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An extremal problem in Banach algebras

by

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Abstract. We study asymptotics of a class of extremal problems $r_n(A,\varepsilon)$ related to norm controlled inversion in Banach algebras. In a general setting we prove estimates that can be considered as quantitative refinements of a theorem of Jan-Erik Björk [1]. In the last section we specialize further and consider a class of analytic Beurling algebras. In particular, a question raised by Jan-Erik Björk in [1] is answered in the negative.

1. Introduction. Let A be a (unitary) topological Banach algebra (see below) with norm $\|\cdot\|$ and denote by $r(f)$ the spectral radius of $f \in A$. In this note we study certain aspects of the extremal problem

$$r_n(A,\varepsilon) = \sup\{\|f^n\| : f \in A,\ \|f\| \le 1,\ r(f) \le \varepsilon\} \quad (n \ge 1,\ 0 < \varepsilon < 1) \tag{1}$$

In particular, we are interested in the limit behavior of (1) as $n \to \infty$. One motivation for the study of this extremal problem is its connection to norm controlled inversion in Banach algebras (see [1], [2], [3], [4] and [6]). (1) can also be found in [5].

In Section 2 we give asymptotic upper bounds for $r_n(A,\varepsilon)$ and the related quantity $r_n(A)$ (see Definition 1, Remark 1 and Proposition 1) introduced by Jan-Erik Björk in [1]. The main results in this section are Theorem 1, Corollary 1 and Theorem 2. In Theorem 1 we give a universal upper bound for the quantity $\lim_{n\to\infty} r_n(A,\varepsilon)^{1/n}$ (for the existence of the limit see Remark 1). In Corollary 1 a corresponding estimate for the quantities $r_n(A)$ is given. Our Theorem 1 and Corollary 1 are quantitative refinements of a theorem of J.-E. Björk in [1] (Theorem 3.1). (See also Remark 2.) In Theorem 2 we give an estimate of $\lim_{n\to\infty} r_n(A,\varepsilon)^{1/n}$ in terms of the quantity $\delta_1(A)$ (Definition 2), connected with a certain quantitative form of Wiener's lemma previously studied in [6], [3], [2] and [4]. The corresponding estimate for $\lim_{n\to\infty} r_n(A)$ is also given. Theorem 2 is an extension of Théorème 3.1 in [2] and has a similar proof. The proofs of these results are completely elementary.

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The material in Section 3 is inspired by [3] and [2]. We consider analytic Beurling algebras A^+_ω corresponding to Banach algebra weights ω on $\mathbb{N}$ (see below) such that $\omega(n) \to c \in [1,\infty)$ as $n \to \infty$. Of course, these algebras are just certain renormings of the well known Wiener algebra A^+ of absolutely convergent Taylor series in the unit disc $\mathbb{D}$. In Theorem 3 we compute the quantities $r_n(A^+_\omega)$ and $K_0(A^+_\omega)$ (see Corollary 1) for such an algebra A^+_ω. Theorem 3 exemplifies a sensitivity of the numbers $r_n(A)$ and the problem of norm controlled inversion for the particular choice of norm in A. In particular, Theorem 3 answers the following question raised in [1] (page 284, line 1): For a commutative semisimple Banach algebra A, does $r_n(A) < 1$ for some $n \geq 2$ imply $r_n(A) \to 0$? In fact, under these circumstances, the limit $\lim_{n\to\infty} r_n(A)$ exists and can be any number in the half-open interval $[0,1)$. Examples are provided by suitable algebras A^+_ω. (See also Remarks 5 and 6.) The proof of Theorem 3 uses ideas from a construction of Y. Katznelson presented in [7] combined with a recent lemma of O. El-Fallah (Lemma 2 in Section 3 below).

