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Stability of the index of a linear relation under compact perturbations

by

Dana Gheorghe (Pitesti and Lille)

Abstract. We prove the stability under compact perturbations of the algebraic index of a Fredholm linear relation with closed range acting between normed spaces. Our main tool is a result concerning the stability of the index of a complex of Banach spaces under compact perturbations.

1. Introduction. The purpose of this paper is to show that the algebraic index of a Fredholm linear relation with closed range acting between normed spaces is stable under compact perturbations.

Let X and Y be two normed spaces. We denote by $\mathcal{B}(X,Y)$ the normed space of bounded linear operators from X into Y and by $\mathcal{K}(X,Y)$ the normed space of compact operators from X into Y. A *linear relation* is a map T from X into $\mathcal{P}(Y) := \{A \subset Y : A \neq \emptyset\}$ such that

$$Tx + Ty = T(x+y), \qquad \alpha Tx = T(\alpha x), \qquad \forall x, y \in X, \forall \alpha \in \mathbb{C} \setminus \{0\}.$$

We denote by $\mathrm{LR}(X,Y)$ the set of all linear relations from X into Y. The *kernel* of $T \in \mathrm{LR}(X,Y)$ is the set $N(T) = \{x \in X : 0 \in Tx\}$ and the *range* of T is the set $R(T) = \bigcup_{x \in X} Tx$. The linear relation T is called *Fredholm* if $\dim N(T) < \infty$ and $\operatorname{codim} R(T) < \infty$. In this case, the integer

$$\operatorname{ind}(T) := \dim N(T) - \operatorname{codim} R(T)$$

is called, as usual, the *index* of T. The main result of this article is the following.

Theorem 1. *Let $T \in \mathrm{LR}(X,Y)$ be a Fredholm linear relation with closed range. If $K \in \mathcal{K}(X,Y)$ is such that the range of $T+K$ is closed, then $T+K$ is Fredholm and* $\operatorname{ind}(T) = \operatorname{ind}(T+K)$.

Note that if T is a continuous Fredholm operator between Banach spaces, then $R(T)$ and $R(T+K)$ are closed linear subspaces of Y. In our case, the

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condition "$R(T+K)$ is closed" becomes a hypothesis because of the possible lack of continuity of T.

Our main tool is a result of Ambrozie [1, Theorem 5] concerning the stability of the index of a complex of Banach spaces under compact perturbations. We state a particular case of this theorem in Section 2. To relate the index of a linear relation between normed spaces to the index of a complex of normed spaces we use an idea of H. Zhang [8]. A similar result for the stability of the algebraic index of a linear relation under small perturbations has been proved in [4].

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2. Notation and preliminaries. Let X_i $(i=1,2,3)$ be normed spaces, and $\alpha_i \in \mathcal{B}(X_i, X_{i+1})$ $(i=1,2)$. The sequence

$$0 \to X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \to 0 \tag{1}$$

is called a *complex* if $R(\alpha_1) \subset N(\alpha_2)$. The complex (1) is said to be *Fredholm* if $\dim N(\alpha_1)$, $\dim N(\alpha_2)/R(\alpha_1)$ and $\operatorname{codim} R(\alpha_2)$ are finite. Define the *index* of (1) by

$$\operatorname{ind}(1) := \dim N(\alpha_1) - \dim N(\alpha_2)/R(\alpha_1) + \operatorname{codim} R(\alpha_2). \tag{2}$$

The next result is a particular case of Theorem 5 in [1] (see also [2]), and will be used to prove our main result.

THEOREM 2. *Let X_i $(i=1,2,3)$ be Banach spaces. Assume that the complex* (1) *is Fredholm and $\beta_i \in \mathcal{B}(X_i, X_{i+1})$ $(i=1,2)$ are such that the sequence*

$$0 \to X_1 \xrightarrow{\beta_1} X_2 \xrightarrow{\beta_2} X_3 \to 0 \tag{3}$$

is a complex. If $\beta_i - \alpha_i \in \mathcal{K}(X_i, X_{i+1})$ $(i=1,2)$, then (3) *is Fredholm and*

$$\operatorname{ind}(3) = \operatorname{ind}(1).$$

REMARK 1. The above theorem is due to A. S. Faĭnshteĭn and V. S. Shul'man. On the other hand, to prove our main result we may also use a result of F.-H. Vasilescu [7] where α_i, β_i $(i=1,2)$ are as above and satisfy the supplementary condition that $(\beta_2-\alpha_2)(\beta_1-\alpha_1)$ is of finite rank.

