

A joint limit theorem for compactly regenerative ergodic transformations

by

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Abstract. We study conservative ergodic infinite measure preserving transformations satisfying a compact regeneration property introduced by the second-named author in *J. Anal. Math.* 103 (2007). Assuming regular variation of the wandering rate, we clarify the asymptotic distributional behaviour of the random vector (Z_n, S_n) , where Z_n and S_n are respectively the time of the last visit before time n to, and the occupation time of, a suitable set Y of finite measure.

1. Introduction. Various interesting classes of conservative (i.e. recurrent) ergodic *infinite measure preserving dynamical systems* (cf. [A0]) exhibit stochastic properties which parallel phenomena known from the probability theory of null-recurrent Markov chains. In fact, systems containing (as an induced map, say) some hyperbolic mechanism generalize these classical processes, while still being closely related to them on a structural level (cf. [Z6]). Lacking the clear-cut dependence structure of Markov chains, their probabilistic analysis depends on identifying ergodic properties which (can be verified and) still entail the desired stochastic features.

Let T be a *measure preserving transformation (m.p.t.)* on the σ -finite measure space (X, \mathcal{A}, μ) with $\mu(X) = \infty$. It is *conservative* and *ergodic (c.e.)* if $\mathbf{S}_n(f) \rightarrow \infty$ a.e. whenever $f \geq 0$ and $\mu(f) = \int f d\mu > 0$, where $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 0$ (cf. Proposition 1.2.2 of [A0]). For specific types of such systems, distributional limit theorems for ergodic sums $\mathbf{S}_n(f)$ with $f \in L_1(\mu)$ (in Darling–Kac type theorems, see [DK]) and for renewal-theoretic variables like $\mathbf{Z}_n(Y) := \max\{k 1_Y \circ T^k : 0 \leq k \leq n\}$, $n \geq 0$ (in Dynkin–Lamperti type arcsine laws, see [D], [L]) are available (see [A0]–[A2], [T2], [TZ], [Z4]). The limit distributions, which depend on a single parameter

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$\alpha \in [0, 1]$ encoding a characteristic return rate of the system, are given in terms of (*normalized*) *Mittag-Leffler variables* \mathcal{M}_α characterized by their moments $\mathbb{E}[\mathcal{M}_\alpha^r] = r!(\Gamma(1 + \alpha))^r/\Gamma(1 + r\alpha)$, $r \geq 0$, and by variables \mathcal{Z}_α with *generalized arcsine laws*,

$$\Pr(\{0 \leq \mathcal{Z}_\alpha \leq t\}) = \frac{\sin \pi \alpha}{\pi} \int_0^t \frac{dx}{x^{1-\alpha}(1-x)^\alpha}, \quad t \in [0, 1],$$

respectively. We recall that $\mathbb{E}[\mathcal{Z}_\alpha^r] = (-1)^r \binom{-\alpha}{r}$, $r \geq 0$.

To fix notation, let ν be a probability measure on (X, \mathcal{A}) and $(R_n)_{n \geq 1}$ a sequence of measurable real functions on X . Then distributional convergence of $(R_n)_{n \geq 1}$ with respect to ν to some random variable R will be denoted by $R_n \xrightarrow{\nu} R$. *Strong distributional convergence* $R_n \xrightarrow{\mathcal{L}(\mu)} R$ on the σ -finite measure space (X, \mathcal{A}, μ) means that $R_n \xrightarrow{\nu} R$ for all probability measures $\nu \ll \mu$.

An approach which only relies on fairly weak conditions regarding the dynamics up to the first visit to a suitable reference set has been developed in [T3], [TZ], and [Z4]. The purpose of the present note is to show that the same method can be used to improve on individual limit theorems by establishing convergence of their joint distributions, thus clarifying the asymptotic dependencies between them. Applied to a prominent system, our main result takes the following form:

EXAMPLE 1.1 (A joint limit theorem for Boole's transformation). The map $T : \mathbb{R} \rightarrow \mathbb{R}$ given by $Tx := x - 1/x$ preserves Lebesgue measure λ and is conservative ergodic (cf. [AW]). Let $Y \subseteq \mathbb{R}$ be some fixed bounded interval. By [A1] and [T2] (or, alternatively, [TZ]),

$$(1.1) \quad \frac{\mathbf{Z}_n(Y)}{n} \xrightarrow{\mathcal{L}(\lambda)} \mathcal{Z} \quad \text{and} \quad \frac{\mathbf{S}_n(1_Y)}{\sqrt{2n/\pi}} \xrightarrow{\mathcal{L}(\lambda)} \lambda(Y)\mathcal{M},$$

with $\mathcal{Z} := \mathcal{Z}_{1/2}$ and $\mathcal{M} := \mathcal{M}_{1/2}$ having densities

$$\mathfrak{z}(s) := \frac{1}{\pi \sqrt{s(1-s)}}, \quad s \in (0, 1), \quad \text{and} \quad \mathfrak{m}(t) := \frac{2}{\pi} e^{-t^2/\pi}, \quad t > 0.$$

