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On spectral continuity of positive elements

by

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Abstract. Let x be a positive element of an ordered Banach algebra. We prove a relationship between the spectra of x and of certain positive elements y for which either $xy \leq yx$ or $yx \leq xy$. Furthermore, we show that the spectral radius is continuous at x, considered as an element of the set of all positive elements $y \geq x$ such that either $xy \leq yx$ or $yx \leq xy$. We also show that the property $\varrho(x+y) \leq \varrho(x) + \varrho(y)$ of the spectral radius ϱ can be obtained for positive elements y which satisfy at least one of the above inequalities.

1. Introduction. The subject of spectral continuity has been studied for more than fifty years, and several authors have contributed; in particular, by providing different types of sufficient conditions for spectral continuity. In her survey paper [2] of 1994, L. Burlando gave an extensive account of these results, and supplied many useful references.

It is well known that if A is a noncommutative Banach algebra, then the spectrum and spectral radius functions are only upper semicontinuous on A, while if A is a commutative Banach algebra, then these functions are uniformly continuous on A. More generally, if $x \in A$, then $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x-y)$ for all $y \in \{x\}^{\mathrm{c}}$ (see [1, Theorem 3.4.1]), and hence $|\varrho(y) - \varrho(x)| \leq \varrho(x-y)$ for all $y \in \{x\}^{\mathrm{c}}$ (where Sp denotes the spectrum, ϱ the spectral radius and $\{x\}^{\mathrm{c}}$ the commutant $\{y \in A : yx = xy\}$ of x), so that the spectral radius is continuous at x, considered as an element of $\{x\}^{\mathrm{c}}$.

In this paper we investigate certain spectral continuity properties of positive elements. Some spectral theory of positive elements in ordered Banach algebras was developed in [8] and [7], and later in [4]–[6]. We recall some of this information in Section 3. In Section 4 we show that if x is a positive element of an ordered Banach algebra, then the results mentioned above can be obtained under the weaker condition that either $xy \leq yx$ or $yx \leq xy$, provided that $x \leq y$ and (for some results) one of a number of additional spectral properties is assumed. In Section 5 we give examples to show that these spectral properties are quite natural.

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Another well known spectral property is that if x and y commute, then $\varrho(x+y) \leq \varrho(x) + \varrho(y)$ and $\varrho(xy) \leq \varrho(x)\varrho(y)$. It is already known (see [8, Proposition 4.4]) that the latter inequality can still be obtained for positive elements x and y if, instead of commuting, they satisfy at least one of the inequalities $xy \leq yx$ and $yx \leq xy$. The problem of finding conditions under which $\varrho(x+y) \leq \varrho(x)+\varrho(y)$ will hold, for positive elements x and y satisfying at least one of the inequalities $xy \leq yx$ and $yx \leq xy$, was investigated in [3] and in [10], where x and y were bounded linear operators on a partially ordered Banach space and on a Banach lattice, respectively. Furthermore, in [11], the same problem was studied for the local spectral radius (instead of the spectral radius) of a bounded linear operator. We show (in Section 4) that if either $xy \leq yx$ or $yx \leq xy$ with x and y positive, then $\varrho(x+y) \leq \varrho(x) + \varrho(y)$ always holds, provided that the algebra cone is normal. The result is applicable, for instance, in the case of the bounded linear operators on the Banach lattice l^p (any p)—see Example 5.2.

2. Preliminaries. Throughout, A will be a complex Banach algebra with unit 1. The spectrum of an element x in A will be denoted by $\mathrm{Sp}(x)$, the spectral radius of x in A by $\varrho(x)$, and the distance $d(0, \mathrm{Sp}(x))$ from 0 to the spectrum of x by $\delta(x)$. We recall that if $\alpha \notin \mathrm{Sp}(x)$, then $d(\alpha, \mathrm{Sp}(x)) = 1/\varrho((\alpha 1 - x)^{-1})$ ([1, Theorem 3.3.5]). If K is a compact set in $\mathbb{C}$ and $r > 0$, then $K + r$ denotes the set $\{z \in \mathbb{C} : d(z, K) \leq r\}$, and $C(0, r)$ the circle in the complex plane with centre 0 and radius r. If $r = 0$, then $C(0, r)$ denotes the one-point set $\{0\}$. Finally, we need the following lemma:

LEMMA 2.1 ([1, proof of Corollary 3.2.10]). *Let a and b be elements of a Banach algebra A, and let $\lambda, \mu \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$. Then*:

(1) $$\Big\|\sum_{k=0}^{n} \binom{n}{k} (\lambda a)^{n-k} (\mu b)^k\Big\|^{1/n} \leq (\lambda + \mu)\Big(\max_{0 \leq k \leq n} \|a^{n-k}\|\,\|b^k\|\Big)^{1/n}.$$

(2) *If $\varrho(a) < 1$, $\varrho(b) < 1$ and $\gamma_n = \max\{\|a^{2^n - k}\|\,\|b^k\| : 0 \leq k \leq 2^n\}$, then there exists an $N \in \mathbb{N}$ such that (γ_n) is decreasing for $n \geq N$.*

3. Ordered Banach algebras. In [8, Section 3] we defined an algebra cone C of a Banach algebra A and showed that C induced on A an ordering which was compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra. We now recall those definitions and also the additional properties that C may have.

Let A be a complex Banach algebra with unit 1. Suppose that A contains a subset C with the following properties:

(1) $C + C \subseteq C$,
(2) $\lambda C \subseteq C$ for all $\lambda \geq 0$,

(3) $C \cdot C \subseteq C$,
(4) $1 \in C$.

Then C is called an *algebra cone* of A, and A, or more specifically (A, C), is called an *ordered Banach algebra* (OBA). We say that A is *ordered by* the algebra cone C. If, in addition, $C \cap -C = \{0\}$, then C is called *proper*.

An algebra cone C of A induces an *ordering* "$\leq$" on A in the following way:

$$x \leq y \quad \text{if and only if} \quad y - x \in C$$

$(x, y \in A)$. This ordering is reflexive and transitive. Furthermore, C is proper if and only if the ordering has the additional property of being antisymmetric. Considering the ordering that C induces we find that $C = \{x \in A : x \geq 0\}$ and therefore we call the elements of C *positive*.

An algebra cone C of A is called *closed* if it is a closed subset of A. Furthermore, C is said to be *normal* if there exists a constant $\alpha > 0$ such that it follows from $0 \leq x \leq y$ in A that $\|x\| \leq \alpha \|y\|$. It is well known that if C is normal, then C is proper. Moreover, C is said to be *inverse-closed* if it has the property that if $x \in C$ and x is invertible, then $x^{-1} \in C$.

The following lemma is immediate:

LEMMA 3.1. *Let (A, C) be an OBA, and let $x, y \in A$ be such that $xy \leq yx$.*

(1) *If x is invertible with $x^{-1} \in C$, then $yx^{-1} \leq x^{-1}y$.*
(2) *If y is invertible with $y^{-1} \in C$, then $y^{-1}x \leq xy^{-1}$.*

The next result follows by induction:

LEMMA 3.2. *Let (A, C) be an OBA, and let $x, y \in C$. If $yx \leq xy$, then*

$$(x + y)^n \leq \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

for every $n \in \mathbb{N} \cup \{0\}$.

Proof. Clearly the statement is true for $n = 0$ (and $n = 1$). So suppose that $(x + y)^m \leq \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k$, where $m \geq 1$. Then, since $yx \leq xy$ implies $yx^{m-k}y^k \leq x^{m-k}y^{k+1}$, it follows that

$$\begin{aligned} (x + y)^{m+1} &\leq (x + y) \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k \\ &\leq \sum_{k=0}^{m} \binom{m}{k} x^{m+1-k} y^k + \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^{k+1} \end{aligned}$$

$$= x^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right] x^{m+1-k} y^k + y^{m+1}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{m+1-k} y^k. \blacksquare$$

We will also need the following results:

THEOREM 3.3 ([8, Theorem 4.1(1)]). *Let (A, C) be an OBA with C normal. If $x, y \in A$ are such that $0 \le x \le y$, then $\varrho(x) \le \varrho(y)$.*

