquad \phi_0(a_1baba_1) = \phi_0(a_1ba_1)(\phi_0(a_1a^{-1}a_1))^{-1}\phi_0(a_1ba_1).$$

Select a positive integer m such that

$$\exp(\|(t_1 + t_2)x\|/2^m) - 1 < 1/2.$$

Clearly,

$$(3.4)\qquad \|\exp(i(t_1 + t_2)x/2^m) - 1\| \le \exp(\|(t_1 + t_2)x\|/2^m) - 1 < 1/2.$$

For $j = 0, 1, \ldots, 2^{m+1}$ let

$$c_j = a^{-1}\exp(ij(t_1 + t_2)x/2^m).$$

Then $c_0 = a^{-1}$, $c_{2^m} = b$, $c_{2^{m+1}} = bab$. It is easy to check by (3.4) that

$$\|a_1c_{j+1}a_1 - a_1c_ja_1\| < 1/2$$

for $j = 0, 1, \ldots, 2^{m+1} - 1$. By the Gelfand–Naimark theorem any C^*-algebra is isometrically *-isomorphic to a C^*-algebra of operators on a Hilbert space. Hence we can apply Proposition 2.4 to infer that

$$\begin{aligned}(3.5)\qquad &\phi_0(a_1c_{j+1}a_1(a_1c_ja_1)^{-1}a_1c_{j+1}a_1) \\ &\qquad = \phi_0(a_1c_{j+1}a_1)(\phi_0(a_1c_ja_1))^{-1}\phi_0(a_1c_{j+1}a_1)\end{aligned}$$

for all $j = 0, 1, \ldots, 2^{m+1} - 2$. By a simple calculation we obtain

$$(a_1c_{j+1}a_1)(a_1c_ja_1)^{-1}(a_1c_{j+1}a_1) = a_1c_{j+2}a_1$$

for every $j = 0, 1, \ldots, 2^{m+1} - 2$. Applying the technical Lemma 7 in [11] for the sequence $\{a_1c_ja_1\}_{j=1}^{2^{m+1}}$, we find that (3.5) implies that the inverted Jordan product of $a_1c_0a_1$, $a_1c_{2^m}a_1$ is also preserved:

$$\phi_0(a_1c_{2^m}a_1(a_1c_0a_1)^{-1}a_1c_{2^m}a_1) = \phi_0(a_1c_{2^m}a_1)(\phi_0(a_1c_0a_1))^{-1}\phi_0(a_1c_{2^m}a_1).$$

Hence we obtain (3.3):

$$\begin{aligned}\phi_0(a_1baba_1) &= \phi_0(a_1c_{2^m}a_1(a_1c_0a_1)^{-1}a_1c_{2^m}a_1)\\ &= \phi_0(a_1c_{2^m}a_1)(\phi_0(a_1c_0a_1))^{-1}\phi_0(a_1c_{2^m}a_1)\\ &= \phi_0(a_1ba_1)(\phi_0(a_1a^{-1}a_1))^{-1}\phi_0(a_1ba_1)\end{aligned}$$

for any x, x_1 and any t_1, t_2. In particular, putting $x_1 = 0$, $t_1 = 0$, we get

$$\phi_0(b^2) = (\phi_0(b))^2 \tag{3.6}$$

as $\phi_0(1) = 1$.

Define $\phi_1 : U_1^0 \to U_2^0$ by $\phi_1(c) = (\phi_0(a_1))^{-1}\phi_0(c)(\phi_0(a_1))^{-1}$. As ϕ_0 is a surjective isometry we infer that ϕ_1 is well defined and also a surjective isometry. By (3.3) we have

$$\begin{aligned}\phi_1(a_1baba_1) &= (\phi_0(a_1))^{-1}\phi_0(a_1baba_1)(\phi_0(a_1))^{-1}\\ &= (\phi_0(a_1))^{-1}\phi_0(a_1ba_1)(\phi_0(a_1a^{-1}a_1))^{-1}\phi_0(a_1ba_1)(\phi_0(a_1))^{-1}\\ &= \big((\phi_0(a_1))^{-1}\phi_0(a_1ba_1)(\phi_0(a_1))^{-1}\big)\big((\phi_0(a_1))^{-1}\phi_0(a_1a^{-1}a_1)(\phi_0(a_1))^{-1}\big)^{-1}\\ &\qquad \times (\phi_0(a_1)^{-1}\phi_0(a_1ba_1)(\phi_0(a_1))^{-1})\\ &= \phi_1(a_1ba_1)(\phi_1(a_1a^{-1}a_1))^{-1}\phi_1(a_1ba_1).\end{aligned} \tag{3.7}$$

We infer from the definition of ϕ_1 that

$$\phi_1(a_1^2) = (\phi_0(a_1))^{-1}\phi_0(a_1^2)(\phi_0(a_1))^{-1} = 1 \tag{3.8}$$

as (3.6) also holds for $x = x_1$ and $t_2 = 1$, $b = a_1$. Substituting $t_2 = 0$ in (3.7) we see by (3.8) that

$$\phi_1(a_1aa_1) = \big((\phi_0(a_1))^{-1}\phi_0(a_1a^{-1}a_1)(\phi_0(a_1))^{-1}\big)^{-1} = (\phi_1(a_1a^{-1}a_1))^{-1}. \tag{3.9}$$

Substituting (3.9) in (3.7) we have

$$\phi_1(a_1baba_1) = \phi_1(a_1ba_1)\phi_1(a_1aa_1)\phi_1(a_1ba_1). \tag{3.10}$$

One can easily deduce from (3.8)–(3.10) that

$$\phi_1(a_1b^la_1) = (\phi_1(a_1ba_1))^l \tag{3.11}$$

for $b, a_1 \in U_1^0$ of the form $b = \exp(itx)$, $a_1 = \exp(ix_1)$ with any $x, x_1 \in A_{1S}$, $t \in \mathbb{R}$, and for every integer l.

Define a map $S_x : \mathbb{R} \to U_2^0$ by

$$S_x(t) = \phi_1(a_1 \exp(itx)a_1), \qquad t \in \mathbb{R}.$$

We assert that S_x is a continuous one-parameter unitary group in A_2. Since ϕ_1 is continuous, we only need to prove that $S_x(t+t') = S_x(t)S_x(t')$ for any real t, t'. First select rational r and r' such that $r = n/m$ and $r' = n'/m'$ with integers m, m', n, n'. We compute, by (3.11),

$$S_x(r+r') = \phi_1\Big(a_1\Big(\exp\Big(i\tfrac{nm'+mn'}{mm'}x\Big)\Big)a_1\Big) = \phi_1\Big(a_1\Big(\exp\Big(i\tfrac{1}{mm'}x\Big)\Big)a_1\Big)^{nm'+mn'}$$
$$= \phi_1\Big(a_1\Big(\exp\Big(i\tfrac{1}{mm'}x\Big)\Big)a_1\Big)^{nm'}\phi_1\Big(a_1\Big(\exp\Big(i\tfrac{1}{mm'}x\Big)\Big)a_1\Big)^{mn'} = S_x(r)S_x(r').$$

Since ϕ_1 is continuous we obtain $S_x(t+t') = S_x(t)S_x(t')$ for all real t, t'.

As already mentioned, we may consider A_1, A_2 as unital C^*-algebras of operators that act on Hilbert spaces H_1, H_2, respectively. Applying Stone's theorem (see [3, Chapter X, Section 5]) for the norm continuous one-parameter unitary group $(S_x(t))_{t\in\mathbb{R}}$, we infer that there exists a unique bounded self-adjoint operator y on H_2 such that $S_x(t) = \exp(ity)$ for every $t \in \mathbb{R}$. Since the generator y can be obtained by differentiating $\exp(ity)$ with respect to t, where the limit of difference quotients is taken in the norm topology, it follows that $y \in A_{2S}$. Defining $f(x) = y$ we obtain a map $f : A_{1S} \to A_{2S}$ for which

$$\phi_1(a_1(\exp(itx))a_1) = S_x(t) = \exp(itf(x)), \qquad t \in \mathbb{R},\ x \in A_{1S}. \tag{3.12}$$

We claim that f is surjective. Define $\psi : U_2^0 \to U_1^0$ by

$$\psi(c) = a_1^{-1}\phi_1^{-1}(c)a_1^{-1}.$$

Then ψ is clearly a surjective isometry. Let $y \in A_{2S}$. Choose any real numbers t_1 and t_2 and set $c = \exp(it_1y)$, $d = \exp(it_2y)$. Select a positive integer m such that

$$\exp(\|(t_1+t_2)y\|/2^m) - 1 < 1/2.$$

For $j = 0, 1, \ldots, 2^{m+1}$ let

$$c_j = c^{-1}\exp(ij(t_1+t_2)y/2^m).$$

Applying Proposition 2.4 as before we see that

$$\psi(c_{j+1}c_j^{-1}c_{j+1}) = \psi(c_{j+1})(\psi(c_j))^{-1}\psi(c_{j+1})$$

for every $j = 0, 1, \ldots, 2^{m+1}-2$ and

$$\begin{aligned}\psi(dcd) &= \psi(c_{2^m}c_0^{-1}c_{2^m}) = \psi(c_{2^m})(\psi(c_0))^{-1}\psi(c_{2^m})\\ &= \psi(d)(\psi(c^{-1}))^{-1}\psi(d).\end{aligned} \tag{3.13}$$

