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Interpolating discrete multiplicity varieties for $A_p^0(\mathbb{C}^n)$

by

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Abstract. A necessary and sufficient condition is obtained for a discrete multiplicity variety to be an interpolating variety for the space $A_p^0(\mathbb{C}^n)$.

1. Introduction. In this paper, we will consider interpolation problems for the space $A_p^0(\mathbb{C}^n)$, which is the ring of entire functions in $\mathbb{C}^n$ with the property that for every $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that $|f(z)| < A_\varepsilon e^{\varepsilon p(z)}$ for all $z \in \mathbb{C}^n$, i.e., $\sup_{z\in\mathbb{C}^n} |f(z)|e^{-\varepsilon p(z)} < \infty$, where p is a weight (see §2). $A_p^0(\mathbb{C}^n)$ is an important class of entire functions. When $p(z) = |z|$, $A_{|z|}^0(\mathbb{C}^n)$ is, via the Fourier–Borel transformation, topologically isomorphic to the ring of infinite order differential operators. The space $A_p^0(\mathbb{C}^n)$ has a natural projective limit structure. This kind of spaces appear naturally in complex, harmonic, and functional analysis.

Let $V = \{(\zeta_k, m_k)\}$ be a discrete multiplicity variety in $\mathbb{C}^n$ (cf. §2) and $A_p^0(V)$ be the space of multi-indexed sequences $\{a_{k,I}\}_{k\in\mathbb{N},\,0\le|I|<m_k}$ of complex numbers with the property that for every $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that

$$\sum_{|I|=0}^{m_k-1} |a_{k,I}| < A_\varepsilon e^{\varepsilon p(\zeta_k)} \quad \text{for all } k \in \mathbb{N},$$

where $I = (i_1, \dots, i_n) \in (\mathbb{Z}^+)^n$ and $|I| = i_1 + \dots + i_n$. If for any sequence $\{a_{k,I}\}_{k\in\mathbb{N},\,0\le|I|<m_k}$ in $A_p^0(V)$, there always exists an entire function f in $A_p^0(\mathbb{C}^n)$ such that

$$\frac{1}{I!}\frac{\partial^{|I|} f(\zeta_k)}{\partial z^I} = a_{k,I} \quad \text{for } k \in \mathbb{N},\ 0 \le |I| < m_k, \tag{1.1}$$

we will say that V is an *interpolating* (*multiplicity*) *variety* for $A_p^0(\mathbb{C}^n)$. Clearly, condition (1.1) means that f has a prescribed finite collection of Taylor coefficients at each ζ_k.

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Note that the constant A_ε in the definition of $A_p^0(\mathbb{C}^n)$ depends on the arbitrarily given ε, which makes the growth condition for $A_p^0(\mathbb{C}^n)$ more restrictive than that for the Hörmander algebra $A_p(\mathbb{C}^n)$, the space of entire functions in $\mathbb{C}^n$ satisfying $|f(z)| < Ae^{Bp(z)}$, $z \in \mathbb{C}^n$, for some $A, B > 0$, or simply $|f(z)| < Ae^{Ap(z)}$, $z \in \mathbb{C}^n$, for some $A > 0$. The space $A_p^0(\mathbb{C}^n)$ has a different topological structure from the one of $A_p(\mathbb{C}^n)$, and there is no weight q such that $A_p^0(\mathbb{C}^n)$ becomes a space $A_q(\mathbb{C}^n)$. We refer to [2, pp. 294–299] for a discussion of relations and differences between $A_p(\mathbb{C}^n)$ and $A_p^0(\mathbb{C}^n)$.

To study problems such as analytic continuation for Dirichlet series and representation of analytic solutions of partial differential equations of infinite order, one needs to consider interpolation problems for the space $A_p^0(\mathbb{C}^n)$ instead of $A_p(\mathbb{C}^n)$ (see e.g. [1], [2], [4] and references therein). In [2], a sufficient interpolation condition for $A_{|z|}^0(\mathbb{C}^n)$, using distribution of points of V in a "tube neighborhood" of V (cf. (2.4) and Remark 2.6 below), was obtained by Berenstein–Kawai–Struppa by expressing $A_{|z|}^0(\mathbb{C}^n)$ as a sort of inductive limit of $A_p(\mathbb{C}^n)$. It however does not provide necessary interpolation conditions. The main purpose of this paper is to give a similar interpolation condition, which is however both necessary and sufficient for interpolation in $A_p^0(\mathbb{C}^n)$ and applies to arbitrary discrete multiplicity varieties in $\mathbb{C}^n$. The proof of the result will use some existing interpolation results and methods, especially those in [5], [11] and [12]. We state the theorem in §2 and give its proof in §3.

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2. Definitions and results. We first fix some notions and notations which will be used throughout the paper.

DEFINITION 2.1. A plurisubharmonic function $p : \mathbb{C}^n \to [0, \infty)$ is called a *weight* (*function*) if it satisfies the following conditions:

$$\log(1 + |z|^2) = o\{p(z)\}, \tag{2.1}$$

$$p(z) = p(|z|), \qquad p(2z) = O\{p(z)\}. \tag{2.2}$$

DEFINITION 2.2. Let $A(\mathbb{C}^n)$ be the ring of all entire functions on $\mathbb{C}^n$. Then

$$A_p^0(\mathbb{C}^n) = \{f \in A(\mathbb{C}^n) : \forall \varepsilon > 0,\ \exists A_\varepsilon > 0 \text{ such that } |f(z)| \le A_\varepsilon \exp(\varepsilon p(z)),\ z \in \mathbb{C}^n\}.$$

A simple but important example of weighted spaces $A_p^0(\mathbb{C}^n)$ is $A_{|z|}^0(\mathbb{C}^n)$, which is the space of entire functions of infraexponential type; it plays important roles in Dirichlet series, Fabry-type gap theorems, etc. (see e.g. [2]).