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Topological Banach algebras. By a *topological Banach algebra* we mean a Banach space A equipped with a multiplication, continuous as a map $A \times A \to A$, in such a way that A becomes a commutative complex algebra with unit. The unit element of A is denoted by e. We assume that the norm of A is normalized by $\|e\| = 1$. The continuity of the multiplication can equivalently be formulated by saying that the inequality

$$\|fg\| \leq C\|f\| \cdot \|g\|, \qquad f, g \in A, \tag{2}$$

holds for some constant $C \in [1,\infty)$. As is well known, every topological Banach algebra becomes a Banach algebra after a suitable renorming (passage to operator norm). In a topological Banach algebra A the spectral radius formula holds in the ordinary sense, i.e.,

$$r(f) = \|\widehat{f}\|_\infty = \lim_{n\to\infty} \|f^n\|^{1/n}, \qquad f \in A, \tag{3}$$

where $\widehat{f}$ denotes the Gelfand transform of f and $\|\cdot\|_\infty$ is the maximum norm on the maximal ideal space of A. The validity of (3) is clear since either side of (3) is unaffected by a renorming of A.

Beurling algebras. By a *Banach algebra weight* ω on $\mathbb{N} = \{0, 1, 2, \ldots\}$ we mean a positive weight function ω such that

$$\omega(0) = 1, \quad \omega(n+m) \le \omega(n)\omega(m), \quad n, m \ge 0.$$

The corresponding analytic Beurling algebra normed by

$$\|f\|_\omega = \sum_{k=0}^{\infty} |a_k|\omega(k), \quad f = \sum_{k=0}^{\infty} a_k z^k,$$

is denoted by A_ω^+. For $\omega \equiv 1$ we write $A^+ = A_\omega^+$ and $\|\cdot\| = \|\cdot\|_\omega$. Most weights in this note are such that $\omega(k)^{1/k} \to 1$. By this normalization the maximal ideal space of A_ω^+ is canonically identified with the closed unit disc $\overline{\mathbb{D}}$ and $r(f) = \|f\|_\infty$, where $\|\cdot\|_\infty$ denotes the maximum norm on $\overline{\mathbb{D}}$.

2. Topological Banach algebras. In the proof of Theorem 1 we use the following lemma:

LEMMA 1. *Let A be a topological Banach algebra. For $f \in A$ with $r(f) < 1$, the following identity holds*:

$$\binom{n+k}{k} f^n = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-in\theta}(e - e^{i\theta} f)^{-k-1}\, d\theta, \quad n \ge 1,\ k \ge 0.$$

Proof. We have the power series expansion

$$\frac{1}{(1-z)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} z^n.$$

Substituting $z = e^{i\theta} f$ and integrating, we obtain the lemma. ■

THEOREM 1. *Assume that the topological Banach algebra A satisfies a bounded inverse formula in the sense that there exist constants $\varepsilon \in (0,1)$ and $K = K(\varepsilon) \in [1, \infty)$ such that*

$$\|(e-f)^{-1}\| \le K \quad \text{if } \|f\| \le 1 \text{ and } r(f) \le \varepsilon. \tag{4}$$

Then

$$\lim_{n\to\infty} r_n(A, \varepsilon)^{1/n} \le 1 - \frac{1}{CK}, \tag{5}$$

where $C \ge 1$ is given by (2). (*For the existence of the limit see Remark* 1.)

Proof. By Lemma 1 and (4) we have

$$\binom{n+k}{n} r_n(A, \varepsilon) \le C^k K^{k+1}, \quad n \ge 1,\ k \ge 0. \tag{6}$$

Using Stirling's formula one verifies that, for $c \in (0, \infty)$,

$$\lim_{\substack{n\to\infty \\ |k/n - c| \le 1/n}} \binom{n+k}{n}^{1/n} = \frac{(1+c)^{1+c}}{c^c}.$$

In the limit as $n \to \infty$, $|k/n - c| \le 1/n$, we deduce from (6) that

$$\lim_{n\to\infty} r_n(A,\varepsilon)^{1/n} \le \frac{1}{1+c}\left(\frac{c}{1+c}\right)^c C^c K^c.$$

Choosing $c = 1/(CK-1)$ in this inequality yields (5). ∎

Definition 1 (J.-E. Björk [1]). Let A be a topological Banach algebra. A sequence $\{f_j\}$ in A is called a *spectral null sequence* if $\|f_j\| \le 1$ and $r(f_j) \to 0$. If $\{f_j\}$ is a spectral null sequence and $n \ge 1$, then $r_n(\{f_j\})$ is defined by $r_n(\{f_j\}) = \limsup_{j\to\infty} \|f_j^n\|^{1/n}$. The number $r_n(A)$ is defined by $r_n(A) = \sup r_n(\{f_j\})$, where the supremum is taken over all spectral null sequences $\{f_j\}$ in A.