Since $\psi(1) = a_1^{-1}\phi_1^{-1}(1)a_1^{-1}$ and $\phi_1(a_1^2) = 1$, we infer that $\psi(1) = 1$ and hence $\psi(c) = (\psi(c^{-1}))^{-1}$ by substituting $d = 1$ in (3.13), so

$$\psi(dcd) = \psi(d)\psi(c)\psi(d).$$

Just as for S_x, we see that the map $T_y : \mathbb{R} \to U_1^0$ defined by $T_y(t) = \psi(\exp(ity))$ is a one-parameter unitary group in A_1 and there is a map $g : A_{2S} \to A_{1S}$ with

$$\psi(\exp(ity)) = T_y(t) = \exp(itg(y)).$$

Then

$$\begin{aligned}\exp(itf(g(y))) &= \phi_1(a_1(\exp(it(g(y))))a_1)\\ &= \phi_1(a_1(\psi(\exp(ity)))a_1) = \phi_1(\phi_1^{-1}(\exp(ity))) = \exp(ity), \quad t \in \mathbb{R}.\end{aligned}$$

It follows that $f(g(y)) = y$ for every $y \in A_{2S}$, which means that f is surjective.

We claim that f is an isometry. Let $x, x' \in A_{1S}$. Since ϕ_1 is an isometry and a_1 is unitary we have

$$\begin{aligned}&\left\|\frac{\exp(itf(x)) - \exp(itf(x'))}{t}\right\|\\ &\qquad = \left\|\frac{\phi_1(a_1(\exp(itx))a_1) - \phi_1(a_1(\exp(itx'))a_1)}{t}\right\|\\ &\qquad = \left\|\frac{a_1(\exp(itx))a_1 - a_1(\exp(itx'))a_1}{t}\right\| = \left\|\frac{\exp(itx) - \exp(itx')}{t}\right\|.\end{aligned}$$

Letting $t \to 0$, we obtain

$$\frac{\exp(itx) - \exp(itx')}{t} = \frac{\exp(itx) - 1}{t} - \frac{\exp(itx') - 1}{t} \to ix - ix',$$

and similarly

$$\frac{\exp(itf(x)) - \exp(itf(x'))}{t} \to if(x) - if(x').$$

Hence $\|f(x) - f(x')\| = \|x - x'\|$ for any $x, x' \in A_{1S}$. This shows that f is an isometry. Since $f(0) = 0$ by (3.12) and (3.8), we infer that f is a surjective real-linear isometry from A_{1S} onto A_{2S}, by the Mazur–Ulam theorem.

Consider the case where $x_1 = 0$. Then $\phi_1 = \phi_0$ by the definition of ϕ_1. By (3.12) we have a surjective isometry f_0 from A_{1S} onto A_{2S} such that

$$\phi_0(\exp itx) = \phi_1(\exp itx) = \exp(itf_0(x)), \quad t \in \mathbb{R},\ x \in A_{1S}. \tag{3.14}$$

We claim $f = f_0$ for any x_1. Since

$$\exp(ix_1) - \exp(-itx) = 0$$

for $t = 1$ and $x = -x_1$, there exists $\varepsilon > 0$ such that

$$\|\exp(ix_1) - \exp(-itx)\| < 1/2$$

for all real t with $|t - 1| < \varepsilon$ and $x \in A_{1S}$ with $\|x + x_1\| < \varepsilon$. By Proposition 2.4 we observe that

$$\begin{aligned}&\phi_0(\exp(ix_1)\exp(itx)\exp(ix_1))\\ &\quad = \phi_0(\exp(ix_1))(\phi_0(\exp(-itx)))^{-1}\phi_0(\exp(ix_1)), \quad |t - 1| < \varepsilon,\ \|x + x_1\| < \varepsilon.\end{aligned}$$

As $\phi_0(1) = 1$ we can easily deduce that

$$(3.15)\qquad \phi_0(\exp(ix_1)\exp(itx)\exp(ix_1)) \\ = \phi_0(\exp(ix_1))\phi_0(\exp(itx))\phi_0(\exp(ix_1)), \quad |t-1| < \varepsilon,\ \|x + x_1\| < \varepsilon.$$

By (3.12) we have

$$(3.16)\quad \phi_0(a_1(\exp(itx))a_1) = \phi_0(a_1)\exp(itf(x))\phi_0(a_1), \quad t \in \mathbb{R},\ x \in A_{1S}.$$

As $a_1 = \exp(ix_1)$ we infer from (3.14)–(3.16) that

$$(3.17)\qquad \exp(itf_0(x)) = \phi_0(\exp(itx)) = \exp(itf(x)), \\ |t-1| < \varepsilon,\ \|x + x_1\| < \varepsilon.$$

In particular,

$$\exp(if_0(x)) = \exp(if(x)), \qquad \|x + x_1\| < \varepsilon.$$

Differentiating both sides of (3.17) at $t = 1$ we get

$$if_0(x)\exp(if_0(x)) = if(x)\exp(if(x)), \qquad \|x + x_1\| < \varepsilon,$$

so that

$$f_0(x) = f(x), \qquad \|x + x_1\| < \varepsilon.$$

Since f_0 and f are surjective real-linear isometries from A_{1S} onto A_{2S} and $f_0 = f$ on a connected open subset $\{x \in A_{1S} : \|x + x_1\| < \varepsilon\}$ of A_{1S}, we infer that $f_0 = f$ on A_{1S}. Then by (3.14) and (3.16),

$$\phi_0(\exp(ix_1)\exp(ix)\exp(ix_1)) = \phi_0(\exp(ix_1))\exp(if_0(x))\phi_0(\exp(ix_1)) \\ = \phi_0(\exp(ix_1))\phi_0(\exp(ix))\phi_0(\exp(ix_1))$$

for every $x \in A_{1S}$. As $x, x_1 \in A_{1S}$ are arbitrary, we conclude that (3.2) holds for $n = 1$.

Suppose that (3.2) holds for $n = k$. We claim it also holds for $n = k+1$. Let $x, x_1, \ldots, x_{k+1} \in A_{1S}$. Choose any real numbers t_1 and t_2 and set $a = \exp(it_1x)$, $b = \exp(it_2x)$, $a_1 = \exp(ix_1)$, $\ldots$, $a_{k+1} = \exp(ix_{k+1})$. Define the product

$$\mathsf{P} = a_1 \cdots a_{k+1}$$

and the reverse product

$$\mathsf{R} = a_{k+1} \cdots a_1$$

just for the simplicity of fomulae. We claim

$$(3.18)\qquad \phi_0(\mathsf{R}bab\mathsf{P}) = \phi_0(\mathsf{R}b\mathsf{P})(\phi_0(\mathsf{R}a^{-1}\mathsf{P}))^{-1}\phi_0(\mathsf{R}b\mathsf{P}).$$

Select a positive integer m such that

$$\exp(\|(t_1 + t_2)x\|/2^m) - 1 < 1/2,$$

and set

$$c_j = a^{-1}\exp(ij(t_1 + t_2)x/2^m).$$

for $j = 0, 1, \ldots, 2^{m+1}$. Then $c_0 = a^{-1}$, $c_{2^m} = b$, $c_{2^{m+1}} = bab$. It is also easy to check that

$$\|\mathsf{R}c_{j+1}\mathsf{P} - \mathsf{R}c_j\mathsf{P}\| < 1/2$$

for every $j = 0, 1, \ldots, 2^{m+1} - 1$ and

$$(\mathsf{R}c_{j+1}\mathsf{P})(\mathsf{R}c_j\mathsf{P})^{-1}(\mathsf{R}c_{j+1}\mathsf{P}) = \mathsf{R}c_{j+2}\mathsf{P}$$

for every $j = 0, 1, \ldots, 2^{m+1}-2$. Applying Proposition 2.4 and [11, Lemma 7] as in the case of $n = 1$, we see that

$$\phi_0\big((\mathsf{R}c_{j+1}\mathsf{P})(\mathsf{R}c_j\mathsf{P})^{-1}(\mathsf{R}c_{j+1}\mathsf{P})\big) = \phi_0(\mathsf{R}c_{j+1}\mathsf{P})(\phi_0(\mathsf{R}c_j\mathsf{P}))^{-1}\phi_0(\mathsf{R}c_{j+1}\mathsf{P})$$

and

$$\begin{aligned}\phi_0(\mathsf{R}bab\mathsf{P}) &= \phi_0\big((\mathsf{R}c_{2^m}\mathsf{P})(\mathsf{R}c_0\mathsf{P})^{-1}(\mathsf{R}c_{2^m}\mathsf{P})\big)\\ &= \phi_0(\mathsf{R}c_{2^m}\mathsf{P})(\phi_0(\mathsf{R}c_0\mathsf{P}))^{-1}\phi_0(\mathsf{R}c_{2^m}\mathsf{P})\\ &= \phi_0(\mathsf{R}b\mathsf{P})(\phi_0(\mathsf{R}a^{-1}\mathsf{P}))^{-1}\phi_0(\mathsf{R}b\mathsf{P}).\end{aligned} \tag{3.19}$$