The result of the present paper determines the asymptotics of the joint distributions, showing that

$$(1.2) \quad \left(\frac{\mathbf{Z}_n(Y)}{n}, \frac{\mathbf{S}_n(1_Y)}{\sqrt{2n/\pi}} \right) \xrightarrow{\mathcal{L}(\lambda)} (\mathcal{Z}, \lambda(Y)\mathcal{M}^* \cdot \sqrt{\mathcal{Z}}),$$

where \mathcal{Z} and \mathcal{M}^* are two independent random variables, \mathcal{Z} being as above, and \mathcal{M}^* having density

$$\mathfrak{m}^*(t) := \frac{2t}{\pi} e^{-t^2/\pi}, \quad t > 0.$$

REMARK 1.2. Statements (1.1) and (1.2) show, in particular, that $\mathcal{M}^* \cdot \sqrt{\mathcal{Z}} \stackrel{d}{=} \mathcal{M}$ (in distribution). We briefly indicate how to check this directly: As \mathcal{Z} has the classical arcsine distribution, we have $\Pr(\{0 \leq \sqrt{\mathcal{Z}} \leq t\}) = 2\pi^{-1} \arcsin t$, so that $\sqrt{\mathcal{Z}}$ has density

$$\tilde{\mathfrak{z}}(s) := \frac{2}{\pi} \frac{1}{\sqrt{1-s^2}}, \quad s \in (0, 1).$$

To explicitly compute the density \mathfrak{d} of $\mathcal{M}^* \cdot \sqrt{\mathcal{Z}}$, that is,

$$\mathfrak{d}(t) := \int_t^\infty \mathfrak{m}^*(s) \tilde{\mathfrak{z}}\left(\frac{t}{s}\right) \frac{ds}{s} = \int_t^\infty \frac{2}{\pi} \frac{sm(s)}{\sqrt{s^2-t^2}} ds, \quad t > 0,$$

let $M(t) := \int_0^t \mathfrak{m}(s) ds$, $t > 0$, and observe that

$$\frac{d}{ds} M(\sqrt{s^2-t^2}) = \frac{2}{\pi} \frac{s}{\sqrt{s^2-t^2}} \frac{\mathfrak{m}(s)}{\mathfrak{m}(t)} \quad \text{for } 0 < t < s.$$

Use this to evaluate the definite integral to obtain $\mathfrak{d}(t) = \mathfrak{m}(t)(M(\infty) - M(0)) = \mathfrak{m}(t)$ for $t > 0$, as required.

2. A joint limit theorem. Let T be a c.e.m.p.t. on the σ -finite measure space (X, \mathcal{A}, μ) . For $Y \in \mathcal{A}$ with $\mu(Y) > 0$ the *first return (entrance) time* of Y is $\varphi(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$. For suitably chosen Y , the asymptotics of the *return probabilities* $f_k(Y) := \mu_Y(Y \cap \{\varphi = k\})$, $k \geq 1$, determines the stochastic properties of the system. Here, $\mu_Y(A) := \mu(Y)^{-1} \mu(Y \cap A)$. Distributional limit theorems frequently depend on regular variation of the *tail probabilities* $q_n(Y) := \sum_{k>n} f_k(Y) = \mu_Y(Y \cap \{\varphi > n\})$, or (slightly more generally) of the *wandering rate* of Y given by $w_N(Y) := \mu(Y) \sum_{n=0}^{N-1} q_n(Y) = \mu(Y^N)$, where $Y^N := \bigcup_{n=0}^{N-1} T^{-n}Y$, $N \geq 1$. For background material about regular variation we refer to Chapter 1 of [BGT]. We will follow the convention that for $a_n, b_n \geq 0$ and $\vartheta \in [0, \infty)$, $a_n \sim \vartheta \cdot b_n$ as $n \rightarrow \infty$ means $\lim_{n \rightarrow \infty} a_n/b_n = \vartheta$, even in case $\vartheta = 0$, where it is interpreted as the usual $a_n = o(b_n)$ as $n \rightarrow \infty$. Analogously for $f(s) \sim \vartheta \cdot g(s)$ as $s \searrow 0$ etc.

The *transfer operator* $\widehat{T} : L_1(\mu) \rightarrow L_1(\mu)$ of T is the positive linear operator characterized by $\int_X (g \circ T) \cdot u d\mu = \int_X g \cdot \widehat{T}u d\mu$ for all $u \in L_1(\mu)$ and $g \in L_\infty(\mu)$.

A good understanding of the long-term behaviour of \widehat{T} when acting on $\mathcal{D}(\mu) := \{u \in L_1(\mu) : u \geq 0, \mu(u) = 1\}$ is often crucial for probabilistic studies. In the course of [T3], [TZ], and [Z4], an approach which only assumes information on \widehat{T} up to the first entrance into a suitable reference set Y has been developed. Let $Y_0 := Y$ and $Y_n := Y^c \cap \{\varphi = n\}$ for $n \geq 1$. We require some control of how large a collection of densities on Y we see when considering (averaged and normalized versions of) $\sum_{k>n} \widehat{T}^k 1_{Y \cap \{\varphi=k\}} = \widehat{T}^n 1_{Y_n}$ for $n \geq 1$. (Note that if our system starts with initial distribution μ_Y , then

$f_k(Y)^{-1}\widehat{T}^k 1_{Y \cap \{\varphi=k\}}$ is its conditional density at time k , given that $\{\varphi = k\}$.) This can easily be verified for several relevant classes of examples. We will say that a collection \mathfrak{H} of densities on Y is *uniformly sweeping* if there is some $K \in \mathbb{N}_0$ such that $\inf_{u \in \mathfrak{H}} \inf_Y \sum_{k=0}^K \widehat{T}^k u > 0$.