We refer to the above property by saying that the spectral radius is *monotone.*

THEOREM 3.4 ([8, Proposition 5.1]). *Let (A, C) be an OBA with C closed and normal. If $x \in C$, then $\varrho(x) \in \operatorname{Sp}(x)$.*

THEOREM 3.5 ([8, Proposition 4.4]). *Let (A, C) be an OBA with C normal. If $x, y \in C$ are such that $xy \le yx$, then $\varrho(xy) \le \varrho(x)\varrho(y)$ and $\varrho(yx) \le \varrho(x)\varrho(y)$.*

PROPOSITION 3.6 ([5, Proposition 4.6)]). *Let (A, C) be an OBA with C closed. If $x \in C$ and $\lambda > \varrho(x)$, then $(\lambda 1 - x)^{-1} \ge 0$.*

Finally, the following lemma follows from Theorem 3.4:

LEMMA 3.7. *Let (A, C) be an OBA with C closed and normal. If $x \in C$ and $\alpha \in \mathbb{R}^+$, then $\varrho(x + \alpha 1) = \varrho(x) + \alpha$.*

We conclude this section with an important example. Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X.

EXAMPLE 3.8. *Let E be a complex Banach lattice and let $C := \{x \in E : x = |x|\}$. If $K := \{T \in \mathcal{L}(E) : TC \subset C\}$, then K is a closed, normal algebra cone of $\mathcal{L}(E)$. Therefore $(\mathcal{L}(E), K)$ is an OBA.*

The nontrivial part of the above example follows from [9, Lemma 3].

4. Spectral continuity. Let (A, C) be an OBA. Define, for each $x \in C$,

$$A(x) = \{y \in A : x \le y,\ xy \le yx \text{ or } yx \le xy,$$
$$\text{and } d(\varrho(y), \operatorname{Sp}(x)) \ge d(\alpha, \operatorname{Sp}(x)) \text{ for all } \alpha \in \operatorname{Sp}(y)\}.$$

Then $x \in A(x)$, $A(x) \subset C$ and $A(0) = C$. In fact, it follows from Lemma 3.7 that if C is closed and normal, then $A(\alpha 1) = C + \alpha 1$ for all $\alpha \in \mathbb{R}^+$.

It is well known that if x is any element of a Banach algebra, then $\operatorname{Sp}(y) \subset \operatorname{Sp}(x) + \varrho(x - y)$ for all y in the commutant $\{x\}^{\mathrm{c}}$ of x ([1, Theorem 3.4.1]). Theorem 4.2 shows that this inclusion continues to hold for positive elements

x of an OBA, if y is an element of the set $A(x)$ rather than of $\{x\}^{\mathrm{c}}$. We need the following lemma:

LEMMA 4.1. *Let A be a Banach algebra, $x, y \in A$ and $\alpha \in \mathbb{C}$. If $\alpha 1 - x$ is invertible and $\varrho((\alpha 1 - x)^{-1}(x - y)) < 1$, then $\alpha 1 - y$ is invertible.*

Proof. If $\varrho((\alpha 1 - x)^{-1}(x - y)) < 1$, then $1 + (\alpha 1 - x)^{-1}(x - y)$ is invertible, and since $\alpha 1 - y = (\alpha 1 - x)[1 + (\alpha 1 - x)^{-1}(x - y)]$, the result follows. ∎

THEOREM 4.2. *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x - y)$ for all $y \in A(x)$.*

Proof. Let $y \in A(x)$. Then $0 \le x \le y$, so that $\varrho(x) \le \varrho(y)$, by Theorem 3.3. If $\varrho(x) = \varrho(y)$, then $d(\varrho(y), \mathrm{Sp}(x)) = 0$, by Theorem 3.4, so that, by the assumption, $d(\alpha, \mathrm{Sp}(x)) = 0$ for all $\alpha \in \mathrm{Sp}(y)$. This implies that $d(\alpha, \mathrm{Sp}(x)) \le \varrho(x - y)$ for all $\alpha \in \mathrm{Sp}(y)$, so that $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x - y)$.