Let $f \not\equiv 0$ be a holomorphic function on an open connected neighborhood of $\zeta \in \mathbb{C}^n$. Then a series $\sum_{j=\nu}^{\infty} \mathcal{P}_j(z-\zeta)$ converges uniformly on some neighborhood of ζ and represents f on this neighborhood. Here $\mathcal{P}_j$ is a homogeneous polynomial of degree j and $\mathcal{P}_\nu \not\equiv 0$. The nonnegative integer ν, uniquely determined by f and ζ, is called the *zero multiplicity*, or *zero divisor*, of f at ζ, denoted by $\mathrm{div}_f(\zeta)$.

Let $V = \{(\zeta_k, m_k)\}$ be a *multiplicity variety*, that is, a discrete set $\{\zeta_k\} \subset \mathbb{C}^n$ with $|\zeta_k| \to \infty$ together with a sequence $\{m_k\}$ of positive integers. We write $V \subseteq f^{-1}(0)$ if $\mathrm{div}_f(\zeta_k) \geq m_k$ for each k, i.e., each ζ_k is a zero of f of multiplicity at least m_k. Associated to V, there is a unique closed ideal in $A(\mathbb{C}^n)$,

$$J = J(V) := \{f \in A(\mathbb{C}^n) : \mathrm{div}_f(\zeta_k) \geq m_k,\ \forall k\}.$$

Two entire functions g, h in $\mathbb{C}^n$ can be identified modulo J if and only if

$$\frac{\partial^{|I|} g(\zeta_k)}{\partial z^I} = \frac{\partial^{|I|} h(\zeta_k)}{\partial z^I}, \qquad 0 \leq |I| < m_k,\ k \in \mathbb{N};$$

here and throughout the paper, we use I to denote a multi-index, that is, $I = (i_1, \ldots, i_n) \in (\mathbb{Z}^+)^n$, $\mathbb{Z}^+ = \{0, 1, \ldots\}$. The quotient space $A(\mathbb{C}^n)/J$ can be identified with the space $A(V)$ of all sequences $\{a_{k,I}\}_{k \in \mathbb{N},\, 0 \leq |I| < m_k}$ of complex numbers. The map

$$f \mapsto \varphi(f) = \left\{ \frac{\partial^{|I|} f(\zeta_k)}{I! \partial z^I} \right\}_{k \in \mathbb{N},\, 0 \leq |I| < m_k} \tag{2.3}$$

is the natural restriction map from $A(\mathbb{C}^n)$ into $A(V)$.

Definition 2.3. Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety on $\mathbb{C}^n$. Then we define

$$A_p^0(V) = \Big\{ a = \{a_{k,I}\} \in A(V) : \forall \varepsilon > 0,\ \exists A_\varepsilon > 0 \ \text{ such that } \sum_{|I|=0}^{m_k - 1} |a_{k,I}| \leq A_\varepsilon \exp(\varepsilon p(\zeta_k)),\ k \in \mathbb{N} \Big\}.$$

Using Cauchy's estimates, it is easy to check that φ is a map from $A_p^0(\mathbb{C}^n)$ to $A_p^0(V)$. But, in general, the space $A_p^0(V)$ is larger.

Definition 2.4. A multiplicity variety $V = \{(\zeta_k, m_k)\}$ is an *interpolating variety* for $A_p^0(\mathbb{C}^n)$ if the restriction map φ is surjective from $A_p^0(\mathbb{C}^n)$ to $A_p^0(V)$.

Clearly, that V is an interpolating variety for $A_p^0(\mathbb{C}^n)$ means that for any multi-indexed sequence $\{a_{k,I}\} \in A_p^0(V)$, there exists an entire function $f \in A_p^0(\mathbb{C}^n)$ such that $\partial^{|I|} f(\zeta_k)/I! \partial z^I = a_{k,I}$ for all $k \in \mathbb{N}$ and $0 \leq |I| < m_k$, i.e., f has a described finite collection of Taylor coefficients.

We obtain the following both necessary and sufficient interpolation condition, which applies to arbitrary multiplicity varieties in $\mathbb{C}^n$.

THEOREM 2.5. *Let $V = \{(\zeta_k, m_k)\}$ be a multiplicity variety in $\mathbb{C}^n$ and $N \geq n$ a positive integer. Then V is an interpolating variety for $A_p^0(\mathbb{C}^n)$ if and only if there exist an entire holomorphic mapping $f = (f_1, \ldots, f_N)$ with $f_j \in A_p^0(\mathbb{C}^n)$ and a positive function $q(z) = o\{p(z)\}$ such that $V \subseteq f^{-1}(0)$ and, for some constants $\varepsilon, C > 0$, each connected component of the set*

$$S_q(f; \varepsilon, C) := \{z \in \mathbb{C}^n : |f(z)| < \varepsilon e^{-Cq(z)}\} \tag{2.4}$$

contains at most one point in V and has diameter at most one.