Remark 1. It is immediate from the definition of the numbers $r_n(A)$ that

$$r_{n+m}(A)^{n+m} \le C r_n(A)^n r_m(A)^m, \quad n,m \ge 1,$$

where C is given by (2). By this submultiplicativity type inequality, $\lim_{n\to\infty} r_n(A)$ exists. The same argument establishes the existence of $\lim_{n\to\infty} r_n(A,\varepsilon)^{1/n}$.

The relation between $r_n(A)$ and the extremal problem (1) is given by the following proposition:

Proposition 1. *In a topological Banach algebra A the following holds*:

$$\lim_{\varepsilon\to 0} r_n(A,\varepsilon)^{1/n} = r_n(A).$$

Proof. Let $\{f_j\}$ be a spectral null sequence in A. For j large we have $\|f_j^n\| \le r_n(A,\varepsilon)$. Taking limits and suprema we get $r_n(A) \le \lim_{\varepsilon\to 0} r_n(A,\varepsilon)^{1/n}$.

Let $\delta > 0$. It is easily seen that there exists an $\varepsilon > 0$ such that $\|f^n\|^{1/n} \le r_n(A) + \delta$ if $\|f\| \le 1$ and $r(f) \le \varepsilon$. From this we have $\lim_{\varepsilon\to 0} r_n(A,\varepsilon)^{1/n} \le r_n(A)$. ∎

Corollary 1. *Let A be a topological Banach algebra. For $\varepsilon > 0$ write*

$$K(\varepsilon, A) = \sup\{\|(e-f)^{-1}\| : f \in A,\ \|f\| \le 1,\ r(f) \le \varepsilon\},$$
$$K_0 = K_0(A) = \lim_{\varepsilon\to 0} K(\varepsilon, A).$$

Then

$$\lim_{n\to\infty} r_n(A) \le 1 - \frac{1}{CK_0}, \tag{7}$$

where C is given by (2).

Proof. By Proposition 1, the corollary follows from Theorem 1 upon letting $\varepsilon \to 0$. ∎

REMARK 2. There is an obvious converse to Theorem 1 and Corollary 1. Assume (5) holds. For $f \in A$, $\|f\| \leq 1$, $r(f) \leq \varepsilon$, we have

$$\|(e-f)^{-1}\| = \Big\| \sum_{n=0}^{\infty} f^n \Big\| \leq \sum_{n=0}^{\infty} r_n(A,\varepsilon) < \infty.$$

Moreover, for any function $g = \sum a_n z^n$ analytic in an open set containing $(1-1/(CK))\overline{\mathbb{D}}$, we have

$$\|g(f)\| \leq \sum_{n=0}^{\infty} |a_n| r_n(A,\varepsilon) < \infty,$$

whenever $\|f\| \leq 1$ and $r(f) \leq \varepsilon$.

In [6], [7], [3], [2] and [4] a somewhat different quantitative Wiener lemma, than the possibility of (4) to hold, is studied. Indeed, given $\|f\| \leq 1$ and $|\widehat{f}| \geq \delta > 0$, one wants to estimate $\|f^{-1}\|$. ($\widehat{f}$ denotes the Gelfand transform of f.) This is formalized in the following definition.