Let X and Y be normed spaces, $X_0 \subset X$ a closed linear subspace and $Y_0 \subset Y$ a linear subspace. Let $T_0 : X \to Y$ be a linear operator such that $T_0(X_0) \subset Y_0$. Then T_0 induces the linear transformation $T : X/X_0 \to Y/Y_0$ defined by

$$T(x+X_0) = T_0x + Y_0, \qquad \forall x \in X.$$

Let $N(T)$ be the kernel of T, $R(T)$ the range of T, $G(T)$ the graph of T, $G_0(T) := \{(x,y) \in X \times Y : T(x+X_0) = y + Y_0\}$ and $R_0(T) :=$

$\{y \in Y : y + Y_0 \in R(T)\}$. Notice that $R(T) = R_0(T)/Y_0$, $G_0(T) = G(T_0) + X_0 \times Y_0$ and $R_0(T) = R(T_0) + Y_0$. The transformation T is said to be *Fredholm* if $\dim N(T)$ and $\operatorname{codim} R(T) := \dim(Y/Y_0)/R(T)$ are finite, and we define

$$\operatorname{ind}(T) = \dim N(T) - \operatorname{codim} R(T)$$

to be the *index* of T.

We associate to T the sequence of normed spaces

$$0 \to X_0 \xrightarrow{i_T} X \times Y_0 \xrightarrow{j_T} Y \to 0, \tag{4}$$

where

$$i_T(x_0) = (x_0, T_0x_0), \quad \forall x_0 \in X_0,$$
$$j_T(x, y_0) = T_0x - y_0, \quad \forall x \in X,\ y_0 \in Y_0.$$

It is easy to see that (4) is a complex.

LEMMA 1. *The linear transformation T is Fredholm iff the complex* (4) *is Fredholm*, *and in this case*,

$$\operatorname{ind}(T) = -\operatorname{ind}(4).$$

Proof. Define

$$\alpha : N(j_T) \to X/X_0, \quad \alpha((x, T_0x)) = x + X_0.$$

Clearly $R(\alpha) = N(T)$ and $N(\alpha) = R(i_T)$, thus $N(j_T)/R(i_T)$ is algebraically isomorphic to $N(T)$. Define

$$\beta : Y \to Y/R_0(T), \quad \beta(y) = y + R_0(T).$$

The map β is surjective and $N(\beta) = R(j_T)$, thus $Y/R(j_T)$ is algebraically isomorphic to $Y/R_0(T) \cong (Y/Y_0)/R(T)$. The proof is complete. ∎

LEMMA 2. *Let X, Y be normed spaces*, *$M \subset X$ a linear subspace of X*, *$A \subset X$ an arbitrary subset of X*, *$T \in \mathcal{B}(X, Y)$ and $T' \in \mathcal{B}(Y', X')$ the adjoint of T. Let B_X and B_Y be the closed unit balls in X and Y*, *respectively. Then*

- (i) *$N(T') = R(T)^\perp$ and $N(T) = R(T')^\perp$* (*where $\perp$ denotes the annihilator or preannihilator of a set*).
- (ii) *$(B_X \cap M)^\circ = B_X^\circ + M^\perp$*, *$(M + A)^\circ = M^\perp \cap A^\circ$*, *$T(B_X)^\circ = T'^{-1}(B_X^\circ)$ and $[T^{-1}(B_Y)]^\circ = T'(B_Y^\circ)$* (*where the circle denotes the polar of a set*).

Proof. The proof of (i) is classical and for the proof of (ii) see [6]. ∎

DEFINITION 1. Let X, Y be normed spaces and $T : X \to Y$ a linear operator. We say that the operator T is *open* if there exists $\varrho > 0$ such that

$$\varrho B_Y \cap R(T) \subset T(B_X). \tag{5}$$

Remark 2. If X, Y are normed spaces and $T \in \mathcal{B}(X,Y)$ then T is open if and only if the map $X \ni x \mapsto Tx \in R(T)$ takes open sets in X into open sets in $R(T)$.