We are going to prove the following limit theorem. The assumptions on Y are exactly those of Theorem 2.1 in [Z4].

THEOREM 2.1 (Joint limit distributions for \mathbf{Z}_n and \mathbf{S}_n). *Let T be a c.e.m.p.t. on the σ -finite measure space (X, \mathcal{A}, μ) , and assume there is some $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$, such that*

$$(2.1) \quad \mathfrak{H}_Y = \left\{ \frac{1}{w_N(Y)} \sum_{n=0}^{N-1} \widehat{T}^n 1_{Y_n} \right\}_{N \geq 1} \quad \begin{array}{l} \text{is precompact in } L_\infty(\mu) \\ \text{and uniformly sweeping} \end{array}$$

and that there is some $\alpha \in [0, 1]$ such that

$$(2.2) \quad (w_N(Y)) \text{ is regularly varying of index } 1 - \alpha.$$

Then, for any $f \in L_1(\mu)$ with $\mu(f) \neq 0$, we have

$$(2.3) \quad \left(\frac{\mathbf{Z}_n(Y)}{n}, \frac{\mathbf{S}_n(f)}{a_n} \right) \xrightarrow{\mathcal{L}(\mu)} (\mathcal{Z}_\alpha, \mu(f) \mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha),$$

where

$$(2.4) \quad a_n := \frac{1}{\mu(Y)} \int_Y \mathbf{S}_n(1_Y) d\mu_Y \sim \frac{1}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \cdot \frac{n}{w_n(Y)}$$

as $n \rightarrow \infty$. Here \mathcal{M}_α^* is a random variable, independent of \mathcal{Z}_α , with density $\mathbf{m}_\alpha^*(t) := t \mathbf{m}_\alpha(t)$, $t \geq 0$, where \mathbf{m}_α denotes the density of \mathcal{M}_α .

The result immediately applies to the collection of examples discussed in [Z4], that is, to a large class of infinite measure preserving interval maps with indifferent fixed points (as in [Z1], generalizing [T1]), to S-unimodal Misiurewicz interval maps with sufficiently flat critical points (as in [Z2]), and to recurrent \mathbb{Z} -extensions of Gibbs–Markov maps (as in §7.3 of [Z4]). We refrain from restating the details here.

REMARK 2.2. Theorem 2.1 in [Z4] asserts that $a_n^{-1} \mathbf{S}_n(f) \xrightarrow{\mathcal{L}(\mu)} \mu(f) \mathcal{M}_\alpha$ under the present assumptions. Hence $\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha \stackrel{d}{=} \mathcal{M}_\alpha$ (compare Remark 1.2). For general α , the \mathbf{m}_α are not known explicitly, but this equality in law can be checked by calculating moments, as was pointed out in [P] (where this observation was attributed to M. Dwass). In fact, our proof below uses the method of moments (see [ST] for a wealth of background information).

REMARK 2.3. If $E \in Y \cap \mathcal{A}$ with $n^{-1} \mathbf{Z}_n(E) \xrightarrow{\mathcal{L}(\mu)} \mathcal{Z}_\alpha$, then, due to $\mathbf{Z}_n(E) \leq \mathbf{Z}_n(Y)$, we necessarily have $\mathbf{Z}_n(E) - \mathbf{Z}_n(Y) \xrightarrow{\mu} 0$, so that Y can be replaced by E in (2.3).

REMARK 2.4. (a) A Markov-chain version of this result can be found in [P].

(b) It would be desirable to establish corresponding results including the other random variables studied in [Z4] (occupation times of suitable infinite-measure sets). Alas, even in a classical Markov-chain setup, no tangible description of the prospective joint limit distributions appears to be available.

3. Analytical tools. Karamata's asymptotic theory of regularly varying functions lies at the heart of many limit theorems for null-recurrent processes, a core result being

LEMMA 3.1 (Karamata's Tauberian theorem, KTT). *Let (b_n) be a sequence in $[0, \infty)$ such that for all $s > 0$, $B(s) := \sum_{n \geq 0} b_n e^{-ns} < \infty$. Let ℓ be slowly varying, and $\rho, \vartheta \in [0, \infty)$. Then*

$$(3.1) \quad B(s) \sim \vartheta \left(\frac{1}{s}\right)^\rho \ell\left(\frac{1}{s}\right) \quad \text{as } s \searrow 0$$

iff

$$(3.2) \quad \sum_{k=0}^{n-1} b_k \sim \frac{\vartheta}{\Gamma(\rho+1)} n^\rho \ell(n) \quad \text{as } n \rightarrow \infty.$$

If (b_n) is eventually monotone and $\rho > 0$, then both conditions are equivalent to

$$(3.3) \quad b_n \sim \frac{\vartheta \rho}{\Gamma(\rho+1)} n^{\rho-1} \ell(n) \quad \text{as } n \rightarrow \infty.$$