REMARK 2.6. The above condition is given by means of distribution of points of V in a "tube neighborhood" $S_q(f; \varepsilon, c)$ of the variety V, which bears a resemblance to the interpolation condition for the Hörmander algebra $A_p(\mathbb{C}^n)$ in [12]. Such a geometric condition has been fundamental in the study of interpolation problems, slowly decreasing ideals, division problems, etc. (see [1], [2], [5], [7], etc. and references therein). A similar sufficient condition was given in [2, Theorem 3.2] when V is the complete intersection of zero sets of some locally slowly decreasing functions in $A_p^0(\mathbb{C}^n)$, which is the main interpolation theorem used to prove the Fabry-type gap theorems in [2, §4]. Note that the multiplicity varieties in Theorem 2.5 are arbitrarily given. If no multiplicities are involved, the condition in Theorem 2.5 is equivalent to an estimate of the Jacobian of the entire holomorphic mapping f in the theorem and was given in [11]. Some other related results can be found in [1]–[5], etc.

We conclude this section by the following corollary, which uses the necessary and sufficient condition of Theorem 2.5. It does not seem trivial to see whether an interpolating variety for $A_{|z|}^0(\mathbb{C}^n)$ is an interpolating variety for $A_q^0(\mathbb{C}^n)$, where $q \geq |z|$ is another weight. This is, however, a trivial consequence of the following general result.

COROLLARY 2.7. *If a multiplicity variety $V = \{(\zeta_k, m_k)\}$ in $\mathbb{C}^n$ is an interpolating variety for $A_p^0(\mathbb{C}^n)$, then it is also an interpolating variety for $A_q^0(\mathbb{C}^n)$ for any weight $q \geq p$.*

Proof. By the *necessary* condition of Theorem 2.5, there exist an entire holomorphic mapping $f = (f_1, \ldots, f_n)$ and a positive function $q_1(z) = o\{p(z)\}$ satisfying the conditions in Theorem 2.5. Since $q \geq p$, we have $A_p^0(\mathbb{C}^n) \subseteq A_q^0(\mathbb{C}^n)$ and $q_1(z) = o\{q(z)\}$. Thus, the conditions in Theorem 2.5 also hold for $A_q^0(\mathbb{C}^n)$. By the *sufficiency* of the condition in Theorem 2.5, V is an interpolating variety for $A_q^0(\mathbb{C}^n)$.

3. Proof of Theorem 2.5. For convenience, in the following proof we will use $0 < \varepsilon < 1, c > 0$ to denote numerical constants, which may depend on n and the actual value of which may vary from one occurrence to the next.

To prove the necessity, we first write down explicitly the projective limit topologies of $A_p^0(\mathbb{C}^n)$ and $A_p^0(V)$ by specifying their neighborhood bases, which will be needed later. For each positive integer m, let $A_m = \{f \in A(\mathbb{C}^n) : \|f\|_{m,\infty} < \infty\}$, where $\|f\|_{m,\infty} := \sup_{z\in\mathbb{C}^n} |f(z)|e^{-p(z)/m}$. The space $A_p^0(\mathbb{C}^n)$ is endowed with the natural projective limit topology (see e.g. [13] for basics of projective limit topology). A neighborhood base of $f \in A_p^0(\mathbb{C}^n)$ is given by all the intersections

$$A_p^0(\mathbb{C}^n) \cap \bigcap_{m\in H} U_m, \tag{3.1}$$

where U_m is any neighborhood of f in A_m with respect to the topology induced by $\|f\|_{m,\infty}$ and H is any finite subset of $\mathbb{N}$. The space $A_p^0(\mathbb{C}^n)$ is metrizable and complete as a projective limit of complete locally convex spaces.

In the same way we set $A_m(V) = \{a = \{a_{k,I}\} : \|a\|_{m,\infty} < \infty\}$, where $\|a\|_{m,\infty} := \sup_{k\in\mathbb{N}} \sum_{|I|=0}^{m_k-1} |a_{k,I}|e^{-p(\zeta_k)/m}$. Then a neighborhood base of $a \in A_p^0(V)$ is given by all the intersections

$$A_p^0(V) \cap \bigcap_{m\in I} V_m, \tag{3.2}$$

where V_m is any neighborhood of a with respect to the topology in $A_m(V)$ induced by $\|a\|_{m,\infty}$ and I is any finite subset of $\mathbb{N}$. The space $A_p^0(V)$ is also metrizable and complete.

Consider now the map $\varphi : A_p^0(\mathbb{C}^n) \to A_p^0(V)$ defined in (2.3). It is surjective since V is an interpolating variety for $A_p^0(\mathbb{C}^n)$. It is easy to check that φ is linear and continuous. Thus, by the open mapping theorem (see e.g. [9, p. 294]), φ maps every neighborhood of 0 in $A_p^0(\mathbb{C}^n)$ onto a neighborhood of 0 in $A_p^0(V)$.