Define $\phi_{k+1} : U_1^0 \to U_2^0$ by

$$\phi_{k+1}(c) = (\phi_0(a_1))^{-1}\cdots(\phi_0(a_{k+1}))^{-1}\phi_0(c)(\phi_0(a_{k+1}))^{-1}\cdots(\phi_0(a_1))^{-1}$$

for $c \in U_1^0$. Then by (3.19) we have

$$\phi_{k+1}(\mathsf{R}bab\mathsf{P}) = \phi_{k+1}(\mathsf{R}b\mathsf{P})(\phi_{k+1}(\mathsf{R}a^{-1}\mathsf{P}))^{-1}\phi_{k+1}(\mathsf{R}b\mathsf{P}). \tag{3.20}$$

By (3.6) and the induction hypothesis we obtain

$$\begin{aligned}\phi_0(\mathsf{R}\,\mathsf{P}) &= \phi_0(a_{k+1})\cdots\phi_0(a_2)\phi_0(a_1^2)\phi_0(a_2)\cdots\phi_0(a_{k+1})\\ &= \phi_0(a_{k+1})\cdots\phi_0(a_2)\phi_0(a_1)\phi_0(a_1)\phi_0(a_2)\cdots\phi_0(a_{k+1}),\end{aligned}$$

so that

$$\phi_{k+1}(\mathsf{R}\,\mathsf{P}) = 1. \tag{3.21}$$

Substituting $t_2 = 0$ (hence $b = 0$) in (3.20) we obtain

$$\begin{aligned}\phi_{k+1}(\mathsf{R}a\mathsf{P}) &= \phi_{k+1}(\mathsf{R}\,\mathsf{P})(\phi_{k+1}(\mathsf{R}a^{-1}\mathsf{P}))^{-1}\phi_{k+1}(\mathsf{R}\,\mathsf{P})\\ &= (\phi_{k+1}(\mathsf{R}a^{-1}\mathsf{P}))^{-1}.\end{aligned} \tag{3.22}$$

Then by (3.20) we obtain

$$\phi_{k+1}(\mathsf{R}bab\mathsf{P}) = \phi_{k+1}(\mathsf{R}b\mathsf{P})\phi_{k+1}(\mathsf{R}a\mathsf{P})\phi_{k+1}(\mathsf{R}b\mathsf{P}). \tag{3.23}$$

By (3.21)–(3.23) one can easily deduce that

$$\phi_{k+1}(\mathsf{R}b^l\mathsf{P}) = (\phi_{k+1}(\mathsf{R}b\mathsf{P}))^l$$

for $b \in U_1^0$ of the form $b = \exp(itx)$ with any $x \in A_{1S}$, $t \in \mathbb{R}$ and for any integer l.

Define $S_x^{(k+1)} : \mathbb{R} \to U_2^0$ by

$$S_x^{(k+1)}(t) = \phi_{k+1}(\mathsf{R}(\exp(itx))\mathsf{P}), \quad t \in \mathbb{R}.$$

Similarly to the case where $n=1$, we see that $S_x^{(k+1)}$ is a continuous one-parameter unitary group in A_2 and there is a unique $y \in A_{2S}$ such that

$$S_x^{(k+1)}(t) = \exp(ity), \quad t \in \mathbb{R}.$$

Then the map $f_{k+1} : A_{1S} \to A_{2S}$ is defined by $f_{k+1}(x) = y$, i.e.,

(3.24)

$$\phi_{k+1}(\mathsf{R}(\exp(itx))\mathsf{P}) = S_x^{(k+1)}(t) = \exp(itf_{k+1}(x)), \quad x \in A_{1S},\ t \in \mathbb{R},$$

hence

$$\begin{aligned}(3.25)\qquad &\phi_0(\mathsf{R}(\exp(itx))\mathsf{P})\\ &= \phi_0(a_{k+1})\cdots\phi_0(a_1)(\exp(itf_{k+1}(x))\phi_0(a_1)\cdots\phi_0(a_{k+1}), \quad x \in A_{1S},\ t \in \mathbb{R}.\end{aligned}$$

We prove that f_{k+1} is surjective. Define $\psi : U_2^0 \to U_1^0$ by

$$\psi(c) = a_1^{-1}\cdots a_{k+1}^{-1}\phi_{k+1}^{-1}(c)a_{k+1}^{-1}\cdots a_1^{-1}.$$

Since ϕ_{k+1} is a surjective isometry, ψ is well defined and is a surjective isometry from U_2^0 onto U_1^0. Choose any $y \in A_{2S}$ and $t_1, t_2 \in \mathbb{R}$. Set $c = \exp(it_1y)$, $d = \exp(it_2y)$. Just as for $n = 1$, applying Proposition 2.4 and [11, Lemma 7] we see that

$$\psi(dcd) = \psi(d)\psi(c)\psi(d).$$

We also see that the map $T_y^{(k+1)} : \mathbb{R} \to U_1^0$ defined by

$$T_y^{(k+1)}(t) = \psi(\exp(ity)), \quad t \in \mathbb{R},$$

is continuous one-parameter unitary group in A_1. Therefore there is a map $g_{k+1} : A_{2S} \to A_{1S}$ with

$$\psi(\exp(ity)) = T_y^{(k+1)}(t) = \exp(itg_{k+1}(y)), \quad y \in A_{2S},\ t \in \mathbb{R}.$$

Then

$$\begin{aligned}\exp(itf_{k+1}(g_{k+1}(y))) &= \phi_{k+1}(\mathsf{R}(\exp(itg_{k+1}(y)))\mathsf{P})\\ &= \phi_{k+1}(\mathsf{R}(\psi(\exp(ity)))\mathsf{P})\\ &= \phi_{k+1}(\phi_{k+1}^{-1}(\exp(ity))) = \exp(ity), \quad y \in A_{2S},\ t \in \mathbb{R}.\end{aligned}$$

It follows that

$$f_{k+1}(g_{k+1}(y)) = y, \quad y \in A_{2S},$$

and so f_{k+1} is a surjection from A_{1S} onto A_{2S}.

We claim that f_{k+1} is an isometry. Let $x, x' \in A_{1S}$. Then as ϕ_{k+1} is an isometry and $a_1, \ldots, a_{k+1} \in U_1^0$, we have

$$\begin{aligned}\left\|\frac{\exp(itf_{k+1}(x))-\exp(itf_{k+1}(x'))}{t}\right\| \\ &= \left\|\frac{\phi_{k+1}(\mathsf{R}(\exp(itx))\,\mathsf{P})-\phi_{k+1}(\mathsf{R}(\exp(itx'))\,\mathsf{P})}{t}\right\| \\ &= \left\|\frac{\exp(itx)-\exp(itx')}{t}\right\|.\end{aligned}$$

Letting $t\to 0$ yields

$$\|f_{k+1}(x)-f_{k+1}(x')\|=\|x-x'\|.$$

Therefore f_{k+1} is a surjective isometry from A_{1S} onto A_{2S}. Since $f_{k+1}(0)=0$ by (3.24) and (3.21), we infer that f_{k+1} is a surjective real-linear isometry from A_{1S} onto A_{2S}, by the Mazur–Ulam theorem.