Proposition 1. *Let X, Y be normed spaces, $T \in \mathcal{B}(X,Y)$ and $K \in \mathcal{K}(X,Y)$. If T is open and* $\operatorname{codim} R(T) < \infty$, *then $T+K$ is open.*

Proof. First of all we show that the adjoint $T' \in \mathcal{B}(Y',X')$ is open. Because T is open, from Lemma 2 it follows that there exists $\varrho > 0$ such that

$$T(B_X)^\circ \subset [\varrho B_Y \cap R(T)]^\circ = R(T)^\perp + \frac{1}{\varrho} B_Y^\circ. \tag{6}$$

Using again Lemma 2 and (6) we deduce that

$$T'^{-1}(B_X^\circ) \subset N(T') + \frac{1}{\varrho} B_Y^\circ.$$

Hence,

$$\varrho B_X^\circ \cap R(T') \subset T'(B_Y^\circ),$$

that is, T' is open. Remark 2 implies that $R(T')$ is closed. From

$$\dim N(T') = \operatorname{codim} \overline{R(T)} \le \operatorname{codim} R(T) < \infty$$

it follows that T' has an index. Applying [5, Corollary V.2.2] we find that $R(T'+K')$ is closed, and because $T'+K'$ is continuous, it follows that $T'+K' : Y' \to R(T'+K')$ is open. This implies that there exists $\varrho > 0$ such that

$$\varrho B_X^\circ \cap R(T'+K') \subset (T'+K')(B_Y^\circ). \tag{7}$$

Using (7), Lemma 2 and the closedness of $R(T'+K')$ we obtain

$$\varrho[B_X + N(T+K)]^\circ \subset [(T+K)^{-1}(B_Y)]^\circ. \tag{8}$$

Taking the polars of both sides in (8) and using the bipolar theorem it follows that

$$\begin{aligned}\varrho(T+K)^{-1}(B_Y) &= \varrho[(T+K)^{-1}(B_Y)]^{\circ\circ} \subset [B_X + N(T+K)]^{\circ\circ} \\ &= \overline{B_X + N(T+K)} \subset 2B_X + N(T+K),\end{aligned}$$

which yields

$$\varrho B_Y \cap R(T+K) \subset 2(T+K)(B_X),$$

that is, $T+K$ is open. ∎

Remark 3. If $T \in \mathcal{B}(X,Y)$ is open then the proof above shows that $R(T')$ is closed.

3. Proof of the main result. The product of two normed spaces X_1 and X_2 will be endowed with the norm

$$\|(x_1, x_2)\| = (\|x_1\|^2 + \|x_2\|^2)^{1/2}, \quad \forall (x_1, x_2) \in X_1 \times X_2.$$

LEMMA 3. *Let X and Y be normed spaces, $X_0 \subset X$ a closed linear subspace and $Y_0 \subset Y$ a linear subspace. Let $T_0 \in \mathcal{B}(X, Y)$ and $K_0 \in \mathcal{K}(X, Y)$ be such that $T_0(X_0) \subset Y_0$ and $K_0(X_0) \subset Y_0$. Suppose that T_0 is open, $K_0|X_0 : X_0 \to Y_0$ is compact, and $R(T_0) + Y_0$, $R(T_0 + K_0) + Y_0$ are closed subspaces of Y. Let $T, K : X/X_0 \to Y/Y_0$ induced by T_0 and K_0 respectively. If T is Fredholm, then $T + K$ is Fredholm and* $\operatorname{ind}(T) = \operatorname{ind}(T + K)$.

Proof. Associate to $T + K$, as in the case of T, the sequence of normed spaces

$$0 \to X_0 \xrightarrow{i_{T+K}} X \times Y_0 \xrightarrow{j_{T+K}} Y \to 0, \tag{9}$$

where

$$i_{T+K}(x_0) = (x_0, (T_0 + K_0)x_0), \quad \forall x_0 \in X_0,$$
$$j_{T+K}(x, y_0) = (T_0 + K_0)x - y_0, \quad \forall x \in X,\ y_0 \in Y_0.$$

It is easy to check that (9) is a complex.

(a) *Some properties of i_T, j_T, i_{T+K} and j_{T+K}.* It is clear that $i_T \in \mathcal{B}(X_0, X \times Y_0)$ and $R(i_T)$ is closed. On the other hand,

$$\|x_0\| \le \|(x_0, T_0 x_0)\| = \|i_T(x_0)\|.$$

It follows that

$$B_{X \times Y_0} \cap R(i_T) \subset i_T(B_{X_0}),$$

hence i_T is open. Clearly, $j_T \in \mathcal{B}(X \times Y_0, Y)$. Because $R(T_0) + Y_0$ is closed, the map β from the proof of Lemma 1 is continuous. As $R(j_T) = N(\beta)$, it follows that $R(j_T)$ is closed. Consider the maps

$$s : X \times Y_0 \to R(T_0) \times Y_0, \quad s(x, y_0) = (T_0 x, y_0),$$
$$t : R(T_0) \times Y_0 \to Y, \quad t(T_0 x, y_0) = T_0 x - y_0.$$

Notice that t is open. Because T_0 is open, it follows easily that s is open. Hence, $j_T = t \circ s$ is open. On the other hand, $\operatorname{codim} R(T_0) = \operatorname{codim} R(T) < \infty$. Hence, Proposition 1 shows that $T_0 + K_0$ is open. So, we can replace T_0 with $T_0 + K_0$ to conclude that i_{T+K} and j_{T+K} have the same properties as i_T and j_T.