For each positive integer m, let $U_m^0 = \{f \in A_p^0(\mathbb{C}^n) : \|f\|_{m,\infty} < L_m\}$, where $L_m > 1$ is a positive number. One can take L_m properly so that

$$\varphi\Big(\bigcap_{j=1}^{m} U_j^0\Big) \supset W_m^0 := \{a = \{a_{k,I}\} \in A_p^0(V) : \|a\|_{l_m,\infty} \le \gamma_m\} \tag{3.3}$$

for some positive numbers l_m, γ_m, with $\gamma_m \ge 1$ (cf. [11]). We include the proof of (3.3) for completeness. In fact, $U_m^0 = U_m \cap A_p^0(\mathbb{C}^n)$, where $U_m = \{f \in A(\mathbb{C}^n) : \|f\|_{m,\infty} < L_m\}$. By (3.1), $\bigcap_{j=1}^m U_j^0$ is a neighborhood of 0 in $A_p^0(\mathbb{C}^n)$. Thus, $\varphi(\bigcap_{j=1}^m U_j^0)$ contains a neighborhood of 0 in $A_p^0(V)$ and so, by (3.2), contains an open set of the form $(\bigcap_{m\in I} V_m) \cap A_p^0(V)$, where V_m is defined as in (3.2) and I is a finite subset of $\mathbb{N}$. We then deduce that there exist an integer $l_m > 0$ and a $\gamma_m > 0$ such that $\varphi(\bigcap_{j=1}^m U_j^0)$ contains the set W_m^0 of (3.3). The positive constant γ_m obtained above might not satisfy the required condition that $\gamma_m \geq 1$. If this happens for some m, we can revise the above sets. Suppose that m is the smallest positive integer so that $\gamma_m < 1$ (m might be 1). We then replace L_j by $(1/\gamma_m)l_j$, U_j^0 by

$$\widehat{U}_j^0 := \frac{1}{\gamma_m} U_j^0 := \left\{ \frac{1}{\gamma_m} f : f \in U_j^0 \right\} = \left\{ f \in A_p^0(\mathbb{C}^n) : \|f\|_{j,\infty} < \frac{1}{\gamma_m} L_j \right\}$$

for $1 \leq j \leq m$, and W_i^0 by $\widehat{W_i^0} := (1/\gamma_m) W_i^0$ for each $1 \leq i \leq m$, which satisfies the desired requirement that $\widehat{\gamma}_i := \gamma_i/\gamma_m \geq 1$ for each $1 \leq i \leq m$. We can continue this process and eventually obtain a sequence of sets in $A_p^0(\mathbb{C}^n)$, still denoted by U_m^0, and a sequence of sets in $A_p^0(V)$, still denoted by W_m^0, which satisfy $\varphi(\bigcap_{j=1}^m U_j^0) \supseteq W_m^0$ and $\gamma_m \geq 1$ for each integer $m \geq 1$. This shows (3.3).

Next, we will use the fact that $\varphi(\bigcap_{j=1}^m U_j^0) \supseteq W_m^0$ to produce a sequence of functions with certain "good" properties, which will help us to construct the mapping desired in the theorem (compare [12]). For each fixed $k \in \mathbb{N}$ and $1 \leq i \leq n$, (3.3) implies that there exists a sequence $\{g_{i,k,m}\}_{m=1}^\infty$ of entire functions such that $g_{i,k,m} \in \bigcap_{j=1}^m U_j^0$ and

$$\varphi(g_{i,k,m}) = \left\{ \frac{\partial^{|I|} g_{i,k,m}(\zeta_l)}{I! \partial z^I} \right\}_{l \in \mathbb{N},\, 0 \leq |I| < m_l} \in W_m^0$$

with all the entries in this sequence being zero except one which is 1, specified as follows:

$$\tag{3.4} \begin{aligned} &\frac{\partial^{|I|} g_{i,k,m}(\zeta_l)}{I! \partial z^I} = 0, \qquad \forall l,\ \forall 0 \leq |I| < m_l \text{ except that} \\ &\frac{\partial^{l_k} g_{i,k,m}(\zeta_k)}{l_k! \partial z_i^{l_k}} = 1, \end{aligned}$$

where $l_k = m_k/2$ if m_k is even and $l_k = (m_k - 1)/2$ if m_k is odd. (This sequence clearly belongs to W_m^0. And it is here where we used the fact that $\gamma_m \geq 1$.) Since $g_{i,k,m} \in \bigcap_{j=1}^m U_j^0$, we have $\|g_{i,k,m}\|_{j,\infty} < L_j$ for $1 \leq j \leq m$ and so

$$\tag{3.5} |g_{i,k,m}(z)| < L_j e^{p(z)/j}, \qquad 1 \leq j \leq m,\ z \in \mathbb{C}^n.$$

In particular, $|g_{i,k,m}(z)| \leq L_1 e^{p(z)}$, $z \in \mathbb{C}^n$. By (2.2), it is easy to check that there are two constants $A, B > 1$ such that

$$\tag{3.6} p(w) \leq A p(z) + B$$

whenever $|w - z| < 2\sqrt{n}$.

Thus, we see that $\{g_{i,k,m}\}_{m=1}^\infty$ is uniformly bounded on compact sets in $\mathbb{C}^n$ and so that $\{g_{i,k,m}\}$ is a normal family by Montel's theorem (see e.g. [8]).