If $t=1$ and $x=-2x_1$, then

$$a_1(\exp(itx))a_1=\exp(ix_1)\exp(itx)\exp(ix_1)=1.$$

Hence there exists an $\varepsilon>0$ such that

$$\|a_1(\exp(itx))a_1-1\|<2$$

for every t with $|t-1|<\varepsilon$ and every $x\in A_{1S}$ with $\|x+2x_1\|<\varepsilon$. Since $a_1(\exp(itx))a_1$ is unitary we infer that for every t with $|t-1|<\varepsilon$ and every $x\in A_{1S}$ with $\|x+2x_1\|<\varepsilon$ there exists $x_t\in A_{1S}$ such that

$$a_1(\exp(itx))a_1=\exp(ix_t).$$

Thus

$$\mathsf{R}(\exp(itx)\,\mathsf{P}=a_{k+1}\cdots a_2(\exp(ix_t))a_2\cdots a_{k+1}.$$

By the induction assumption we infer that

$$\begin{aligned}\phi_0(\mathsf{R}(\exp(itx))\,\mathsf{P})&=\phi_0(a_{k+1}\cdots a_2(\exp(ix_t))a_2\cdots a_{k+1})\\ &=\phi_0(a_{k+1})\cdots\phi_0(a_2)\phi_0(\exp(ix_t))\phi_0(a_2)\cdots\phi_0(a_{k+1}).\end{aligned}$$

We already know that (3.2) holds for $n=1$, hence

$$\phi_0(\exp(ix_t))=\phi_0(\exp(ix_1))\phi_0(\exp(itx))\phi_0(\exp(ix_1)),$$

so

$$\begin{aligned}&\phi_0(\mathsf{R}(\exp(itx))\,\mathsf{P})\\ &=\phi_0(\exp(ix_{k+1}))\cdots\phi_0(\exp(ix_1))\phi_0(\exp(itx))\phi_0(\exp(ix_1))\cdots\phi_0(\exp(ix_{k+1})),\\ &\qquad |t-1|<\varepsilon,\ \|x+2x_1\|<\varepsilon.\end{aligned}\tag{3.26}$$

Recall that the isometry f_0 satisfies (3.14). Hence, by (3.25) and (3.26),

$$\exp(itf_0(x))=\phi_0(\exp(itx))=\exp(itf_{k+1}(x)),\quad |t-1|<\varepsilon,\ \|x+2x_1\|<\varepsilon.$$

As in the case where $n=1$ we find that $f_0=f_{k+1}$ on A_{1S}. Then by (3.14) and (3.25),

$$
\begin{aligned}
&\phi_0(\exp(ix_{k+1})\cdots\exp(ix_1)\exp(ix)\exp(ix_1)\cdots\exp(ix_{k+1})\\
&\ = \phi_0(\exp(ix_{k+1}))\cdots\phi_0(\exp(ix_1)(\exp(if_0(x)))\phi_0(\exp(ix_1))\cdots\phi_0(\exp(ix_{k+1}))\\
&\ = \phi_0(\exp(ix_{k+1}))\cdots\phi_0(\exp(ix_1)\phi_0(\exp(ix))\phi_0(\exp(ix_1))\cdots\phi_0(\exp(ix_{k+1})).
\end{aligned}
$$

As $x, x_1, \ldots, x_{k+1} \in A_{1S}$ are arbitrary, we conclude that (3.2) holds for $n = k+1$. ■

Proof of Theorem 3.1. Suppose that ϕ is a surjective isometry. Let $a \in U_1^0$. Then by Lemma 3.2 there are a finite number of points $x_0, x_1, \ldots, x_n \in A_{1S}$ with

$$a = \exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n).$$

Recall that f_0 is a surjective isometry from A_{1S} onto A_{2S} defined by (3.14). The form of such an isometry is already known by [15, Theorem 2]: there exists a central projection $p \in A_2$ and a Jordan *-isomorphism J from A_1 onto A_2 such that $f_0(x) = (2p-1)J(x)$ for every $x \in A_{1S}$. Looking at this isometry we obtain the partial form of ϕ_0. Since $(p-(1-p))^n = p+(-1)^n(1-p)$ for all positive integers n, we can compute in the same way as in [12, Theorem 1] that

$$
\begin{aligned}
\phi_0(\exp ix) &= \exp(if_0(x)) = \exp(i(2p-1)J(x))\\
&= \exp(i(p-(1-p))J(x)) = \sum_{n=0}^{\infty}\frac{((i(p-(1-p)))J(x))^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{i^n(p-(1-p))^nJ(x)^n}{n!} = \sum_{n=0}^{\infty}\frac{i^n(p+(-1)^n(1-p))J(x^n)}{n!}\\
&= pJ\left(\sum_{n=0}^{\infty}\frac{(ix)^n}{n!}\right) + (1-p)J\left(\sum_{n=0}^{\infty}\frac{(-ix)^n}{n!}\right)\\
&= pJ(\exp(ix)) + (1-p)J(\exp(ix))^*
\end{aligned}
$$

for every $x \in A_{1s}$. Then by (3.2) and the properties of Jordan *-algebras,

$$
\begin{aligned}
&\phi_0(a) = \phi_0(\exp(ix_n))\cdots\phi_0(\exp(ix_1))\phi_0(\exp(ix_0))\phi_0(\exp(ix_1))\cdots\phi_0(\exp(ix_n))\\
&= (pJ(\exp(ix_n))+(1-p)J(\exp(ix_n))^*)\cdots(pJ(\exp(ix_1))+(1-p)J(\exp(ix_1))^*)\\
&\qquad\qquad \times(pJ(\exp(ix_0))+(1-p)J(\exp(ix_0))^*)\\
&\times(pJ(\exp(ix_1))+(1-p)J(\exp(ix_1))^*)\cdots(pJ(\exp(ix_n))+(1-p)J(\exp(ix_n))^*)\\
&\qquad = pJ(\exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n))\\
&\qquad\quad + (1-p)J(\exp(ix_n)\cdots\exp(ix_1)\exp(ix_0)\exp(ix_1)\cdots\exp(ix_n))^*\\
&\qquad = pJ(a) + (1-p)J(a)^*.
\end{aligned}
$$

Suppose conversely that (3.1) holds for a central projection p and a Jordan *-isomorphism J. Define an extension $\widetilde{\phi}$ of ϕ by

$$\widetilde{\phi}(a) = \phi(1)(pJ(a) + (1-p)J(a)^*), \qquad a \in A_1.$$

One can easily check that $\widetilde{\phi}$ is a surjective isometry from A_1 onto A_2, by using the properties of central projections and Jordan *-isomorphisms. Since $J(U_1) = U_2$ [5, Lemma 6.2.4, Theorem 6.2.5] and U_j^0 is a connected component of U_j for $j = 1, 2$, we have $J(U_1^0) = U_2^0$. We infer that $\widetilde{\phi}(U_1^0) = U_2^0$. As $\widetilde{\phi}|_{U_1^0} = \phi$, we conclude that ϕ is a surjective isometry from U_1^0 onto U_2^0. ∎

Recall that Λ_j denotes the quotient group U_j/U_j^0 for the unitary group U_j of a unital C^*-algebra A_j and the principal component U_j^0 of U_j. For $a \in U_j$, the coset $\{au : u \in U_j^0\}$ is denoted by $[a]$.

Lemma 3.4. *Let λ and λ' be different elements in Λ_j. If $a \in \lambda$ and $a' \in \lambda'$, then $\|a - a'\| = 2$.*

Proof. Suppose that $\|a - a'\| \neq 2$. Then $\|a - a'\| < 2$, hence $\|a'^{-1}a - 1\| < 2$. As in the proof of Lemma 3.2, we see that $a = a' \exp(ix)$ for some $x \in A_{jS}$. This is a contradiction as $a \in \lambda$ and $\lambda \neq \lambda'$. ∎

Since U_j^0 is closed and open and connected, U_j can be written as the disjoint union of the connected sets of the form uU_j^0, where each u is taken in a different coset in Λ_j; hence the connected components of U_j are exactly the cosets in Λ_j.

Corollary 3.5. *Let A_j be a unital C^*-algebra and $u_j \in U_j$ for $j = 1, 2$. Let ϕ: $u_1U_1^0 \to u_2U_2^0$. Then ϕ is a surjective isometry if and only if there exists a central projection $p \in A_2$, a Jordan *-isomorphism J and a $u \in u_2(pJ(u_1^{-1}) + (1-p)J(u_1^{-1})^*)U_2^0$ such that*

$$\phi(a) = u(pJ(a) + (1-p)J(a)^*), \quad a \in u_1U_1^0. \tag{3.27}$$

Proof. Suppose that ϕ is of the form (3.27). Then the natural extension defined by $\widetilde{\phi}(a) = u(pJ(a) + (1-p)J(a)^*)$, $a \in A_1$, is a surjective isometry from A_1 onto A_2 since p is a central projection and J is a surjective isometry. The map $pJ + (1-p)J^*$ is also an isometry from A_1 onto A_2 and $(pJ + (1-p)J^*)(U_1) = U_2$ as $J(U_1) = U_2$. Thus $(pJ + (1-p)J^*)(u_1U_1^0)$ is a connected component of U_2 which contains $pJ(u_1) + (1-p)J(u_1)^*$. Hence

$$(pJ + (1-p)J^*)(u_1U_1^0) = (pJ(u_1) + (1-p)J(u_1)^*)U_2^0.$$

As $u \in u_2(pJ(u_1^{-1}) + (1-p)J(u_1^{-1})^*)U_2^0$ and U_2^0 is a normal subgroup of U_2, we have $u \in u_2U_2^0(pJ(u_1^{-1}) + (1-p)J(u_1^{-1})^*)$. Hence we infer that

$$u(pJ + (1-p)J^*)(u_1U_1^0) = u_2U_2^0.$$

Thus ϕ is a surjective isometry from $u_1U_1^0$ onto $u_2U_2^0$.