(b) *The adjoints of complexes* (4) *and* (9). We consider the sequences

$$0 \to Y' \xrightarrow{j'_T} X' \times Y'_0 \xrightarrow{i'_T} X'_0 \to 0, \tag{10}$$

$$0 \to Y' \xrightarrow{j'_{T+K}} X' \times Y'_0 \xrightarrow{i'_{T+K}} X'_0 \to 0. \tag{11}$$

Using Lemma 2 and Remark 3 it follows that (10), (11) are complexes, (10) is Fredholm, and

$$\operatorname{ind}(4) = \operatorname{ind}(10). \tag{12}$$

Moreover, (9) is Fredholm iff (11) is Fredholm, and in this case,

$$\operatorname{ind}(9) = \operatorname{ind}(11). \tag{13}$$

We will use Theorem 2 to prove that (11) is Fredholm and $\operatorname{ind}(10) = \operatorname{ind}(11)$. To do this, we write (11) as a compact perturbation of (10) as follows. Consider the compact operators

$$\begin{aligned} i : X_0 \to X \times Y_0, &\quad i(x_0) = (0, K_0 x_0), \\ j : X \times Y_0 \to Y, &\quad j(x, y_0) = K_0 x. \end{aligned}$$

Note that $i_{T+K} = i_T + i$ and $j_{T+K} = j_T + j$. Because i and j are compact it follows that $i'_{T+K} - i'_T$ and $j'_{T+K} - j'_T$ are compact. Applying Theorem 2 we deduce that (11) is Fredholm and

$$\operatorname{ind}(10) = \operatorname{ind}(11). \tag{14}$$

(c) *End of proof.* Using Lemma 1 and (12)–(14) we find that

$$\operatorname{ind}(T) = \operatorname{ind}(T + K). \ \blacksquare$$

Let X be a normed space and $M := \{x \in X : \|x\| = 1\}$. Define

$$l_0^1(M) := \{\lambda : M \to \mathbb{C} : \operatorname{supp} \lambda \text{ is finite}\},$$

where $\operatorname{supp} \lambda = \{x \in M : \lambda(x) \neq 0\}$. We endow the vector space $l_0^1(M)$ with the norm $\|\lambda\| = \sum_{x \in M} |\lambda(x)|$. Consider the linear operator

$$S_0 : l_0^1(M) \to X, \quad S_0 \lambda = \sum_{x \in M} \lambda(x) x.$$

Obviously, $S_0 \in \mathcal{B}(l_0^1(M), X)$ and S_0 is surjective. Define the linear operator

$$S : l_0^1(M)/N(S_0) \to X, \quad S(\lambda + N(S_0)) = S_0 \lambda.$$

Then S is continuous and bijective.

Let $T \in \operatorname{LR}(X, Y)$ be a linear relation and $q_T : Y \to Y/T(0)$ be the canonical surjection. Associate to T, as in [3, Section I.6], the linear transformation

$$q_T T : X \to Y/T(0), \quad (q_T T)(x) = y + T(0),$$

where $y \in Tx$ is arbitrarily chosen. We see that $R_0(q_T T) = R(T)$ and

$$T \text{ is Fredholm iff } q_T T \text{ is Fredholm}, \quad \text{and} \quad \operatorname{ind}(T) = \operatorname{ind}(q_T T). \tag{15}$$

Proof of Theorem 1. We denote $N(S_0)$ by X_0 and $T(0)$ by Y_0. Consider the linear relation $TS \in \operatorname{LR}(l_0^1(M)/X_0, Y)$. Because S is bijective, we have $T(0) = (TS)(0)$, and hence (15) shows that $\operatorname{ind}(T) = \operatorname{ind}(q_T T) = \operatorname{ind}(q_T(TS)) = \operatorname{ind}(Q)$, where $Q = q_T TS$. Thus

$$\operatorname{ind}(Q) = \operatorname{ind}(T). \tag{16}$$

Consider

$$q_1 : G_0(Q) \to l_0^1(M), \quad q_1(\lambda, y) = \lambda,$$
$$q_2 : G_0(Q) \to Y, \quad q_2(\lambda, y) = y.$$