We will also rely on parts of the somewhat technical Propositions 3.2 and 3.3 of [Z4]. For the reader's convenience, we restate the relevant assertions:

LEMMA 3.2. *Let T be a c.e.m.p.t. on the σ -finite measure space (X, \mathcal{A}, μ) and $Y \in \mathcal{A}$, $0 < \mu(Y) < \infty$. Suppose that $(R_n)_{n \geq 0}$ is a sequence of measurable functions $R_n : X \rightarrow [0, \infty)$ such that*

$$(3.4) \quad R_n \leq R_{n-m} \circ T^m \quad \text{on } Y_m \quad \text{for } n \geq m \geq 0,$$

and

$$(3.5) \quad \|(R_n \circ T - R_n) \cdot u\|_1 \rightarrow 0 \quad \begin{array}{l} \text{for all } u \in L_\infty(\mu) \text{ supported} \\ \text{on } Y^M \text{ for some } M = M(u), \end{array}$$

and

$$(3.6) \quad \{1_{Y^M} \cdot R_n\}_{n \geq 0} \text{ is weakly precompact in } L_1(\mu) \text{ for each } M \geq 1.$$

Moreover, let $v_n : Y \rightarrow [0, \infty)$, $n \geq 0$, be bounded measurable functions with $\int_Y v_n d\mu > 0$, and $(b_n)_{n \geq 0}$ be a sequence in $[0, \infty)$ such that $\sum_{n \geq 0} b_n e^{-ns} =:$

$B(s) \in (0, \infty)$ for $s > 0$. Assume that

$$(3.7) \quad \left\{ \frac{\sum_{k=0}^n v_k}{\sum_{k=0}^n \int_Y v_k d\mu} \right\}_{n \geq 0} \text{ is precompact in } L_\infty(\mu),$$

and that for some $\vartheta \in [0, \infty)$,

$$(3.8) \quad \sum_{k=0}^n \int_Y v_k d\mu \sim \vartheta \cdot \sum_{k=0}^n b_k \quad \text{as } n \rightarrow \infty.$$

Let $(\gamma_n)_{n \geq 0}$ be a sequence in $[0, \infty)$ with $\sum_{n \geq 0} \gamma_n = \infty$ and such that $G(s) := \sum_{n \geq 0} \gamma_n e^{-ns} \in (0, \infty)$ for $s > 0$, and consider the weighted Laplace transform $R_\gamma(s) := \sum_{n \geq 0} R_n \gamma_n e^{-ns}$.

(a) Suppose that for some $\kappa \in [0, \infty)$,

$$(3.9) \quad \int_Y \left(\sum_{n \geq 0} v_n e^{-ns} \right) \cdot R_\gamma(s) d\mu \sim \kappa \vartheta \cdot B(s) G(s) \quad \text{as } s \searrow 0.$$

If $\vartheta > 0$, then, for any $u \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ with $\int_Y u d\mu = 1$,

$$(3.10) \quad \int_X R_\gamma(s) \cdot u d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0.$$

(b) Assume that $r = 0$, or that $r \in \mathbb{N}$ and $B \in \mathcal{R}_{-\rho}(0)$ for some $\rho \in [0, \infty)$. Suppose also that for some $\kappa \in [0, \infty)$ and some $u \in \mathcal{D}(\mu) \cap L_\infty(\mu)$ with $\int_Y u d\mu = 1$,

$$(3.11) \quad \int_X R_\gamma(s) \cdot u d\mu \sim \kappa \cdot G(s) \quad \text{as } s \searrow 0.$$

Then, as $s \searrow 0$,

$$(3.12) \quad \int_Y \left(\sum_{n \geq 0} n^r v_n e^{-ns} \right) \cdot R_\gamma(s) d\mu \sim \kappa \vartheta \cdot (-1)^r r! \binom{-\rho}{r} \left(\frac{1}{s} \right)^r B(s) G(s).$$

(c) If, in the situation of (b), $v_n \searrow 0$ a.e. on Y as $n \rightarrow \infty$, so that $v_n = \sum_{k > n} w_k$ with $w_n \geq 0$, $n \geq 1$, measurable, then, for all $r \geq 1$, as $s \searrow 0$,

$$(3.13) \quad \int_Y \left(\sum_{n \geq 1} n^r w_n e^{-ns} \right) \cdot R_\gamma(s) d\mu \\ \sim \kappa \vartheta \cdot (-1)^{r-1} r! \binom{1-\rho}{r} \left(\frac{1}{s} \right)^{r-1} B(s) G(s).$$

4. Proof of the theorem. The argument to follow uses the machinery developed in [TZ] and [Z4], and determines the asymptotic behaviour of mixed moments of the variables under consideration.

Proof of Theorem 2.1. We can assume that $\mu(Y) = 1$.