Suppose conversely that $\phi : u_1U_1^0 \to u_2U_2^0$ is a surjective isometry. Define $\phi' : U_1^0 \to U_2^0$ by $\phi'(v) = u_2^{-1}\phi(u_1 v)$, $v \in U_1^0$. It is clear that ϕ' is a surjective isometry. Then by Theorem 3.1 there is a central projection p in A_2 and a Jordan *-isomorphism J' from A_1 onto A_2 such that

$$\phi'(v) = u_2^{-1}\phi(u_1)(pJ'(v) + (1-p)J'(v)^*), \quad v \in U_1^0,$$

hence

$$\phi(a) = \phi(u_1)(pJ'(u_1^{-1}a) + (1-p)J'(u_1^{-1}a)^*), \quad a \in u_1U_1^0.$$

Define $J : A_1 \to A_2$ by

$$J(x) = pJ'(u_1)J'(u_1^{-1}x) + (1-p)J'(u_1^{-1}x)J'(u_1), \quad x \in A_1.$$

Then $J(1) = 1$ and J is complex-linear. Since J' is surjective, a simple calculation shows that J is surjective. As every Jordan *-isomorphism is an isometry, so is J'. Since p is a central projection and J' is an isometry,

$$\begin{aligned}\|J(x)\| &= \max\{\|pJ'(u_1)J'(u_1^{-1}x)\|, \|(1-p)J'(u_1^{-1}x)J'(u_1)\|\}\\ &= \max\{\|pJ'(u_1^{-1}x)\|, \|(1-p)J'(u_1^{-1}x)\|\}\\ &= \|pJ'(u_1^{-1}x) + (1-p)J'(u_1^{-1}x)\| = \|J'(u_1^{-1}x)\| = \|x\|\end{aligned}$$

for every $x \in A_1$. By the theorem of Kadison [14], J is a Jordan *-isomorphism. Put

$$u = \phi(u_1)(pJ(u_1^{-1}) + (1-p)J(u_1^{-1})^*).$$

Since U_2^0 is a normal subgroup of U_2, and $\phi(u_1)$ and u_2 are in the same coset of U_2, we infer that

$$u \in u_2(pJ(u_1^{-1}) + (1-p)J(u_1^{-1})^*)U_2^0.$$

As $J(u_1) = J'(u_1)$ we have

$$\begin{aligned}u(pJ(a) + (1-p)J(a)^*) &= u(pJ'(u_1)J'(u_1^{-1}a) + (1-p)(J'(u_1^{-1}a)J'(u))^*)\\ &= u(pJ'(u_1)J'(u_1^{-1}a) + (1-p)(J'(u)^*J'(u_1^{-1}a)^*)\\ &= u(pJ'(u_1) + (1-p)J'(u_1)^*)(pJ'(u_1^{-1}a) + (1-p)J'(u_1^{-1}a)^*)\\ &= \phi(u_1)(pJ'(u_1^{-1}a) + (1-p)J'(u_1^{-1}a)^*) = \phi(a), \quad a \in u_1U_1^0.\end{aligned}$$

Therefore (3.27) holds. ∎

4. Isometries between unitary groups. Let A_j be a unital C^*-algebra for $j = 1, 2$. Suppose that $[\phi] : \Lambda_1 \to \Lambda_2$ is a bijection, and $\phi_\lambda : \lambda \to [\phi](\lambda)$ is a surjective isometry for each $\lambda \in \Lambda_1$. Then ϕ_λ is of the form (3.27). If $\phi : U_1 \to U_2$ is defined by

$$\phi(a) = \phi_\lambda(a), \quad a \in \lambda,\ \lambda \in \Lambda_1,$$

then ϕ is a surjective isometry by Lemma 3.4. We will show that any surjective isometry from U_1 onto U_2 has this form, obtaining a complete description of the surjective isometries between unitary groups. Note that the corresponding result for commutative C^*-algebras has been proved in [12, Theorem 7]. Observe that Theorem 1 in [12] gives a partial description of surjective isometries between the unitary groups of unital C^*-algebras. Recall that by an isometry we merely mean a distance preserving transformation.

THEOREM 4.1. *Let A_j be a unital C^*-algebra and U_j its unitary group, $j = 1, 2$. Suppose that ϕ is a map from U_1 into U_2. Then ϕ is a surjective isometry from U_1 onto U_2 if and only if the following hold. First, for each $\lambda \in \Lambda_1$ there exists a unitary $u_\lambda \in U_2$, a central projection $p_\lambda \in A_2$, and a Jordan $*$-isomorphism J_λ from A_1 onto A_2 such that*

$$\phi(a) = u_\lambda(p_\lambda J_\lambda(a) + (1 - p_\lambda)J_\lambda(a)^*), \quad a \in \lambda. \tag{4.1}$$

Second, the map from Λ_1 to Λ_2 defined by $[a] \mapsto [\phi(a)]$ is well defined and bijective. In this case, u_λ, p_λ, and J_λ are unique for each $\lambda \in \Lambda$.

Proof. Assume first that ϕ is of the form (4.1) and the map $[a] \mapsto [\phi(a)]$ is well defined and bijective. One can easily check that for each λ in Λ_1, the natural extension $\widetilde{\phi}_\lambda$ of ϕ_λ defined by

$$\widetilde{\phi}_\lambda(a) = u_\lambda(p_\lambda J_\lambda(a) + (1 - p_\lambda)J_\lambda(a)^*), \quad a \in A_1,$$

is a surjective isometry from A_1 onto A_2, by using the properties of central projections and Jordan *-isomorphisms. Then $\widetilde{\phi}_\lambda|_\lambda = \phi|_\lambda$ is a surjective isometry from λ onto $\phi(\lambda)$ for every $\lambda \in \Lambda$. As $[a] \mapsto [\phi(a)]$ is a bijection from Λ_1 onto Λ_2 we infer that ϕ is a surjective isometry from U_1 onto U_2, by Lemma 3.4.

Assume conversely that ϕ is a surjective isometry from U_1 onto U_2. As the connected components of U_1 are exactly the cosets in Λ_1, the map $[a] \mapsto [\phi(a)]$, $a \in U_1$, is a well defined bijective map from Λ_1 onto Λ_2. Let $\lambda \in \Lambda_1$. Then $\phi|_\lambda$ is a surjective isometry from λ onto $\phi(\lambda)$. By Corollary 3.5 there exists a unitary $u_\lambda \in U_2$, a central projection $p_\lambda \in A_2$, a Jordan *-isomorphism J_λ such that

$$\phi(a) = u_\lambda(p_\lambda J_\lambda(a) + (1 - p_\lambda)J_\lambda(a)^*), \quad a \in \lambda. \tag{4.2}$$

To prove the uniqueness of this representation, assume that $\lambda \in \Lambda_1$ and

$$u_\lambda\big(p_\lambda J_\lambda(a)+(1-p_\lambda)J_\lambda(a)^*\big) = u'_\lambda\big(p'_\lambda J'_\lambda(a)+(1-p'_\lambda)J'_\lambda(a)^*\big), \quad a \in \lambda, \tag{4.3}$$

for $u_\lambda, u'_\lambda \in U_2$, central projections p_λ, p'_λ, and Jordan *-isomorphisms J_λ, J'_λ. Multiplying (4.3) by the central element ip_λ we have

$$u_\lambda(ip_\lambda J_\lambda(a)) = u'_\lambda\big(ip_\lambda p'_\lambda J'_\lambda(a) + ip_\lambda(1 - p'_\lambda)J'_\lambda(a)^*\big)$$

for any $a \in \lambda$. Substituting ia instead of a in (4.3) and then multiplying by p_λ we obtain

$$u_\lambda(ip_\lambda J_\lambda(a)) = u'_\lambda\big(ip_\lambda p'_\lambda J'_\lambda(a) - ip_\lambda(1 - p'_\lambda)J'_\lambda(a)^*\big)$$

for any $a\in\lambda$. Hence $p_\lambda(1-p'_\lambda)=0$. In the same way we see that $p'_\lambda(1-p_\lambda)=0$. It follows that $p_\lambda=p'_\lambda$, and by (4.3),

$$u_\lambda\bigl(p_\lambda J_\lambda(a)+(1-p_\lambda)J_\lambda(a)^*\bigr)=u'_\lambda\bigl(p_\lambda J'_\lambda(a)+(1-p_\lambda)J'_\lambda(a)^*\bigr),\qquad a\in\lambda.$$