By passing to a subsequence we can assume that $\{g_{i,k,m}\}$ converges to a function $g_{i,k}$ in $\mathbb{C}^n$ as $m \to \infty$, which is an entire function in $\mathbb{C}^n$ by the Weierstrass theorem. Clearly, $g_{i,k}$ also satisfies

$$\begin{aligned} &\frac{\partial^{|I|} g_{i,k}(\zeta_l)}{\partial z^I} = 0, \qquad \forall l,\ \forall 0 \le |I| < m_l \text{ except that} \\ &\frac{\partial^{l_k} g_{i,k}(\zeta_k)}{l_k! \partial z_i^{l_k}} = 1, \end{aligned} \tag{3.7}$$

since each $g_{i,k,m}$ satisfies (3.4). From the fact that $\lim_{m\to\infty} g_{i,k,m}(z) = g_{i,k}(z)$ and from (3.5) it follows that for each m and each $z \in \mathbb{C}^n$ there exists an integer $m_0 > m$ such that

$$\begin{aligned} |g_{i,k}(z)| &\le |g_{i,k}(z) - g_{i,k,m_0}(z)| + |g_{i,k,m_0}(z)| \\ &\le 1 + |g_{i,k,m_0}(z)| \le 1 + L_j e^{p(z)/j} \end{aligned}$$

for each $1 \le j \le m_0$. In particular,

$$|g_{i,k}(z)| \le 1 + L_m e^{p(z)/m} \le 2L_m e^{p(z)/m}.$$

Note that this inequality is true for each m. We have

$$|g_{i,k}(z)| \le \exp(\inf_m \{\log(2L_m) + p(z)/m\}) =: \exp(q_1(z)), \tag{3.8}$$

where $q_1(z) = \inf_m\{\log(2L_m) + p(z)/m\}$. Clearly, $q_1(z) = o\{p(z)\}$.

Take a large positive number K and define, for each fixed integer i ($1 \le i \le n$),

$$f_i(z) = \sum_{k=1}^{\infty} h_{i,k}(z) \frac{1}{(1+|\zeta_k|)^{K+1}} \exp(-2nAq_1(\zeta_k)), \qquad z \in \mathbb{C}^n \tag{3.9}$$

where $h_{i,k} = g_{i,k}^2$ if m_k is even and $h_{i,k} = (z_i - \zeta_{k,i}) g_{i,k}^2$ if m_k is odd; here $z = (z_1, \ldots, z_n)$, $\zeta_k = (\zeta_{k,1}, \ldots, \zeta_{k,n})$, and A is the number in (3.6). We will prove that $f_i \in A_p^0(\mathbb{C}^n)$. We denote by $f_{i,k}$ the general term of the series in (3.9). We then have, by virtue of (3.8),

$$\begin{aligned} |f_{i,k}(z)| &\le (|z| + |\zeta_k|) e^{2q_1(z)} \frac{1}{(1+|\zeta_k|)^{K+1}} \exp(-2nAq_1(\zeta_k)) \\ &\le (1+|z|) e^{2q_1(z)} \frac{1}{(1+|\zeta_k|)^{K}} \exp(-2nAq_1(\zeta_k)). \end{aligned} \tag{3.10}$$

Write $d_k = \min\{1, \inf_{l\ne k} |\zeta_l - \zeta_k|\}$, and $\mathcal{D}_k = B(\zeta_k, d_k/2)$, the ball centered at ζ_k with radius $d_k/2$. Then $d_k \le 1$ and $\mathcal{D}_k \cap \mathcal{D}_l = \emptyset$ for $k \ne l$. By (3.6), when $|z - w| \le 2\sqrt{n}$, we have

$$q_1(z) \le A \inf_m \{\log(2L_m) + p(w)/m\} + B = Aq_1(w) + B, \tag{3.11}$$

where A and B are the numbers in (3.6). If $d_k < 1$, then there is a $\zeta_j \in V \cap \mathcal{B}(\zeta_k, 1)$ such that $\zeta_j \neq \zeta_k$ and $d_k = |\zeta_j - \zeta_k|$. By (3.7) we know that

$$\frac{\partial^{l_j} g_{i,j}(\zeta_j)}{l_j!\partial z_i^{l_j}} = 1 \quad \text{and} \quad \frac{\partial^{l_j} g_{i,j}(\zeta_k)}{l_j!\partial z_i^{l_j}} = 0.$$

Also, by Cauchy's estimate, we know that

$$\frac{\partial^{l_j} g_{i,j}(z)}{l_j!\partial z_i^{l_j}} \leq c \max_{w\in\mathbb{C}^n\,:\,|w-z|\leq 1} |g_{i,j}(w)| \leq c e^{Aq_1(\zeta_k)+B},$$

in view of (3.8) and (3.11). Thus, by the Schwarz lemma ([8, p. 7]), we have

$$\left|\frac{\partial^{l_j} g_{i,j}(z)}{\partial z_i^{l_j}}\right| \leq c e^{Aq_1(\zeta_k)+B}|z-\zeta_k| \quad \text{ for } |z-\zeta_k| < 1,$$

and in particular,

$$1 = \left|\frac{\partial^{l_j} g_{i,j}(\zeta_j)}{\partial z_i^{l_j}}\right| \leq c e^{Aq_1(\zeta_k)+B}|\zeta_j-\zeta_k|,$$

or $d_k = |\zeta_j - \zeta_k| \geq \varepsilon e^{-Aq_1(\zeta_k)}$. This inequality is obviously also true if $d_k = 1$. Therefore in any case the volume of the ball $\mathcal{D}_k$ satisfies $\operatorname{vol}\mathcal{D}_k = \pi^n (d_k/2)^{2n}/n! \geq \varepsilon e^{-2nAq_1(\zeta_k)}$. We thus deduce, by (3.10), that

$$\begin{aligned} |f_{i,k}(z)| &\leq (1+|z|)e^{2q_1(z)} \frac{1}{\operatorname{vol}\mathcal{D}_k} \int_{\mathcal{D}_k} \frac{1}{(1+|\zeta_k|)^K} \exp(-2nAq_1(\zeta_k))\, d\sigma \\ &\leq c(1+|z|)e^{2q_1(z)} \int_{\mathcal{D}_k} \frac{1}{(1+|\zeta_k|)^K}\, d\sigma, \end{aligned} \tag{3.12}$$