DEFINITION (N. K. Nikolski [3, 4]). Let A be a topological Banach algebra. For $0 < \delta \leq 1$ we define

$$\begin{aligned} c_1(A,\delta) &= \sup\{\|1/f\| : f \in A,\ \|f\| \leq 1,\ |\widehat{f}| \geq \delta\}, \\ \delta_1(A) &= \inf\{\delta \in (0,1] : c_1(A,\delta) < \infty\}. \end{aligned}$$

(We use the convention that $\inf \emptyset = \infty$.)

The following theorem is an extension of Théorème 3.1 of [2].

THEOREM 2. *Let A be a topological Banach algebra and define*

$$r(A) = \lim_{n\to\infty} r_n(A).$$

Assume $\delta_1(A) < 1$. Then

$$\lim_{n\to\infty} r_n(A,\varepsilon)^{1/n} \leq \frac{\varepsilon + \delta_1(A)}{1-\delta_1(A)}, \tag{8}$$

$$r(A) \leq \frac{\delta_1(A)}{1-\delta_1(A)}, \qquad \frac{r(A)}{1+r(A)} \leq \delta_1(A). \tag{9}$$

Proof. It is straightforward to check that $r(A) \leq \delta_1(A)/(1-\delta_1(A))$ follows from (8) by letting $\varepsilon \to 0$ and that the two inequalities in (9) are equivalent. Hence, it suffices to prove (8).

Let $f \in A$, $\|f\| \leq 1$ and $r(f) \leq \varepsilon$. Let $z \in \mathbb{C}$. Since the element $(e-zf)/(1+|z|)$ is of norm ≤ 1 and has Gelfand transform of minimal modulus $\geq (1-|z|\varepsilon)/(1+|z|)$, we have

$$(1+|z|)\|(e-zf)^{-1}\| \leq c_1\Big(A, \frac{1-|z|\varepsilon}{1+|z|}\Big) < \infty \quad \text{provided} \quad \frac{1-|z|\varepsilon}{1+|z|} > \delta_1(A).$$

Write $z = re^{i\theta}$ and assume $r > 0$ is such that $(1-r\varepsilon)/(1+r) > \delta_1(A)$. Since $r\varepsilon < 1$, we have $(e - re^{i\theta} f)^{-1} = \sum_{n=0}^{\infty} e^{in\theta} r^n f^n$ in A, and

$$r^n f^n = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-in\theta} (e - re^{i\theta} f)^{-1} \, d\theta.$$

By an obvious estimate we have

$$r^n \|f^n\| \le \frac{1}{1+r} c_1 \left(A, \frac{1-r\varepsilon}{1+r} \right)$$

so that

$$r^n r_n(A, \varepsilon) \le \frac{1}{1+r} c_1 \left(A, \frac{1-r\varepsilon}{1+r} \right).$$

Taking the nth roots and passing to the limit we obtain

$$\lim_{n\to\infty} r_n(A,\varepsilon)^{1/n} \le 1/r \quad \text{ if } \frac{1-r\varepsilon}{1+r} > \delta_1(A).$$

Letting $(1 - r\varepsilon)/(1 + r) \to \delta_1(A)$ yields (8). ∎

REMARK 3. In all cases known to the author, equality holds in (9).

3. Analytic Beurling algebras. In the present section $\|\cdot\|$ always denotes the norm of absolutely convergent Taylor series on $\mathbb{D}$ (see Section 1). We begin with some preliminary lemmas needed in the proof of Theorem 3.

DEFINITION 3 (O. El-Fallah [2]). Let ω be a Banach algebra weight on $\mathbb{N}$. For positive integers n, k the following quantities are considered:

$$\begin{aligned} a(k,n,\omega) &= \sup\left\{ \left(\frac{\omega(m_1 + \ldots + m_n)}{\omega(m_1)\ldots\omega(m_n)} \right)^{1/n} : m_j \ge k,\ j = 1,\ldots,n \right\}, \\ a(n,\omega) &= \lim_{k\to\infty} a(k,n,\omega). \end{aligned}$$

LEMMA 2 (O. El-Fallah [2], Lemme 5.3). *Let ω and a be as in Definition 3. Assume $\omega(k)^{1/k} \to 1$ as $k \to \infty$. Then, for $f \in A_\omega^+$ with $\|f\|_\omega \le 1$, the following inequality holds*:

$$\|f^n\|_\omega \le r(f) n \sum_{m=0}^{k-1} \omega(m) + a(k,n,\omega)^n.$$

In particular, $r_n(A_\omega^+) \le a(n,\omega)$.