Let $a_\lambda\in\lambda$. Then

$$\begin{aligned}(4.4)\qquad & u_\lambda\bigl(p_\lambda J_\lambda(a_\lambda b)+(1-p_\lambda)J_\lambda(a_\lambda b)^*\bigr)\\ &\qquad=u'_\lambda\bigl(p_\lambda J'_\lambda(a_\lambda b)+(1-p_\lambda)J'_\lambda(a_\lambda b)^*\bigr),\qquad b\in U_1^0.\end{aligned}$$

In particular,

$$(4.5)\quad u_\lambda\bigl(p_\lambda J_\lambda(a_\lambda)+(1-p_\lambda)J_\lambda(a_\lambda)^*\bigr)=u'_\lambda\bigl(p_\lambda J'_\lambda(a_\lambda)+(1-p_\lambda)J'_\lambda(a_\lambda)^*\bigr).$$

Putting

$$\begin{aligned}J_1(b)&=p_\lambda J_\lambda(a_\lambda^{-1})J_\lambda(a_\lambda b)+(1-p_\lambda)J_\lambda(a_\lambda b)J_\lambda(a_\lambda^{-1}),\qquad b\in A_1,\\ J'_1(b)&=p_\lambda J'_\lambda(a_\lambda^{-1})J'_\lambda(a_\lambda b)+(1-p_\lambda)J'_\lambda(a_\lambda b)J'_\lambda(a_\lambda^{-1}),\qquad b\in A_1,\end{aligned}$$

as in the proof of Corollary 3.5 we see that J_1 and J'_1 are Jordan *-isomorphisms from A_1 onto A_2 and

$$\begin{aligned}(4.6)\qquad & u_\lambda(p_\lambda J_\lambda(a_\lambda b)+(1-p_\lambda)J_\lambda(a_\lambda b)^*)\\ &=u_\lambda(p_\lambda J_\lambda(a_\lambda)+(1-p_\lambda)J_\lambda(a_\lambda)^*)(p_\lambda J_1(b)+(1-p_\lambda)J_1(b)^*),\quad b\in A_1,\end{aligned}$$

and

$$\begin{aligned}(4.7)\qquad & u'_\lambda(p_\lambda J'_\lambda(a_\lambda b)+(1-p_\lambda)J'_\lambda(a_\lambda b)^*)\\ &=u'_\lambda(p_\lambda J'_\lambda(a_\lambda)+(1-p_\lambda)J'_\lambda(a_\lambda)^*)(p_\lambda J'_1(b)+(1-p_\lambda)J'_1(b)^*),\quad b\in A_1.\end{aligned}$$

Thus by (4.4), (4.6) and (4.7) we have

$$\begin{aligned}(4.8)\qquad & u_\lambda(p_\lambda J_\lambda(a_\lambda)+(1-p_\lambda)J_\lambda(a_\lambda)^*)(p_\lambda J_1(b)+(1-p_\lambda)J_1(b)^*)\\ &=u'_\lambda(p_\lambda J'_\lambda(a_\lambda)+(1-p_\lambda)J'_\lambda(a_\lambda)^*)(p_\lambda J'_1(b)+(1-p_\lambda)J'_1(b)^*),\quad b\in U_1^0.\end{aligned}$$

From (4.5) we infer

$$p_\lambda J_1(b)+(1-p_\lambda)J_1(b)^*=p_\lambda J'_1(b)+(1-p_\lambda)J'_1(b)^*,\quad b\in U_1^0,$$

and hence $J_1=J'_1$ on U_1^0. Furthermore, $J_1=J'_1$ on A_1: To prove this, take an arbitrary x in A_{1S}. Then

$$\begin{aligned}n\biggl(\exp\Bigl(\frac{iJ_1(x)}{n}\Bigr)-1\biggr)&=n\biggl(J_1\Bigl(\exp\Bigl(\frac{ix}{n}\Bigr)\Bigr)-1\biggr)\\ &=n\biggl(J'_1\Bigl(\exp\Bigl(\frac{ix}{n}\Bigr)\Bigr)-1\biggr)=n\biggl(\exp\Bigl(\frac{iJ'_1(x)}{n}\Bigr)-1\biggr)\end{aligned}$$

for $\exp(ix/n)\in U_1^0$. Letting $n\to\infty$ we obtain $iJ_1(x)=iJ'_1(x)$. Since $x\in A_{1S}$ is arbitrary and J_1, J'_1 are complex-linear, we infer that $J_1=J'_1$ on A_1.

Putting $b = a_\lambda^{-1}$ in (4.6)–(4.7) we get

$$1 = \big(p_\lambda J_\lambda(a_\lambda) + (1-p_\lambda)J_\lambda(a_\lambda)^*\big)\big(p_\lambda J_1(a_\lambda^{-1}) + (1-p_\lambda)J_1(a_\lambda^{-1})^*\big),$$
$$1 = \big(p_\lambda J'_\lambda(a_\lambda) + (1-p_\lambda)J'_\lambda(a_\lambda)^*\big)\big(p_\lambda J'_1(a_\lambda^{-1}) + (1-p_\lambda)J'_1(a_\lambda^{-1})^*\big).$$

As $J_1(a_\lambda^{-1}) = J'_1(a_\lambda^{-1})$, we obtain

$$p_\lambda J_\lambda(a_\lambda) + (1-p_\lambda)J_\lambda(a_\lambda)^* = p_\lambda J'_\lambda(a_\lambda) + (1-p_\lambda)J'_\lambda(a_\lambda)^*,$$

hence $J_\lambda(a_\lambda) = J'_\lambda(a_\lambda)$. From (4.5) we infer that $u_\lambda = u'_\lambda$. Hence by (4.6) and (4.7) we have

$$p_\lambda J_\lambda(a_\lambda b) + (1-p_\lambda)J_\lambda(a_\lambda b)^* = p_\lambda J'_\lambda(a_\lambda b) + (1-p_\lambda)J'_\lambda(a_\lambda b)^*, \quad b \in A_1.$$

Therefore $J_\lambda = J'_\lambda$ on A_1. This completes the proof of the uniqueness of the representation (4.2) of ϕ on each $\lambda \in \Lambda$. ■

5. Extensibility. In this section we exhibit a necessary and sufficient condition so that the isometries between the unitary groups of two unital C^*-algebras can be extended to isometries between these C^*-algebras. Note that a corresponding result for commutative C^*-algebras is proved in [12, Corollary 8]. Roughly speaking, a surjective isometry $\phi : U_1 \to U_2$ extends to an isometry between the corresponding C^*-algebras if and only if u_λ, p_λ and J_λ which appear in the representation (4.1) coincide with each other for any $\lambda \in \Lambda_1$. More precisely, we have the following.

Corollary 5.1. *Let $\phi : U_1 \to U_2$ be a surjective isometry. Consider the representation of ϕ given in Theorem* 4.1, *i.e., for every $\lambda \in \Lambda_1$ take the unitary u_λ, the central projection p_λ and the Jordan *-isomorphism J_λ such that*

$$\phi(a) = u_\lambda(p_\lambda J_\lambda(a) + (1-p_\lambda)J_\lambda(a)^*), \quad a \in \lambda. \tag{5.1}$$

The map ϕ can be extended to a surjective isometry from A_1 onto A_2 if and only if all u_λ's coincide with $\phi(1)$ and all p_λ's as well as all J_λ's coincide. Moreover, denoting $p = p_\lambda$ and $J = J_\lambda$, the map

$$\widetilde{\phi}(a) = \phi(1)(pJ(a) + (1-p)J(a)^*), \quad a \in A_1,$$

extends ϕ.