where $d\sigma$ is the Euclidean volume element in $\mathbb{C}^n$. Note that if $z \in \mathcal{D}_k$,

$$1+|z| < 1+|z-\zeta_k|+|\zeta_k| < 2+|\zeta_k| < 2(1+|\zeta_k|).$$

Therefore, in view of the fact that $\mathcal{D}_k \cap \mathcal{D}_l = \emptyset$ for $k \neq l$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{\mathcal{D}_k} \frac{1}{(1+|\zeta_k|)^K}\, d\sigma &\leq \sum_{k=1}^{\infty} \int_{\mathcal{D}_k} \frac{2^K}{(1+|z|)^K}\, d\sigma \\ &\leq 2^K \int_{\mathbb{C}^n} \frac{1}{(1+|z|)^K} d\sigma =: 2^K L < \infty \end{aligned} \tag{3.13}$$

if we take K sufficiently large. Also, by (3.11), $q(w) \leq Aq_1(z) + B$ whenever $|w - z| < 1$.

We have thus showed that the series $f_i = \sum_{k=1}^{\infty} f_{i,k}$ converges uniformly on compact sets in $\mathbb{C}^n$ and so that f_i is an entire function in $\mathbb{C}^n$. Moreover, by virtue of (3.12) and (3.13), we have

$$|f_i(z)| \leq c(1+|z|)e^{2q_1(z)}. \tag{3.14}$$

But $(1+|z|)e^{2q_1(z)} = e^{\log(1+|z|)+2q_1(z)} = e^{o\{p(z)\}}$ by (2.1). We thus conclude that $f_i \in A_p^0(\mathbb{C}^n)$.

Let $f = (f_1, \dots, f_n)$. It is obvious that $V \subseteq f^{-1}(0)$ by the construction of each f_i (see (3.7) and (3.9)). Next we will find a positive function q such that a tube neighborhood $S_q(f; \varepsilon, C)$ satisfies the conditions in the theorem. By (3.9) and (3.7) one can check that f_i, $1 \le i \le n$, can be expanded into the following power series at each ζ_k:

$$\begin{aligned} f_i(z) &= c_k(z_i - \zeta_{k,i})^{m_k} \\ &+ \sum_{i_1+\cdots+i_n \ge m_k+n_k}^{\infty} C_{i_1,\dots,i_n}(\zeta_1 - \zeta_{k,1})^{i_1} \cdots (z_i - \zeta_{k,i})^{i_j} \cdots (\zeta_n - \zeta_{k,n})^{i_n}, \end{aligned} \tag{3.15}$$

where

$$c_k = \frac{1}{(1+|\zeta_k|)^{K+1}} \exp(-2nAq_1(\zeta_k)), \tag{3.16}$$

$C_{i_1,\dots,i_n}$'s are complex numbers, and $n_k = m_k/2$ if m_k is even and $n_k = (m_k+1)/2$ if m_k is odd.

Next, we let $u = (u_1, \dots, u_n)$ be a unit vector in $\mathbb{C}^n$. Then there exists an i ($1 \le i \le n$) such that $u_i \ge 1/\sqrt{n}$. For this fixed i, we see, by (3.15), that for $w \in \mathbb{C}$,

$$\begin{aligned} F_i(w) &:= f_i(\zeta_k + \sqrt{n}\,uw) \\ &= (\sqrt{n})^{m_k} c_k u_i^{m_k} w^{m_k} + \eta_k w^{s_k} + \sum_{j > s_k} b_j w^j, \end{aligned} \tag{3.17}$$

where $s_k \ge 3m_k/2$ is an integer, and η_k and b_j are complex numbers.

Let $G_i(w) = F_i(w)/w^{m_k}$. Then $G_i(0) = (\sqrt{n})^{m_k} c_k u_j^{m_k} \ge c_k$. By (3.14) and (3.11) we have $|G_i(w)| \le c(1+|\zeta_k|)e^{2Aq_1(\zeta_k)}$ for $|w| = 1$, which is also true in $|w| \le 1$ by the maximum modulus theorem. Also let $H_i(w) = G_i(w) - G_i(0)$. Then by (3.17), we see that $H_i(w)$ has a zero at $w = 0$ of order at least $m_k/2$. Note that $|H_i(w)| \le 2c(1+|\zeta_k|)e^{2Aq_1(\zeta_k)}$ on $|w| \le 1$. We have, by the Schwarz lemma, $|H_i(w)| \le 2c(1+|\zeta_k|)e^{2Aq_1(\zeta_k)}|w|^{m_k/2}$ on $|w| \le 1$. Thus, if $a \ne 0$ is a zero of $F_i(w)$ in $|w| \le 1$, then $G_i(a) = 0$ and thus

$$2c(1+|\zeta_k|)e^{2Aq_1(\zeta_k)}|a|^{m_k/2} \ge |H_i(a)| = |G_i(0)| = (\sqrt{n})^{m_k} c_k u_i^{m_k} \ge c_k,$$

or by (3.16),

$$|a|^{m_k/2} \ge (2c)^{-1}(1+|\zeta_k|)^{-K-2} e^{-2A(n+1)q_1(\zeta_k)} \ge \varepsilon(1+|\zeta_k|)^{-c} e^{-cq_1(\zeta_k)}$$

for some $\varepsilon, c > 0$. If we let $d_u = \min\{1, \operatorname{dist}(0, F_i^{-1}(0) \setminus \{0\})\}$, we have showed that

$$d_u^{m_k} \ge \varepsilon(1+|\zeta_k|)^{-c} e^{-cq_1(\zeta_k)} := (2\sqrt{n}\,\delta_k)^{m_k}. \tag{3.18}$$