Proof. Since

$$\begin{aligned} f^n = \Big(\sum_{m=0}^{k-1} a_m z^m \Big) \Big[f^{n-1} + f^{n-2} \Big(\sum_{m=k}^{\infty} a_m z^m \Big) + \ldots \\ + f \Big(\sum_{m=k}^{\infty} a_m z^m \Big)^{n-2} + \Big(\sum_{m=k}^{\infty} a_m z^m \Big)^{n-1} \Big] + \Big(\sum_{m=k}^{\infty} a_m z^m \Big)^n, \end{aligned}$$

we have

$$\|f^n\|_\omega \le n\Big\|\sum_{m=0}^{k-1} a_m z^m\Big\|_\omega + \Big\|\Big(\sum_{m=k}^{\infty} a_m z^m\Big)^n\Big\|_\omega.$$

The first term is estimated by

$$\Big\|\sum_{m=0}^{k-1} a_m z^m\Big\|_\omega \le r(f)\sum_{m=0}^{k-1}\omega(m)$$

and the last term is estimated by

$$\Big\|\Big(\sum_{m=k}^{\infty} a_m z^m\Big)^n\Big\|_\omega \le \sum_{m_j\ge k} |a_{m_1}|\ldots|a_{m_n}|\omega(m_1+\ldots+m_n) \le a(k,n,\omega)^n. \ \blacksquare$$

LEMMA 3. *Let n be a positive integer and $\varepsilon > 0$. Then there exists $f \in A^+$ with $\|f\| = \|f^2\| = \ldots = \|f^n\| = 1$ and $\|f\|_\infty < \varepsilon$. (In fact, f can be chosen to be a polynomial.) In particular, $r_n(A^+,\varepsilon) = 1$ for all $n \ge 1$ and $\varepsilon \in (0,1)$.*

Proof. Let $g(z) = (1+z)(1-z^{n+1})/4$. Now, g is a polynomial with $\|g\| = \|g^2\| = \ldots = \|g^n\| = 1$ and $\|g\|_\infty < 1$. Setting $f(z) = g(z)g(z^{n_1})\ldots g(z^{n_r})$ for some $1 \ll n_1 \ll \ldots \ll n_r$ we achieve $\|f\| = \|f^2\| = \ldots = \|f^n\| = 1$ and $\|f\|_\infty < \varepsilon$. $\blacksquare$

REMARK 4. In the above form, Lemma 3 is due to H. S. Shapiro and G. Ryd, and has been communicated to the author by A. Dahlner. In [7] a somewhat weaker version of Lemma 3 was used.

The following lemma is well known.

LEMMA 4. *Let $f_k \in A^+$, $k = 0,1,\ldots,$ be such that $\sum \|f_k\| < \infty$. Then*

$$\lim_{N\to\infty}\Big\|\sum_{k=0}^{\infty} z^{kN} f_k\Big\| = \sum_{k=0}^{\infty}\|f_k\|.$$

THEOREM 3. *Let ω be a Banach algebra weight on $\mathbb{N}$ such that $\omega(k) \to c \in [1,\infty)$ as $k \to \infty$. Then, for the corresponding analytic Beurling algebra A_ω^+, the following holds:*

$$r_n(A_\omega^+) = a(n,\omega) = c^{1/n-1} \quad \text{for } n \ge 1, \tag{10}$$

$$K_0(A_\omega^+) := \lim_{\varepsilon\to 0} K(\varepsilon, A_\omega^+) = 1 + c/(c-1) = (2c-1)/(c-1). \tag{11}$$

For $c = 1$ the right hand side of (11) *is to be interpreted as $+\infty$.*

In (11) we have written

$$K(\varepsilon, A_\omega^+) = \sup\Big\{\Big\|\frac{1}{1-f}\Big\|_\omega : \|f\|_\omega \le 1,\ r(f) \le \varepsilon\Big\}.$$

In the proof below this quantity is denoted by $K(\varepsilon)$.