Clearly q_1, q_2 are continuous and q_2 is open. On the other hand, the map q_1 takes open sets in $G_0(Q)$ to open sets in $R(q_1)$, hence

$$\gamma(q_1) := \sup\{\delta > 0 : \delta d(\xi, N(q_1)) \leq \|q_1\xi\|, \forall \xi \in G_0(Q)\} > 0.$$

Let

$$Q_1 : G_0(Q)/(X_0 \times Y_0) \to l_0^1(M)/X_0, \quad Q_2 : G_0(Q)/(X_0 \times Y_0) \to Y/Y_0$$

be induced by q_1 and q_2. It follows easily that Q_1 is bijective and $Q = Q_2 \circ Q_1^{-1}$. For $x \in M$, let $e_x \in l_0^1(M)$ be such that $e_x(x) = 1$ and $\operatorname{supp} e_x = \{x\}$. Using the fact that q_1 is surjective and $\gamma(q_1) > 0$ we deduce that for all $x \in M$ there exists $\xi_x = (e_x, y_x) \in G_0(Q)$ such that $\|\xi_x\| \leq r$, where $r > \gamma(q_1)^{-1}$. Define the linear operator

$$q_0 : l_0^1(M) \to G_0(R), \quad q_0(\lambda) = \sum_{x\in M} \lambda(x)\xi_x = \Big(\lambda, \sum_{x\in M} \lambda(x) y_x\Big).$$

From the choice of ξ_x it follows that q_0 is continuous and a simple computation shows that q_0 is open. Let

$$T_0 : l_0^1(M) \to Y, \quad T_0 = q_2 \circ q_0.$$

The linear operator T_0 is continuous open and because q_2 induces Q_2 and q_0 induces Q_1^{-1}, we deduce that T_0 induces Q. Using

$$R(T) = R(TS) = R_0(Q) = R(T_0) + Y_0$$

and the closedness of $R(T)$ it follows that $R(T_0) + Y_0$ is closed.

Let $\tau : l_0^1(M) \to l_0^1(M)/X_0$ be the canonical surjection and

$$K_0 : l_0^1(M) \to Y, \quad K_0 = KS\tau.$$

Note that K_0 is compact and $K_0|X_0$ is identically zero. Let

$$\widetilde{Q} = q_{(T+K)S}(T + K)S = q_T(T + K)S.$$

We have

$$\begin{aligned}\widetilde{Q}(\lambda + X_0) &= [q_T(T + K)S](\lambda + X_0) = Q(\lambda + X_0) + q_T KS\tau(\lambda)\\ &= T_0(\lambda) + Y_0 + q_T K_0(\lambda) = T_0(\lambda) + K_0(\lambda) + Y_0,\end{aligned}$$

that is, $\widetilde{Q}$ is induced by $T_0 + K_0$. From

$$R(T + K) = R((T + K)S) = R_0(\widetilde{Q}) = R(T_0 + K_0) + Y_0$$

and the closedness of $R(T + K)$ it follows that $R(T_0 + K_0) + Y_0$ is closed.

We are now in a position to apply Lemma 3 to the operators T_0 and K_0. It follows that $\widetilde{Q}$ is Fredholm and

$$\operatorname{ind}(Q) = \operatorname{ind}(\widetilde{Q}). \tag{17}$$

On the other hand, using (15) we see that $(T+K)S$ is Fredholm iff $\widetilde{Q}$ is Fredholm, and

$$\operatorname{ind}((T+K)S) = \operatorname{ind}(\widetilde{Q}). \tag{18}$$

From the bijectivity of S and (16)–(18) the conclusion of the theorem follows. ■

Remark 4. Let X be a normed space. If $K \in \mathcal{K}(X)$, then $\operatorname{ind}(I-K)=0$. In particular $I-K$ is injective iff $I-K$ is surjective.

Remark 5. Let X, Y be normed spaces, $T : X \to Y$ a linear bijection and $K \in \mathcal{K}(X,Y)$. Suppose that the equation

$$Tx + Kx = 0$$

has at least one nontrivial solution. Then there exists $y \in Y$ such that the equation

$$Tx + Kx = y$$

has no solution.

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Facultatea de matematica-informatica
Universitatea din Pitesti
Str. Targul din Vale, nr. 1
110040 Pitesti, Arges, Romania
E-mail: gheorghedana@yahoo.com

Université des Sciences et Technologies de Lille
59655 Villeneuve d'Ascq Cedex, France

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