So suppose that $\varrho(x) < \varrho(y)$, and suppose there exists an $\alpha \in \mathrm{Sp}(y)$ such that $d(\alpha, \mathrm{Sp}(x)) > \varrho(x - y)$. By Theorem 3.4, $\varrho(y) \in \mathrm{Sp}(y)$ and hence, by the assumption, we may take $\alpha \in \mathbb{R}^+$ with $\alpha > \varrho(x)$. Therefore

$$\varrho((\alpha 1 - x)^{-1})\varrho(x - y) < 1, \tag{4.3}$$

with $\alpha \in \mathbb{R}^+$ and $\alpha > \varrho(x)$. It follows from Proposition 3.6 that $(\alpha 1 - x)^{-1} \in C$.

If $xy \le yx$, then $(y - x)(\alpha 1 - x) \le (\alpha 1 - x)(y - x)$, so $(\alpha 1 - x)^{-1}(y - x) \le (y - x)(\alpha 1 - x)^{-1}$, by Lemma 3.1. It now follows from Theorem 3.5 that $\varrho((\alpha 1 - x)^{-1}(y - x)) \le \varrho((\alpha 1 - x)^{-1})\varrho(y - x)$. A similar argument yields the result in case $yx \le xy$.

This together with 4.3 implies $\varrho((\alpha 1 - x)^{-1}(y - x)) < 1$. It follows from Lemma 4.1 that $\alpha \notin \mathrm{Sp}(y)$, a contradiction. Therefore $d(\alpha, \mathrm{Sp}(x)) \le \varrho(x - y)$ for all $\alpha \in \mathrm{Sp}(y)$, so that $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x - y)$. ∎

If x and y are commuting elements of a Banach algebra, then $\varrho(x + y) \le \varrho(x) + \varrho(y)$. In an OBA we have the following result:

COROLLARY 4.4. *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then $\varrho(x + y) \le \varrho(x) + \varrho(y)$ for all $y \in A$ such that $x + y \in A(x)$.*

Proof. If $\lambda \in \mathrm{Sp}(x + y)$, then $d(\lambda, \mathrm{Sp}(x)) = |\lambda - \mu_\lambda|$ for some $\mu_\lambda \in \mathrm{Sp}(x)$. It follows from Theorem 4.2 that $|\lambda| \le |\lambda - \mu_\lambda| + |\mu_\lambda| \le \varrho(y) + \varrho(x)$ for all $\lambda \in \mathrm{Sp}(x + y)$, so that the result follows. ∎

Note that $x + y \in A(x)$ if and only if $y \in C$, $xy \le yx$ or $yx \le xy$ and $d(\varrho(x + y), \mathrm{Sp}(x)) \ge d(\alpha, \mathrm{Sp}(x))$ for all $\alpha \in \mathrm{Sp}(x + y)$.

COROLLARY 4.5. *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then $\varrho(y) \le \varrho(x) + \varrho(y - x)$ for all $y \in A(x)$.*

COROLLARY 4.6. *Let (A, C) be an OBA with C closed and normal, and let $x \in C$. Then the spectral radius is continuous at x, considered as an element of $A(x)$.*

Proof. If $y \in A(x)$, then $\varrho(x) \leq \varrho(y)$, so that it follows from Corollary 4.5 that $|\varrho(y) - \varrho(x)| \leq \varrho(y - x) \leq \|y - x\|$. ∎

The previous three corollaries can be strengthened. In fact, the following theorem illustrates that Corollary 4.4 continues to hold under omission of the spectral inequality in the definition of the set $A(x)$.

THEOREM 4.7. *Let (A, C) be an OBA with C normal, and let $x, y \in C$ be such that either $xy \leq yx$ or $yx \leq xy$. Then $\varrho(x + y) \leq \varrho(x) + \varrho(y)$.*

Proof. Let $a = \lambda^{-1}x$ and $b = \mu^{-1}y$, where $\lambda > \varrho(x)$ and $\mu > \varrho(y)$. Then $\varrho(a) < 1$ and $\varrho(b) < 1$. Hence, by Lemma 3.2,

$$0 \leq (x + y)^{2^n} \leq \sum_{k=0}^{2^n} \binom{2^n}{k} z^{2^n - k} w^k,$$

with $z = x$ and $w = y$ or vice versa. Since C is normal, there exists an $\alpha > 0$ such that

$$\|(x + y)^{2^n}\| \leq \alpha \left\| \sum_{k=0}^{2^n} \binom{2^n}{k} z^{2^n - k} w^k \right\|,$$

and hence

$$\|(x + y)^{2^n}\|^{1/2^n} \leq \alpha^{1/2^n} \left\| \sum_{k=0}^{2^n} \binom{2^n}{k} z^{2^n - k} w^k \right\|^{1/2^n}.$$