(i) Let $S_n := \sum_{j=1}^n 1_Y \circ T^j$, and $Z_n := \mathbf{Z}_n(Y)$, $n \geq 0$. Distributional convergence

$$(4.1) \quad \left(\frac{Z_n}{n}, \frac{S_n}{a_n} \right) \xrightarrow{\mu_Y} (\mathcal{Z}_\alpha, \mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha),$$

with \mathcal{M}_α^* and \mathcal{Z}_α independent, will be established by the method of moments, i.e. we prove that, for all integers $l, r \geq 0$,

$$(4.2) \quad \int_Y \left(\frac{Z_n}{n} \right)^r \left(\frac{S_n}{a_n} \right)^l d\mu \rightarrow \mathbb{E}[\mathcal{Z}_\alpha^r (\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)^l] \quad \text{as } n \rightarrow \infty.$$

The moments on the right-hand side have been identified in Corollary 3.2 of [P] (the result being attributed to M. Dwass), where it is shown that

$$(4.3) \quad \mathbb{E}[\mathcal{Z}_\alpha^r (\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)^l] = (-1)^r \frac{r! l! (\Gamma(1 + \alpha))^l}{\Gamma(1 + r + \alpha l)} \binom{-\alpha(l + 1)}{r} \quad \text{for } l, r \geq 0.$$

It is not hard to see that these moments determine the distribution (see [ST]) of the random vector $(\mathcal{Z}_\alpha, \mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)$ (compare the proof of Theorem 3.1 of [P]).

(ii) Due to KTT and our assumption on the wandering rate, we have, for $s > 0$,

$$(4.4) \quad \begin{aligned} Q_Y(s) &:= \sum_{n \geq 0} q_n(Y) e^{-ns} = \left(\frac{1}{s} \right)^{1-\alpha} \ell \left(\frac{1}{s} \right) \quad \text{for } s > 0, \\ w_n(Y) &\sim \frac{n^{1-\alpha} \ell(n)}{\Gamma(2 - \alpha)} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

with some fixed slowly varying function ℓ . Via the crucial structural assumption (2.1), this information on the return distribution was exploited in [Z4] to obtain information about the moment asymptotics for the individual sequences (S_n) and (Z_n) . Indeed, the dissection identities

$$(4.5) \quad Z_n = \begin{cases} k + Z_{n-k} \circ T^k & \text{on } Y \cap \{\varphi = k\}, 1 \leq k \leq n, \\ 0 & \text{on } Y \cap \{\varphi > n\}, \end{cases} \quad \text{for } n \geq 0,$$

$$(4.6) \quad S_n = \begin{cases} 1 + S_{n-k} \circ T^k & \text{on } Y \cap \{\varphi = k\}, 1 \leq k \leq n, \\ 0 & \text{on } Y \cap \{\varphi > n\}, \end{cases} \quad \text{for } n \geq 0$$

were used to show (see the proofs of Theorems 2.1 and 2.3 of [Z4]) that for

all $r, l \geq 0$,

$$(\spadesuit_r) \quad \sum_{n \geq 0} \left(\int_Y Z_n^r d\mu \right) e^{-ns} \sim (-1)^r r! \binom{-\alpha}{r} \left(\frac{1}{s} \right)^{r+1} \quad \text{as } s \searrow 0,$$

$$(\diamondsuit_l) \quad \sum_{n \geq 0} \left(\int_Y S_n^l d\mu \right) e^{-ns} \sim l! \left(\frac{1}{s} \right)^{1+\alpha l} \ell \left(\frac{1}{s} \right)^{-l} \quad \text{as } s \searrow 0.$$

As $\int_Y S_n d\mu \sim a_n$ is non-decreasing (since (S_n) is), KTT shows that (\diamondsuit_1) entails (2.4). Analogously, by KTT and monotonicity of $(\int_Y Z_n^r d\mu)_{n \geq 1}$ and $(\int_Y S_n^l d\mu)_{n \geq 0}$, the relations (\spadesuit_r) and (\diamondsuit_l) are equivalent to

$$(4.7) \quad \int_Y \left(\frac{Z_n}{n} \right)^r d\mu \rightarrow \mathbb{E}[\mathcal{Z}_\alpha^r] \quad \text{and} \quad \int_Y \left(\frac{S_n}{a_n} \right)^l d\mu \rightarrow \mathbb{E}[\mathcal{M}_\alpha^l]$$

as $n \rightarrow \infty$, proving distributional convergence of each coordinate variable in (4.1).

(iii) We are going to extend this argument to deal with the joint moments in (4.2), observing first that (4.4) and (2.4) give

$$a_n \sim \frac{n^\alpha}{\Gamma(1+\alpha)\ell(n)} \quad \text{as } n \rightarrow \infty.$$

This implies that for $l, r \geq 0$,

$$(4.8) \quad G^{(r,l)}(s) := \sum_{n \geq 0} n^r a_n^l e^{-ns} \sim \frac{\Gamma(1+r+\alpha l)}{\Gamma(1+\alpha)^l} \left(\frac{1}{s} \right)^{1+r+\alpha l} \ell \left(\frac{1}{s} \right)^{-l}$$

as $s \searrow 0$. Now, since each of the sequences $(n^r a_n^l)_{n \geq 0}$ and $(\int_Y Z_n^r S_n^l d\mu)_{n \geq 0}$ is non-decreasing, KTT ensures that, for arbitrary $l, r \geq 0$, our claim (4.2) is equivalent to

$$\sum_{n \geq 0} \left(\int_Y Z_n^r S_n^l d\mu \right) e^{-ns} \sim \mathbb{E}[\mathcal{Z}_\alpha^r (\mathcal{M}_\alpha^* \cdot \mathcal{Z}_\alpha^\alpha)^l] \cdot G^{(r,l)}(s)$$

as $s \searrow 0$. Substituting the explicit expression (4.3) for the moments, and (4.8), we thus see that our goal is to prove, for all $l, r \geq 0$, that

$$\begin{aligned} (\dagger_{r,l}) \quad & \sum_{n \geq 0} \left(\int_Y Z_n^r S_n^l d\mu \right) e^{-ns} \\ & \sim (-1)^r r! l! \binom{-\alpha(l+1)}{r} \left(\frac{1}{s} \right)^{1+r+\alpha l} \ell \left(\frac{1}{s} \right)^{-l} \quad \text{as } s \searrow 0. \end{aligned}$$