Note that $G_i(w)$ has no zero in $|w| \le 2\delta_k \le d_u \le 1$ by the definition of d_u. Recall the following result from the Carathéodory theorem (see e.g. [10]): If h is holomorphic and has no zero in $|w| \le R$ with $h(0) = 1$, then

$$\log|h(w)| \ge -\frac{2r}{R-r}\log\max_{|w|=R}|h(w)| \quad \text{for } |w| \le r < R.$$

Applying it to $G_i(w)$ in $|w| \le 2\delta_k$ we deduce that for $|w| \le \delta_k$,

$$\log\left|\frac{G_i(w)}{G_i(0)}\right| \ge -2\log\left(\max_{|w|=2\delta_k}\left|\frac{G_i(w)}{G_i(0)}\right|\right),$$

which implies that

$$\log|G_i(w)| \ge -2\log(\max_{|w|=2\delta_k}|G_i(w)|) + 3\log|G_i(0)|$$

and so that

$$\begin{aligned}|G_i(w)| &\ge (\max_{|w|=2\delta_k}|G_i(w))^{-2}|G_i(0)|^3 \\ &\ge c_k^3 c^{-1}(1+|\zeta_k|)^{-2}e^{-4Aq_1(\zeta_k)} \ge \varepsilon(1+|\zeta_k|)^{-c}e^{-cq_1(\zeta_k)}\end{aligned}$$

for some $\varepsilon, c > 0$. By (3.18) we have, for $|w| = \delta_k$,

$$|F_i(w)| = |w^{m_k}G_i(w)| = \delta_k^{m_k}|G_i(w)| \ge (1/2\sqrt{n})^{m_k}\varepsilon(1+|\zeta_k|)^{-c}e^{-cq_1(\zeta_k)}.$$

On the other hand, by (3.15), (3.14), (3.6), and by the Cauchy estimates,

$$\begin{aligned}c_k &= \left|\frac{1}{(m_k)!}\,\frac{\partial^{m_k} f_i(\zeta_k)}{\partial z_i^{m_k}}\right| \\ &\le \frac{c}{(2\sqrt{n})^{m_k}}\max_{P_k}|f_i(z)| \le \frac{c}{(2\sqrt{n})^{m_k}}\,(1+|\zeta_k|)e^{2Aq_1(\zeta_k)},\end{aligned}$$

where $P_k = \{z \in \mathbb{C}^n : |z_j - \zeta_{k,j}| < 2\sqrt{n},\ 1 \le j \le n\}$. Hence, $(1/2\sqrt{n})^{m_k} \ge \varepsilon(1+|\zeta_k|)^{-c}e^{-cq_1(\zeta_k)}$ and we finally conclude that on $|w| = \delta_k$, $|f_i(\zeta_k + \sqrt{n}\,uw)| = |F_i(w)| \ge \varepsilon e^{-cq(\zeta_k)}$ for some $\varepsilon, c > 0$ independent of u and k, where $q(z) = \log(1+|z|) + q_1(z) = o\{p(z)\}$ by (2.1) and the fact that $q_1(z) = o\{p(z)\}$.

Since the above u is an arbitrary unit vector, we have thus showed that

$$|f(\zeta_k + z)| \ge |f_i(\zeta_k + z)| \ge \varepsilon e^{-cq(\zeta_k)}$$

for $|z| = \sqrt{n}\,\delta_k$. By virtue of (3.11) we have $|f(z)| \ge \varepsilon e^{-Cq(z)}$ for some $\varepsilon, C > 0$ on $|z - \zeta_k| = \sqrt{n}\,\delta_k$. Note that $\sqrt{n}\,\delta_k \le \frac{1}{2}d_u \le \frac{1}{2}$ in view of (3.18). We have thus showed that the connected component U_k of $S(f;\varepsilon,c) = \{z \in \mathbb{C}^n : |f(z)| < \varepsilon e^{-Cq(\zeta_k)}\}$ containing ζ_k must be completely contained in the ball $|z - \zeta_k| = \sqrt{n}\,\delta_k$, which has diameter at most 1 and does not contain any other points of V. This shows the necessity of the theorem when $N = n$. If $N > n$, we can easily add $N - n$ entire functions $f_{n+1}, \dots, f_N \in A_p^0(\mathbb{C}^n)$ satisfying $V \subset f_j^{-1}(0)$, $n+1 \le j \le N$. Let $F = (f_1, \dots, f_N)$. Then

$S_q(F;\varepsilon,C) \subseteq S_q(f;\varepsilon,C)$. Thus, the mapping F satisfies the conclusion of the theorem.