It follows from Lemma 2.1(1) that

$$\|(x + y)^{2^n}\|^{1/2^n} \leq \alpha^{1/2^n} (\lambda + \mu) \gamma_n^{1/2^n},$$

where $\gamma_n = \max\{\|a^{2^n - k}\| \, \|b^k\| : 0 \leq k \leq 2^n\}$. By Lemma 2.1(2) there exists an $N \in \mathbb{N}$ such that $\gamma_n^{1/2^n} \leq \gamma_N^{1/2^n}$ for all $n \geq N$. Therefore

$$\varrho(x + y) = \lim_{n \to \infty} \|(x + y)^{2^n}\|^{1/2^n} \leq \lim_{n \to \infty} (\alpha^{1/2^n} (\lambda + \mu) \gamma_N^{1/2^n}) = \lambda + \mu.$$

Since this holds for every $\lambda > \varrho(x)$ and $\mu > \varrho(y)$, the result follows. ∎

COROLLARY 4.8. *Let (A, C) be an OBA with C normal, and let $x, y \in C$ be such that $x \leq y$ and either $xy \leq yx$ or $yx \leq xy$. Then $\varrho(y) \leq \varrho(x) + \varrho(y - x)$.*

Proof. The condition $xy \leq yx$ or $yx \leq xy$ implies that $x(y-x) \leq (y-x)x$ or $(y - x)x \leq x(y - x)$. Hence $\varrho(y) = \varrho(x + (y - x)) \leq \varrho(x) + \varrho(y - x)$, since $y - x \in C$. ∎

Corollary 4.9. *Let (A, C) be an OBA with C normal, and let $x \in C$. Then the spectral radius is continuous at x, considered as an element of the set*

$$\{y \in A : x \leq y, \text{ and } xy \leq yx \text{ or } yx \leq xy\}.$$

Unlike for the spectral radius function, continuity of the spectrum function $\mathrm{Sp} : A \to K(\mathbb{C})$ (where $K(\mathbb{C})$ denotes the set of compact subsets of $\mathbb{C}$) does not follow from Theorem 4.2, since x and y cannot be interchanged in this theorem. In order to obtain continuity of the spectrum, further spectral conditions need to be imposed. This problem will not be investigated in the present note.

For each $x \in C$, consider the set

$$D(x) = \{y \in A : x \leq y,\ xy \leq yx \text{ or } yx \leq xy, \text{ and } \delta(y) \geq \varrho(x)\}.$$

Then $D(x) \subset C$, but $x \in D(x)$ if and only if $\mathrm{Sp}(x) \subset C(0, \varrho(x))$. Furthermore, $D(0) = C$, and if $x \in C$ is such that $C(0, \varrho(x)) \subset \mathrm{Sp}(x)$, then $D(x) \subset A(x)$. We thus have the following corollary:

Corollary 4.10. *Let (A, C) be an OBA with C closed and normal, and let $x \in C$ be such that $C(0, \varrho(x)) \subset \mathrm{Sp}(x)$. Then*:

(1) $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x - y)$ *for all* $y \in D(x)$.
(2) $\varrho(x + y) \leq \varrho(x) + \varrho(y)$ *for all* $y \in A$ *such that* $x + y \in D(x)$.
(3) $\varrho(y) \leq \varrho(x) + \varrho(y - x)$ *for all* $y \in D(x)$.
(4) *If* $C(0, \varrho(x)) = \mathrm{Sp}(x)$, *then the spectral radius is continuous at* x, *considered as an element of* $D(x)$.