The special cases $(\dagger_{0,l})$ and $(\dagger_{r,0})$ of this assertion coincide with (\diamondsuit_l) and (\spadesuit_r) , respectively.

(iv) It has been shown in [Z4] that, for any $l, r \geq 0$, the sequences

$$(4.9) \quad (R_n) = \left(\left(\frac{Z_n}{n} \right)^r \right) \text{ and } (R_n) = \left(\left(\frac{S_n}{a_n} \right)^l \right) \text{ satisfy (3.4)–(3.6).}$$

To prepare the analytic argument below, we need to improve on this, and show that for any $l, r \geq 0$, the sequence given by

$$R_n = R_n^{(r,l)} := \left(\frac{Z_n}{n} \right)^r \left(\frac{S_n}{a_n} \right)^l, \quad n \geq 1,$$

also satisfies the assumptions of Lemma 3.2. Note first that $0 \leq Z_n/n \leq 1$ and

$$(4.10) \quad \left| \left(\frac{Z_n}{n} \right)^r \circ T - \left(\frac{Z_n}{n} \right)^r \right| \leq r \frac{|Z_n \circ T - Z_n|}{n} \xrightarrow{\mu} 0 \quad \text{as } n \rightarrow \infty$$

(by the mean-value theorem and Lemma 1 of [T2]). Next,

$$(4.11) \quad Z_n \leq Z_{n-m} \circ T^m \text{ and } S_n = S_{n-m} \circ T^m \quad \text{on } Y_m \quad \text{for } n \geq m \geq 0,$$

and as (a_n) is non-decreasing, we see that (3.4) is satisfied.

Due to (4.7), each moment sequence $(\int_Y (S_n/a_n)^{l+1} d\mu)_{n \geq 1}$ is bounded, so that $(1_Y (S_n/a_n)^l)_{n \geq 1}$ is uniformly integrable for every l . Now (4.11) proves that in fact

$$(4.12) \quad \left(1_{Y^M} \left(\frac{S_n}{a_n} \right)^l \right)_{n \geq 1} \text{ is uniformly integrable for all } M \geq 1,$$

since indeed $\int_{Y_m \cap \{S_n^l > ta_n^l\}} (S_n/a_n)^l d\mu \leq \int_{Y \cap \{S_{n-m}^l > ta_{n-m}^l\}} (S_{n-m}/a_{n-m})^l d\mu$. Boundedness of the Z_n/n thus guarantees that $(1_{Y^M} R_n)_{n \geq 1}$ is uniformly integrable for all $M \geq 1$, which is (3.6).

We finally have to check (3.5). Fixing M and u , we have

$$\begin{aligned} \|(R_n \circ T - R_n) \cdot u\|_1 &\leq \left\| \left(\left(\frac{S_n}{a_n} \right)^l \circ T - \left(\frac{S_n}{a_n} \right)^l \right) \cdot u \right\|_1 \\ &\quad + r \|u\|_\infty \int_{Y^M} \left(\frac{S_n}{a_n} \right)^l \frac{|Z_n \circ T - Z_n|}{n} d\mu. \end{aligned}$$

The first expression on the right-hand side tends to zero by (4.9). For the second bit, note that for any $\delta > 0$,

$$\begin{aligned} \int_{Y^M} \left(\frac{S_n}{a_n} \right)^l \frac{|Z_n \circ T - Z_n|}{n} d\mu &\leq 2 \int_{Y^M \cap \{|Z_n \circ T - Z_n| > \delta n\}} \left(\frac{S_n}{a_n} \right)^l d\mu \\ &\quad + \delta \int_{Y^M} \left(\frac{S_n}{a_n} \right)^l d\mu. \end{aligned}$$

Take any $\varepsilon > 0$. By (4.12), we have $\sup_{n \geq 1} \int_{Y^M} (S_n/a_n)^l d\mu < \infty$, so the rightmost expression here is less than $\varepsilon/2$ for $\delta = \delta_\varepsilon$ small enough. Fixing such a δ_ε , we have $\mu(Y^M \cap \{|Z_n \circ T - Z_n| > \delta_\varepsilon n\}) \rightarrow 0$ as $n \rightarrow \infty$ by (4.10), and hence $2 \int_{Y^M \cap \{|Z_n \circ T - Z_n| > \delta_\varepsilon n\}} (S_n/a_n)^l d\mu < \varepsilon/2$ for $n \geq n_\varepsilon$, again by the uniform integrability (4.12). This proves (3.5).