Proof. Suppose that ϕ extends to a surjective isometry $\widehat{\phi}$ from A_1 onto A_2. Then by the celebrated Mazur–Ulam theorem $\widehat{\phi}_0 = \widehat{\phi} - \widehat{\phi}(0)$ is real-linear. Applying (5.1) for $\lambda = [1]$ we infer by a simple calculation that for every positive integer n,

$$\begin{aligned}\widehat{\phi}_0\Big(n\Big(\exp\Big(\tfrac{iy}{n}\Big)-1\Big)\Big) &= n\Big(\widehat{\phi}_0\Big(\exp\Big(\tfrac{iy}{n}\Big)\Big)-\widehat{\phi}_0(1)\Big) = n\Big(\phi\Big(\exp\Big(\tfrac{iy}{n}\Big)\Big)-\phi(1)\Big)\\ &= n\Big\{\phi(1)\Big(p_{[1]}J_{[1]}\Big(\exp\Big(\tfrac{iy}{n}\Big)\Big)+(1-p_{[1]})J_{[1]}\Big(\exp\Big(\tfrac{iy}{n}\Big)\Big)^*\Big)-\phi(1)\Big\}\\ &= \phi(1)\Big\{p_{[1]}J_{[1]}\Big(n\Big(\exp\Big(\tfrac{iy}{n}\Big)-1\Big)\Big)+(1-p_{[1]})J_{[1]}\Big(n\Big(\exp\Big(\tfrac{iy}{n}\Big)-1\Big)\Big)^*\Big\},\ y\in A_{1S}.\end{aligned}$$

As $\widehat{\phi}(i)=\phi(i)=\phi(1)(ip_{[1]}-i(1-p_{[1]}))$ we also have

$$\begin{aligned}&\widehat{\phi}_0\bigg(ni\bigg(\exp\bigg(\frac{ix}{n}\bigg)-1\bigg)\bigg)\\ &=\phi(1)\bigg\{p_{[1]}J_{[1]}\bigg(ni\bigg(\exp\bigg(\frac{ix}{n}\bigg)-1\bigg)\bigg)+(1-p_{[1]})J_{[1]}\bigg(ni\bigg(\exp\bigg(\frac{ix}{n}\bigg)-1\bigg)\bigg)^*\bigg\},\\ &\hspace{30em} x\in A_{1S}.\end{aligned}$$

Letting $n\to\infty$ for each of the above equations we get

$$\widehat{\phi}_0(iy)=\phi(1)\{p_{[1]}J_{[1]}(iy)+(1-p_{[1]})J_{[1]}(iy)^*\},\quad y\in A_{1S}, \tag{5.2}$$

and

$$\begin{aligned}-\widehat{\phi}_0(x)=\widehat{\phi}(i^2x)&=\phi(1)\{p_{[1]}J_{[1]}(i^2x)+(1-p_{[1]})J_{[1]}(i^2x)^*\}\\ &=-\phi(1)\{p_{[1]}J_{[1]}(x)+(1-p_{[1]})J_{[1]}(x)^*\},\quad x\in A_{1S}.\end{aligned} \tag{5.3}$$

Since $\widehat{\phi}_0$ is real-linear we observe from (5.2) and (5.3) that

$$\widehat{\phi}_0(a)=\phi(1)(p_{[1]}J_{[1]}(a)+(1-p_{[1]})J_{[1]}(a)^*),\quad a\in A_1.$$

It follows that $\widehat{\phi}(0)=0$ and

$$\phi(1)(p_{[1]}J_{[1]}(a)+(1-p_{[1]})J_{[1]}(a)^*)=u_\lambda(p_\lambda J_\lambda(a)+(1-p_\lambda)J_\lambda(a)^*),\quad a\in\lambda,$$

for every $\lambda\in\Lambda$ since $\widehat{\phi}$ is an extension of ϕ. Due to Theorem 4.1 this representation is unique for each $\lambda\in\Lambda$, we have $\phi(1)=u_\lambda$, $p_{[1]}=p_\lambda$, and $J_{[1]}=J_\lambda$ for every $\lambda\in\Lambda$.

Conversely, assume $\phi(1)=u_\lambda$, $p=p_\lambda$, $J=J_\lambda$ for every $\lambda\in\Lambda$. Then the map defined by

$$\widetilde{\phi}(a)=\phi(1)(pJ(a)+(1-p)J(a)^*),\quad a\in A_1$$

clearly extends ϕ, and $\widetilde{\phi}$ is a surjective isometry since p is a central projection. ■

Corollary 5.2. *Let A_j be a unital C^*-algebra such that $U_j=U_j^0$, $j=1,2$. A map $\phi:U_1\to U_2$ is a surjective isometry if and only if there is a central projection p in A_2 and a Jordan *-isomorphism $J:A_1\to A_2$ such that*

$$\phi(a)=\phi(1)\big(pJ(a)+(1-p)J(a)^*\big),\quad a\in U_1. \tag{5.4}$$

The proof is straightforward from Theorem 3.1. Note that the unitary group U_M coincides with $\exp(iM_s)$ for any von Neumann algebra M. Hence $U_M = U_M^0$.

6. An application and a problem. We say that two unital C^*-algebras A_1 and A_2 are *real-linear* (resp. *complex-linear, conjugate-linear*) *-*algebra isomorphic* if there is a real-linear (resp. complex-linear, conjugate-linear) bijection from A_1 onto A_2 which preserves multiplication and the *-operation.

Al-Rawashdeh, Booth and Giordano [2] proved that two unital AH-algebras of slow dimension growth and of real rank zero are complex-linear *-algebra isomorphic or conjugate-linear *-algebra isomorphic if and only if their unitary groups are isomorphic as topological groups. They also showed that two unital Kirchberg algebras are complex-linear *-algebra isomorphic or conjugate-linear *-algebra isomorphic if and only if their unitary groups are isomorphic as abstract groups.

In general there exists a pair of unital commutative C^*-algebras whose unitary groups are topologically isomorphic while the C^*-algebras themselves are not isomorphic as real algebras. Let X be a compact Hausdorff space. We denote by $C(X)$ (resp. $C_{\mathbb{R}}(X)$) the Banach algebra (resp. real Banach algebra) of all complex-valued (resp. real-valued) continuous functions on X. Then $C(X)$ is a unital commutative C^*-algebra. By the Gelfand–Naimark theorem any unital commutative C^*-algebra is isometrically complex-linear *-algebra isomorphic to $C(X)$ for some X. The unitary group of $C(X)$ is denoted by $UC(X)$.

The following example is essentially due to Żelazko [24, Remark 7.8].

Example 6.1. Let $X_1 = [0,1]$ be the closed unit interval and $X_2 = \{(x,y) \in \mathbb{R}^2 : x \in [0,2/3],\ y = 0\} \cup \{(x,y) \in \mathbb{R}^2 : x = 1/3,\ y \in [0,1/3]\}$. Let $\phi : UC(X_1) \to UC(X_2)$ be defined as

$$\phi(f)(x,y) = \begin{cases} f(x), & 0 \le x \le 2/3,\ y = 0, \\ \dfrac{f(1/3)}{f(2/3)} f(y+2/3), & x = 1/3,\ 0 < y \le 1/3, \end{cases}$$

for every $f \in UC(X_1)$. By a simple calculation we have

$$\tfrac{1}{3}\|f-g\| \le \|\phi(f)-\phi(g)\| \le 3\|f-g\|, \tag{6.1}$$

and hence ϕ and ϕ^{-1} are continuous group isomorphisms. On the other hand, ϕ cannot be extended to a real-algebra isomorphism from $C(X_1)$ onto $C(X_2)$. The reason is as follows. The maximal ideal space of $C(X_j)$ is homeomorphic to X_j for $j = 1,2$ while X_1 and X_2 are not homeomorphic to each other. Therefore $C(X_1)$ is not isomorphic to $C(X_2)$ as a real Banach algebra (cf. [10, Theorem 3.1]). Note that the first cohomotopy group on X_j is isomorphic to the first Čech cohomology group on X_j with integer coefficients [7, 7.4. Corollary, p. 91] and it vanishes. It follows that $\exp iC_{\mathbb{R}}(X_i) = UC(X_i)$ for $i = 1, 2$. Note also that the constant 3 in (6.1) is the best possible in the sense that if $\frac{1}{K}\|f - g\| \le \|\phi(f) - \phi(g)\| \le K\|f - g\|$ for all $f, g \in C(X_1)$, then $K \ge 3$. The reason is as follows. Let $0 < \theta \le \pi/3$. Choose $f \in C(X_1)$ such that $f(X_1) \subset \{z = \exp it : t \in \mathbb{R}, |t| \le \theta\}$, $f(1/3) = \exp i\theta$, $f(2/3) = \exp(-i\theta)$, and $f(1) = \exp i\theta$. Put $g = 1$. Then $\|f - g\| = |\exp i\theta - 1|$ and $\|\phi(f) - \phi(g)\| = |\exp 3i\theta - 1|$. The constant θ can be arbitrarily small, hence $K \ge 3$.

As a corollary of Theorem 3.1 we will prove the following (cf. [12]).

COROLLARY 6.2. *Let A_j be a unital C^*-algebra for $j = 1, 2$. The following are equivalent:*

(1) *A_1 is Jordan *-isomorphic to A_2,*
(2) *U_1 is isometric to U_2 as a metric space,*
(3) *U_1^0 is isometric to U_2^0 as a metric space.*

Proof. Suppose that (1) holds. Let $J : A_1 \to A_2$ be a Jordan *-isomorphism. Then J is a surjective isometry and $J(U_1) = U_2$ (cf. [5, Lemma 6.2.4, Theorem 6.2.5]), hence U_1 is isometric to U_2, so (2) holds.

Suppose that (2) holds. Let $\phi : U_1 \to U_2$ be a surjective isometry. Then ϕ_0 defined by $\phi_0(\cdot) = (\phi(1))^{-1}\phi(\cdot)$ is also a surjective isometry from U_1 onto U_2 such that $\phi_0(1) = 1$. Hence $\phi_0(U_1^0) = U_2^0$ as U_j^0 is the connected component of U_j which contains 1, for $j = 1, 2$. Thus U_1^0 is isometric to U_2^0, and (3) holds.