However, besides being included in Corollary 4.9, Corollary 4.10(4) is also a consequence of the following property:

Proposition 4.11. *Let A be a Banach algebra and let $x \in A$ be such that $\mathrm{Sp}(x) \subset C(0, \varrho(x))$. Then the spectral radius is continuous at x.*

Proof. Let $\varepsilon > 0$, and $G_\varepsilon = \{\lambda \in \mathbb{C} : \varrho(x) - \varepsilon < |\lambda| < \varrho(x) + \varepsilon\}$. Then $\mathrm{Sp}(x) \subset G_\varepsilon$. If $x_n \to x$, then by the upper semicontinuity of the spectrum, $\mathrm{Sp}(x_n) \subset G_\varepsilon$ for all $n \geq N$, say. Since $\varrho(x_n) = |\lambda_n|$ for some $\lambda_n \in \mathrm{Sp}(x_n)$, it follows that $\varrho(x) - \varepsilon < \varrho(x_n) < \varrho(x) + \varepsilon$, i.e. $|\varrho(x) - \varrho(x_n)| < \varepsilon$ for all $n \geq N$. ∎

To continue the discussion, define, for each $x \in C$,

$$B(x) = \{y \in A : x \leq y,\ xy \leq yx \text{ or } yx \leq xy, \text{ and } (\alpha 1 - x)^{-1} \in C \text{ for all } \alpha \in \mathrm{Sp}(y) \setminus \mathrm{Sp}(x)\}.$$

Then $x \in B(x)$, $B(x) \subset C$ and $B(0) = \{y \in C : \alpha 1 \in C \text{ for all } \alpha \in \mathrm{Sp}(y)\}$ if C is inverse-closed. As in the case of $A(x)$ we have the following theorem:

Theorem 4.12. *Let (A, C) be an OBA with C normal, and let $x \in C$. Then $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x - y)$ for all $y \in B(x)$.*

Proof. Let $y \in B(x)$. Suppose there exists an $\alpha \in \mathrm{Sp}(y)$ such that $d(\alpha, \mathrm{Sp}(x)) > \varrho(x-y)$. Then

$$\varrho((\alpha 1 - x)^{-1})\varrho(x-y) < 1. \tag{4.13}$$

If $xy \leq yx$, then $(y-x)(\alpha 1-x) \leq (\alpha 1-x)(y-x)$, so that $(\alpha 1-x)^{-1}(y-x) \leq (y-x)(\alpha 1-x)^{-1}$, by Lemma 3.1. Since $y \in B(x)$, we have $y-x \in C$ and $(\alpha 1-x)^{-1} \in C$, so Theorem 3.5 shows that $\varrho((\alpha 1-x)^{-1}(y-x)) \leq \varrho((\alpha 1-x)^{-1})\varrho(y-x)$. A similar argument yields the result in case $yx \leq xy$. This together with (4.13) implies $\varrho((\alpha 1-x)^{-1}(y-x)) < 1$. By Lemma 4.1, $\alpha \notin \mathrm{Sp}(y)$, which is a contradiction, and hence the result follows. ∎

Finally, for each $x \in C$ set

$$E(x) = \{y \in A : x \leq y,\ xy \leq yx \text{ or } yx \leq xy,\ \text{and } x \leq \alpha 1 \text{ for all } \alpha \in \mathrm{Sp}(y) \setminus \mathrm{Sp}(x)\}.$$

Then $x \in E(x)$ and $E(x) \subset C$. Furthermore, $E(0) = \{y \in C : \alpha 1 \in C \text{ for all } \alpha \in \mathrm{Sp}(y)\}$, and if C is inverse-closed, then $E(x) = B(x)$.

In conclusion, we have the following corollary:

COROLLARY 4.14. *Let (A, C) be an OBA with C normal, and let $x \in C$. If either C is inverse-closed or $E(x) \subset B(x)$, then* $\mathrm{Sp}(y) \subset \mathrm{Sp}(x) + \varrho(x-y)$ *for all $y \in E(x)$.*

5. Examples. As mentioned before, the results in Section 4 are known to hold for elements commuting with x. Therefore, to show that these results are indeed applicable, we supply some examples showing that the sets defined in Section 4 contain elements which do not commute with x.