It is a trivial matter to check that (3.4)–(3.6) remain valid with $R_n^{(r,l)}$ replaced by some $R_n^{*(r,l)} := (Z_n/n)^r (\Phi_n/a_n)^l$ with measurable functions Φ_n satisfying $S_n \leq \Phi_n \leq 1 + S_n$, $n \geq 1$.

Observe also that due to $S_n \nearrow \infty$ and $1 + S_n \sim S_n$ a.e. we have $\int_Y Z_n^j \Phi_n^l d\mu \sim \int_Y Z_n^j S_n^l d\mu$ as $n \rightarrow \infty$ for such Φ_n . The Laplace transforms of these sequences then satisfy

$$(4.13) \quad \int_Y \left(\sum_{n \geq 0} Z_n^j \Phi_n^l e^{-ns} \right) d\mu \sim \int_Y \left(\sum_{n \geq 0} Z_n^j S_n^l e^{-ns} \right) d\mu \quad \text{as } s \searrow 0.$$

(v) To validate $(\dagger_{r,l})$, we are going to use an inductive argument. Since the statements $(\dagger_{0,l})$, $l \geq 0$, and $(\dagger_{r,0})$, $r \geq 0$, are already contained in (\blacklozenge_l) , $l \geq 0$, and (\spadesuit_r) , $r \geq 0$, respectively, we need only justify the inductive step. Proving that for any pair (r, l) with $r, l \geq 1$,

$$(4.14) \quad \left. \begin{array}{l} (\dagger_{k,i}) \text{ for } 0 \leq i < l \text{ and } k \geq 0, \text{ and} \\ (\dagger_{j,l}) \text{ for } 0 \leq j < r, \end{array} \right\} \text{ together imply } (\dagger_{r,l}),$$

will therefore complete the proof of (4.1).

To this end, we need to understand the relationship between mixed moments of different orders. The dissection identities (4.5) and (4.6) imply that for any l and r , and any $n \geq 1$,

$$1_Y Z_n^r S_n^l = \sum_{k=1}^n 1_{Y \cap \{\varphi=k\}} ((1 + S_{n-k})^l \circ T^k) \sum_{j=0}^r \binom{r}{j} k^{r-j} \cdot (Z_{n-k}^j \circ T^k).$$

Integrating and taking the $j = r$ terms to the left-hand side, we obtain

$$(4.15) \quad \int_Y \left(Z_n^r S_n^l - \sum_{k=1}^n \widehat{T}^k 1_{Y \cap \{\varphi=k\}} (1 + S_{n-k})^l Z_{n-k}^r \right) d\mu \\ = \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \sum_{k=1}^n k^{r-j} \widehat{T}^k 1_{Y \cap \{\varphi=k\}} \cdot (1 + S_{n-k})^l Z_{n-k}^j d\mu.$$

(vi) To tackle (4.14), we consider the Laplace transform $\sum_{n \geq 0} \dots e^{-ns}$ of the expression in (4.15), regarded as a sequence in n . Starting from the bit on the right-hand side of (4.15), we obtain the transform

$$(4.16) \quad \sum_{j=0}^{r-1} \binom{r}{j} \int_Y \left(\sum_{n \geq 1} n^{r-j} \widehat{T}^n 1_{Y \cap \{\varphi=n\}} e^{-ns} \right) \left(\sum_{n \geq 0} (1 + S_n)^l Z_n^j e^{-ns} \right) d\mu.$$

Take any $j \in \{0, 1, \dots, r-1\}$. From (4.13) we see (taking $\Phi_n := 1 + S_n$) that

$$(4.17) \quad \int_Y \left(\sum_{n \geq 0} (1 + S_n)^l Z_n^j e^{-ns} \right) d\mu \sim \int_Y \left(\sum_{n \geq 0} Z_n^j S_n^l e^{-ns} \right) d\mu \quad \text{as } s \searrow 0,$$

the asymptotics of which is given by $(\dagger_{j,l})$. We can now apply Lemma 3.2(c) with $R_n := R_n^{*(j,l)}$ obtained from $\Phi_n := 1 + S_n$, $\gamma_n := a_n^l n^j$ (which, by step (iv) above, satisfies all assumptions), and $w_n := \widehat{T}^n 1_{Y \cap \{\varphi=n\}}$ (which due to (2.1) has the required properties), $b_n := q_n(Y)$, $\vartheta := 1$, $B := Q_Y$, $\rho := 1 - \alpha$, and $G := G^{(j,l)}$, to obtain

$$\begin{aligned} & \binom{r}{j} \int_Y \left(\sum_{n \geq 1} n^{r-j} \widehat{T}^n 1_{Y \cap \{\varphi=n\}} e^{-ns} \right) \left(\sum_{n \geq 0} (1 + S_n)^l Z_n^j e^{-ns} \right) d\mu \\ & \sim (-1)^{r+1} r! l! \binom{-\alpha(l+1)}{j} \binom{\alpha}{r-j} \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell \left(\frac{1}{s} \right)^{1-l} \end{aligned}$$

as $s \searrow 0$. Summing over j , we see that the expression in (4.16) is asymptotically equivalent, as $s \searrow 0$, to

$$(4.18) \quad (-1)^r r! l! \left[\binom{-\alpha(l+1)}{r} - \binom{-\alpha l}{r} \right] \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell \left(\frac{1}{s} \right)^{1-l}.$$