To prove the sufficiency, let $\{a_{k,I}\}_{k\in\mathbb{N},\,0\le|I|<m_k}$ be any given sequence in $A_p^0(V)$. For any integer m there exists a $c_m \ge 1$ such that $\sum_{|I|=0}^{m_k-1}|a_{k,I}| < c_m e^{p(\zeta_k)/m}$ for each $k\in\mathbb{N}$ since $\{a_{k,I}\}\in A_p^0(V)$ and, meanwhile, $|f(z)| < c_m e^{p(\zeta_k)/m}$ for each $z\in\mathbb{C}^n$, since $f\in A_p^0(\mathbb{C}^n)$. Thus,

$$\sum_{|I|=0}^{m_k-1}|a_{k,I}| < \exp(\inf_m\{\log c_m + p(\zeta_k)/m\}) \tag{3.19}$$

for each k and

$$|f(z)| < \exp(\inf_m\{\log c_m + p(z)/m\}) \tag{3.20}$$

for $z\in\mathbb{C}^n$. Define

$$g_a(z) = \inf_m\{\log c_m + p(z)/m\}, \qquad \alpha(z) = \max\{q(z), g_a(z)\}.$$

Then $\alpha(z) = o\{p(z)\}$.

We recall the following theorem [6, 1.7 and 1.8]: For any continuous and increasing function $\omega(r)$, if $\omega(r)$ satisfies (2.1) and (2.2), and $\omega(e^r)$ is convex, then for any function $h(r):[0,\infty)\to[0,\infty)$ satisfying $h(r)=o(\omega(r))$ there exists an increasing function $g(r)$ such that g satisfies (2.1) and (2.2), $g(e^r)$ is convex, and $h(r)=o\{g(r)\}$ and $g(r)=o\{\omega(r)\}$. Applying this result with $\omega=p$ and $h=\alpha$ we obtain an increasing function $q_a(r)$ such that (2.1) and (2.2) are satisfied, $q_a(e^r)$ is convex, and $\alpha(r)=o\{q_a(r)\}$ and $q_a(r)=o\{p(r)\}$. Then $q_a(|z|)=q_a(e^{\ln|z|})$ is plurisubharmonic and thus a weight. Also, there exists a $c>0$ such that $\alpha(z)\le q_a(|z|)+c$. We see, by (3.19) and (3.20), that $\sum_{|I|=0}^{m_k-1}|a_{k,I}| < e^{q_a(|\zeta_k|)+c}$ for each k and some $c>0$, and $|f(z)|<e^{q_a(|z|)+c}$ for each $z\in\mathbb{C}^n$, which implies that $f\in A_{q_a}(\mathbb{C}^n)\subset A_p^0(\mathbb{C}^n)$. Also, it is easy to see that $S_{q_a}(f;\varepsilon_0,C)\subseteq S_q(f;\varepsilon,C)$, where $\varepsilon_0=e^{-cC}\varepsilon$. Thus, by the hypotheses of the theorem, each connected component of $S_{q_a}(f;\varepsilon_0,C)$ contains at most one point in V and such a component has diameter at most 1. Then the proof of the sufficiency can be finished using the interpolation for the space $A_{q_a}(\mathbb{C}^n)$, or by the following argument.

Let U_k be the connected component of $S_{q_a}(f;\varepsilon_0,C)$ containing ζ_k. We define an analytic function $\lambda: S_{q_a}(f;\varepsilon_0,C)\to\mathbb{C}$ by

$$\lambda(z) = \begin{cases} \sum_{|I|=0}^{m_k-1} a_{k,I}(z-\zeta_k)^I & \text{if } z\in U_k,\\ 0 & \text{if } z\in S_{q_a}(f;\varepsilon_0,C)\setminus\bigcup_{k\in\mathbb{N}}U_k.\end{cases}$$

Then it is clear that

$$\frac{1}{I!}\frac{\partial^{|I|}\lambda(\zeta_k)}{\partial z^I} = a_{k,I}$$

for all $k \in \mathbb{N}$ and all $0 \leq |I| < m_k$. Moreover, on U_k we have $|z - \zeta_k| \leq 1$, since the diameter of U_k is at most 1, and thus

$$|\lambda(z)| \leq \sum_{|I|=0}^{m_k-1} |a_{k,I}| \leq e^{q_a(\zeta_k)} \leq e^{Aq_a(z)+B} \tag{3.21}$$

for some $A, B > 0$ by virtue of the property (2.2) of a weight, which implies that $q_a(w) \leq Aq_a(z) + B$ whenever $|w - z| \leq 1$. By the definition of λ, estimate (3.21) holds for all z in $S_{q_a}(f; \varepsilon_0, C)$. We then use the following result [5, Theorem 2.2]: If λ is analytic and satisfies $|\lambda(z)| \leq e^{Aq(z)+B}$ for some $A, B > 0$ on $S_q(f; \varepsilon, C)$, where q is a weight and $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ is an entire holomorphic mapping with $f_j \in A_q(\mathbb{C}^n)$, then there exists an entire function $F \in A_q(\mathbb{C}^n)$ such that $F(z) = \lambda(z)$ on the variety $f(z) = 0$. Applying this result to our function λ, we obtain a function $F \in A_{q_a}(\mathbb{C}^n) \subset A_p^0(\mathbb{C}^n)$ such that $F(z) = \lambda(z)$ on $f^{-1}(0) \supseteq V$. In particular,

$$\frac{1}{I!}\frac{\partial^{|I|}F(\zeta_k)}{\partial z^I} = \frac{1}{I!}\frac{\partial^{|I|}\lambda(\zeta_k)}{\partial z^I} = a_{k,I}$$

for all $k \in \mathbb{N}$ and all $0 \leq |I| < m_k$. This shows that V is an interpolating variety for $A_p^0(\mathbb{C}^n)$, and thus concludes the proof.

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