EXAMPLE 5.1. *Let A be the set of upper triangular 2×2 complex matrices, $l^\infty(A)$ the set*

$$\{x = (x_1, x_2, \ldots) : x_i \in A \text{ for all } i \in \mathbb{N} \text{ and } \|x_i\|_A \leq K_x \text{ for all } i \in \mathbb{N}\},$$

and C the set

$$\{(c_1, c_2, \ldots) \in l^\infty(A) : c_i \text{ has only nonnegative entries for all } i \in \mathbb{N}\}.$$

Then $(l^\infty(A), C)$ is an (infinite-dimensional) OBA, C is closed and normal, and for at least some $x \in C$ the sets $E(x)$, $B(x)$ and $A(x)$ contain elements which do not commute with x.

Proof. A proof of the fact that $(l^\infty(A), C)$ is an OBA with C normal was given in [4, Example 4.16]. Closedness of C follows easily from the definition of C. Let

$$x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right).$$

Then $x \in C$ and $\mathrm{Sp}(x) = \{0, 1\}$. Let

$$y = \left(\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \ldots \right).$$

Then $x \leq y$ and $\mathrm{Sp}(y) = \{1, 2\}$. Since

$$xy = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ldots \right), \quad yx = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right),$$

we have $yx \leq xy$. The only element of $\mathrm{Sp}(y) \setminus \mathrm{Sp}(x)$ is 2, and

$$(2 \cdot 1 - x)^{-1} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \ldots \right) \in C.$$

Therefore $y \in B(x)$. In fact, $x \leq 2 \cdot 1$, so that $y \in E(x)$. Moreover, although C is not inverse-closed, $E(x) = B(x)$.

Since $d(\varrho(y), \mathrm{Sp}(x)) = 1$ and $\{d(\alpha, \mathrm{Sp}(x)) : \alpha \in \mathrm{Sp}(y)\} = \{0, 1\}$, it follows that $y \in A(x)$ as well. Furthermore, it is easily checked that $x + y \in A(x) \cap E(x) \subset B(x)$. ∎

Consequently, the following results apply to $(l^\infty(A), C)$: Theorem 4.2, Corollaries 4.4, 4.5 and 4.6, Theorem 4.7, Corollaries 4.8 and 4.9, Theorem 4.12 and Corollary 4.14. (Alternatively, for $l^\infty(A)$, Corollaries 4.6 and 4.9 follow directly from the fact that the spectrum of every element of $l^\infty(A)$ is totally disconnected. This is Newburgh's Theorem [1, Corollary 3.4.5].)

EXAMPLE 5.2. *Consider, for any p with $1 \leq p \leq \infty$, the complex Banach lattice l^p, and let $A = \mathcal{L}(l^p)$, $C = \{x \in l^p : x = |x|\}$ and $K = \{T \in \mathcal{L}(l^p) : TC \subset C\}$. Then (A, K) is an OBA with K closed and normal, and for some $S \in K$ such that $\mathrm{Sp}(S) = C(0, \varrho(S))$, the sets $D(S)$ and $A(S)$ contain elements which do not commute with S.*

Proof. The first statement follows from Example 3.8. Let $S(\xi_1, \xi_2, \ldots) = (0, \xi_1/1, \xi_2/2, \ldots)$. Then $S \in K$ and $\mathrm{Sp}(S) = \{0\} = C(0, \varrho(S))$. Let $T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$. Then $S \leq T$ and $\mathrm{Sp}(T) = \overline{D}(0, 1)$, so that $\delta(T) = 0 = \varrho(S)$. Since we have $(ST)(\xi_1, \xi_2, \ldots) = (0, 0, \xi_1/2, \xi_2/3, \ldots)$ and $(TS)(\xi_1, \xi_2, \ldots) = (0, 0, \xi_1/1, \xi_2/2, \ldots)$, it follows that $ST \leq TS$, and therefore $T \in D(S)$. In addition, clearly $S + T \in D(S)$. (Hence, also $T \in A(S)$ and $S + T \in A(S)$.) ∎

In the case of (A, K) of the above example, the applicable results are: Theorem 4.2, Corollaries 4.4, 4.5 and 4.6, Theorem 4.7 and Corollaries 4.8, 4.9 and 4.10.

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