(vii) Turning to the left-hand side of (4.15), we again take the Laplace transform, which equals

$$\begin{aligned} (4.19) \quad & \int_Y \left(\sum_{n \geq 0} Z_n^r S_n^l e^{-ns} - \left(\sum_{n \geq 1} \widehat{T}^n 1_{Y \cap \{\varphi=n\}} e^{-ns} \right) \left(\sum_{n \geq 0} (1 + S_n)^l Z_n^r e^{-ns} \right) \right) d\mu \\ & = \int_Y \left(1 - \sum_{n \geq 1} \widehat{T}^n 1_{Y \cap \{\varphi=n\}} e^{-ns} \right) \left(\sum_{n \geq 0} Z_n^r S_n^l e^{-ns} \right) d\mu \\ & \quad - l \int_Y \left(\sum_{n \geq 0} Z_n^r \Phi_{l,n}^{l-1} e^{-ns} \right) \left(\sum_{n \geq 1} \widehat{T}^n 1_{Y \cap \{\varphi=n\}} e^{-ns} \right) d\mu. \end{aligned}$$

Here, for any $l, n \geq 1$, we let $\Phi_{l,n} : X \rightarrow [0, \infty)$ denote the measurable function with $(1 + S_n)^l - S_n^l = l \Phi_{l,n}^{l-1}$ and $S_n \leq \Phi_{l,n} \leq S_n + 1$, which is provided by the mean-value theorem. Focusing on the rightmost term, we find that

$$\begin{aligned} (4.20) \quad & l \int_Y \left(\sum_{n \geq 0} Z_n^r \Phi_{l,n}^{l-1} e^{-ns} \right) \left(\sum_{n \geq 1} \widehat{T}^n 1_{Y \cap \{\varphi=n\}} e^{-ns} \right) d\mu \\ & \sim l \int_Y \left(\sum_{n \geq 0} Z_n^r \Phi_{l,n}^{l-1} e^{-ns} \right) d\mu \sim l \int_Y \left(\sum_{n \geq 0} Z_n^r S_n^{l-1} e^{-ns} \right) d\mu \\ & \sim (-1)^r r! l! \binom{-\alpha l}{r} \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell \left(\frac{1}{s} \right)^{1-l} \quad \text{as } s \searrow 0, \end{aligned}$$

where we first use the fact that

$$\sum_{n \geq 1} \widehat{T}^n 1_{Y \cap \{\varphi = n\}} e^{-ns} \nearrow 1_Y \quad \text{uniformly as } s \searrow 0$$

(which, under the present assumptions, has been proved as statement (4.5) in [Z4]), and then (4.13) above. The third step uses $(\dagger_{r,l-1})$, which holds by assumption.

(viii) Recall that (4.16) and (4.19) coincide (being the Laplace transforms of the two expressions in (4.15)). Since (4.20) is of the same order as (4.18), we can combine these two observations to conclude that

$$(4.21) \quad s \int_Y \left(\sum_{n \geq 1} \widehat{T}^n 1_{Y_n} e^{-ns} \right) \left(\sum_{n \geq 0} Z_n^r S_n^l e^{-ns} \right) d\mu \\ \sim (-1)^r r! l! \binom{-\alpha(l+1)}{r} \cdot \left(\frac{1}{s} \right)^{1+r+\alpha(l-1)} \ell \left(\frac{1}{s} \right)^{1-l} \quad \text{as } s \searrow 0.$$

Here we have used identity (5.3) of [TZ], which states that

$$1_Y - \sum_{n \geq 1} \widehat{T}^n 1_{Y \cap \{\varphi = n\}} e^{-ns} = (1 - e^{-s}) \sum_{n \geq 0} \widehat{T}^n 1_{Y_n} e^{-ns} \quad \text{a.e.}$$

Letting $R_n := R_n^{(r,l)} = (Z_n/n)^r (S_n/a_n)^l$, $\gamma_n := n^r a_n^l$, $G := G^{(r,l)}$, $v_n := \widehat{T}^n 1_{Y_n}$, and b_n, B, ϑ, ρ be as before, we can now apply Lemma 3.2(a) to the integral on the left-hand side of (4.21) to obtain $(\dagger_{r,l})$. This completes the proof of (4.1).

(ix) As a straightforward consequence of Hopf's ratio ergodic theorem (see [A0] or [Z3]), we may replace S_n by $\mu(f)^{-1} \mathbf{S}_n(f)$ in (4.1) whenever $f \in L_1(\mu)$ and $\mu(f) \neq 0$.

According to Theorem 1 of [Z5], the now established distributional convergence $\xrightarrow{\mu_Y}$ with respect to μ_Y entails strong distributional convergence $\xrightarrow{\mathcal{L}(\mu)}$ if

$$\left| \left(\frac{Z_n}{n}, \frac{\mathbf{S}_n(f)}{a_n} \right) \circ T - \left(\frac{Z_n}{n}, \frac{\mathbf{S}_n(f)}{a_n} \right) \right| \xrightarrow{\mu} 0,$$

and the latter is immediate from the corresponding asymptotic invariance statements for (Z_n) and (S_n) already recorded in [Z4] (cf. (4.9) above). This completes the proof of the theorem. ■

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