Suppose that (3) holds. We see at once that A_1 is Jordan *-isomorphic to A_2 by Theorem 3.1, so (1) holds. ∎

From Theorem 3.1 we will deduce the following.

COROLLARY 6.3. *Let A_j be a unital C^*-algebra for $j = 1, 2$. The following are equivalent:*

(1) *there exists a central projection p in A_2 and a (complex-linear) Jordan *-isomorphism J from A_1 onto A_2 such that pJ is multiplicative and $(1 - p)J$ is anti-multiplicative,*
(2) *U_1 is isometrically isomorphic to U_2 as a metric group,*
(3) *U_1^0 is isometrically isomorphic to U_2^0 as a metric group.*

Proof. Suppose that there exists a central projection p in A_2 and a Jordan *-isomorphism J from A_1 onto A_2 such that pJ is multiplicative and $(1-p)J$ is anti-multiplicative. Put $\widetilde{\phi} = pJ + (1-p)J^*$. It is well known that $J(U_1) = U_2$, hence $\widetilde{\phi}(U_1) = U_2$. As $(1 - p)J$ is anti-multiplicative and $1 - p$ is a central projection, we infer that $(1 - p)J^*$ is multiplicative. Thus $\widetilde{\phi}$ is an real-linear *-algebra isomorphism. Since p is a central projection and J is an isometry we deduce that $\widetilde{\phi}$ is also an isometry. It follows that $\widetilde{\phi}|_{U_1}$ is an isometrical isomorphism from U_1 onto U_2.

Suppose next that (2) holds and ϕ is an isometrical isomorphism from U_1 onto U_2. Then $\phi(1) = 1$ ensures that $\phi(U_1^0) = U_2^0$, since U_j^0 is the connected component which contains 1, and ϕ is an isometry. Thus U_1^0 is isometrically isomorphic to U_2^0.

Suppose that (3) holds and ϕ is an isometrical isomorphism from U_1^0 onto U_2^0. We claim (1) holds. As ϕ is a surjective isometry, Theorem 3.1 ensures that there exists a central projection p in A_2 and a Jordan *-isomorphism J from A_1 onto A_2 such that $\phi(u) = pJ(u) + (1-p)(J(u))^*$ for every $u \in U_1^0$. Let $\widetilde{\phi}$ be defined by

$$\widetilde{\phi}(a) = pJ(a) + (1-p)(J(a))^*, \quad a \in A_1,$$

which is an extension of ϕ to A_1. Let t be a non-zero real number and $x, y \in A_{1S}$. Since $\exp(itx), \exp(ity) \in U_1^0$, and $\widetilde{\phi}$ is real-linear, and

$$\begin{aligned}\widetilde{\phi}(\exp(itx)\exp(ity)) &= \phi(\exp(itx)\exp(ity)) \\ &= \phi(\exp(itx))\phi(\exp(ity)) = \widetilde{\phi}(\exp(itx))\widetilde{\phi}(\exp(ity)),\end{aligned}$$

a calculation yields

$$\widetilde{\phi}\left(\frac{(\exp(itx)-1)(\exp(ity)-1)}{t^2}\right) = \widetilde{\phi}\left(\frac{\exp(itx)-1}{t}\right)\widetilde{\phi}\left(\frac{\exp(ity)-1}{t}\right).$$

Similarly,

$$\begin{aligned}\widetilde{\phi}\left(\frac{i(\exp(itx)-1)(\exp(ity)-1)}{t^2}\right) &= \widetilde{\phi}\left(\frac{i(\exp(itx)-1)}{t}\right)\widetilde{\phi}\left(\frac{\exp(ity)-1}{t}\right), \\ \widetilde{\phi}\left(\frac{(\exp(itx)-1)i(\exp(ity)-1)}{t^2}\right) &= \widetilde{\phi}\left(\frac{\exp(itx)-1}{t}\right)\widetilde{\phi}\left(\frac{i(\exp(ity)-1)}{t}\right), \\ \widetilde{\phi}\left(\frac{i(\exp(itx)-1)i(\exp(ity)-1)}{t^2}\right) &= \widetilde{\phi}\left(\frac{i(\exp(itx)-1)}{t}\right)\widetilde{\phi}\left(\frac{i(\exp(ity)-1)}{t}\right).\end{aligned}$$

Letting $t \to 0$ in the above four equations we get

$$\begin{aligned}\widetilde{\phi}(ixiy) &= \widetilde{\phi}(ix)\widetilde{\phi}(iy), \\ \widetilde{\phi}(xiy) &= \widetilde{\phi}(x)\widetilde{\phi}(iy), \\ \widetilde{\phi}(ixy) &= \widetilde{\phi}(ix)\widetilde{\phi}(y), \\ \widetilde{\phi}(xy) &= \widetilde{\phi}(x)\widetilde{\phi}(y).\end{aligned}$$

Since $\widetilde{\phi}$ is real-linear, we obtain

$$\widetilde{\phi}(ab) = \widetilde{\phi}(a)\widetilde{\phi}(b), \quad a, b \in A_1.$$

Thus $pJ = p\widetilde{\phi}$ and $(1-p)J^* = (1-p)\widetilde{\phi}$ are multiplicative. Hence $(1-p)J$ is anti-multiplicative, for $(1-p)J = (J^*(1-p)^*)^* = ((1-p)J^*)^*$. Consequently, (1) holds. ∎

Let A_j be a unital C^*-algebra for $j = 1,2$. Suppose that $\phi : U_1^0 \to U_2^0$ is a surjective isometrical group isomorphism. As is shown in the proof of Corollary 6.3, ϕ can be extended to a real-linear *-algebra isomorphism from A_1 onto A_2. Note that a surjective isometrical group isomorphism from U_1 onto U_2 need not extend to an isometry from A_1 onto A_2. To give an example let $\mathbb{T}$ be the unit circle in the complex plane and $A_1 = A_2 = C(\mathbb{T})$. Then $U_j = \{z^n \exp(if) : n \in \mathbb{Z},\ f \in C_{\mathbb{R}}(\mathbb{T})\}$ and $\Lambda_j = \mathbb{Z}$ for $j = 1,2$, where $\mathbb{Z}$ denotes the additive group of all integers. Let $\phi : U_1 \to U_2$ be defined as $\phi(z^n \exp(if)) = z^{-n} \exp(if)$ for every $z^n \exp(if) \in U_1$. Then ϕ is a surjective isometrical group isomorphism from U_1 onto U_2. But ϕ cannot be extended to an isometry from A_1 onto A_2 by Corollary 5.1.

For a von Neumann algebra M the unitary group U coincides with the principal component U^0 of U. Hence every surjective isometrical group isomorphism between two unitary groups of von Neumann algebras can be extended to a real-linear *-algebra isomorphism between these von Neumann algebras.

COROLLARY 6.4. *Let M_j be a von Neumann algebra for $j = 1,2$. The following are equivalent:*

(1) *there exists a central projection p in M_2 and a (complex-linear) Jordan *-isomorphism J from M_1 onto M_2 such that pJ is multiplicative and $(1-p)J$ is anti-multiplicative,*
(2) *M_1 is Jordan *-isomorphic to M_2,*
(3) *U_1 is isometric to U_2 as a metric space,*
(4) *U_1 is isometrically isomorphic to U_2 as a metric group.*

Proof. We have already proved that (2) and (3) are equivalent. (1) and (4) are also equivalent by Corollary 6.3. It is apparent that (1) implies (2).

Suppose that (2) holds and $J : M_1 \to M_2$ is a Jordan *-isomorphism. Then by a theorem of Kadison [14, Theorem 10], J is a direct sum of a multiplicative part and an anti-multiplicative part, that is, there is a central projection p in M_2 such that pJ is multiplicative and $(1-p)J$ is anti-multiplicative; thus (1) holds. ∎

Note that Sakai [23] proved that topological group isomorphisms between two AW^*-factors are implemented by complex-linear *-algebra isomorphisms or conjugate-linear *-algebra isomorphisms of the factors.

We conclude the paper with a problem: for which constant K, the existence of a group isomorphism $\phi : U_1 \to U_2$ (resp. $U_1^0 \to U_2^0$) with

$$\frac{1}{K}\|a-b\| \leq \|\phi(a)-\phi(b)\| \leq K\|a-b\|, \quad a,b \in U_1 \text{ (resp. } U_1^0)$$

ensures that A_1 is real-linear *-algebra isomorphic to A_2? This is the case for $K = 1$ by Corollary 6.3, but due to Example 6.1 it is not the case for $K \geq 3$. The author does not know whether the statement holds or not for $1 < K < 3$ even if the C^*-algebras are commutative.

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