Proof. We first prove (10). Since

$$\frac{\omega(m_1+\ldots+m_n)}{\omega(m_1)\ldots\omega(m_n)} \to \frac{c}{c^n}$$

as $m_j \to \infty$, by Lemma 2 we have $r_n(A^+_\omega)^n \le a(n,\omega)^n = c/c^n$. Next we prove $r_n(A^+_\omega)^n \ge c/c^n$. By Lemma 3 we can choose a sequence $\{f_j\} \subset A^+$ such that

$$\|f_j\| = \ldots = \|f_j^j\| = 1 \quad \text{and} \quad r(f_j) \to 0. \tag{12}$$

Now $\|(f_j/\|f_j\|_\omega)\|_\omega = 1$ and $r(f_j/\|f_j\|_\omega) \to 0$, whence

$$\limsup_{j\to\infty} \|(f_j/\|f_j\|_\omega)^n\|_\omega \le r_n(A^+_\omega)^n.$$

Observe that $\|f_j^n\|_\omega \to c$ as $j \to \infty$. Since

$$\|(f_j/\|f_j\|_\omega)^n\|_\omega = \frac{1}{\|f_j\|_\omega^n}\|f_j^n\|_\omega \to \frac{c}{c^n} \quad \text{as } j \to \infty,$$

we have $r_n(A^+_\omega)^n \ge c/c^n$.

Next we prove (11). Let $\|f\|_\omega \le 1$, $r(f) \le \varepsilon$. Since

$$\frac{1}{1-f} = \sum_{k=0}^\infty f^k,$$

we have

$$\left\|\frac{1}{1-f}\right\|_\omega \le \sum_{k=0}^\infty \|f^k\|_\omega \le 1 + \sum_{k=1}^\infty r_k(A^+_\omega, \varepsilon).$$

Hence

$$K(\varepsilon) \le 1 + \sum_{k=1}^\infty r_k(A^+_\omega, \varepsilon).$$

Passing to the limit as $\varepsilon \to 0$, using (10) and Proposition 1, we get $K_0 \le 1 + \sum_{k=1}^\infty r_k(A^+_\omega)^k = (2c-1)/(c-1)$.

Now we prove $K_0 \ge (2c-1)/(c-1)$. Let $\{f_j\} \subset A^+$ be a sequence satisfying (12). For j large we have

$$K(\varepsilon) \ge \left\|\left(1 - \frac{z^N f_j}{\|z^N f_j\|_\omega}\right)^{-1}\right\|_\omega = \left\|\sum_{k=0}^\infty \frac{1}{\|z^N f_j\|_\omega^k} z^{kN} f_j^k\right\|_\omega.$$

Next we compute the limit as $N \to \infty$ of the right hand side in this inequality. Since

$$\sum_{k=0}^\infty \left(\frac{1}{\|z^N f_j\|_\omega^k} - \frac{1}{c^k}\right) z^{kN} f_j^k \to 0 \quad \text{in } A^+_\omega, \qquad N \to \infty,$$

we have

$$\Big\| \sum_{k=0}^{\infty} \frac{1}{\|z^N f_j^k\|_\omega^k} z^{kN} f_j^k \Big\|_\omega = \Big\| \sum_{k=0}^{\infty} z^{kN} (f_j/c)^k \Big\|_\omega + o(1).$$

Now

$$\Big\| \sum_{k=0}^{\infty} z^{kN} (f_j/c)^k \Big\|_\omega = 1 + c \Big\| \sum_{k=1}^{\infty} z^{kN} (f_j/c)^k \Big\| + o(1) = 1 + c \sum_{k=1}^{\infty} \frac{1}{c^k} \|f_j^k\| + o(1),$$

where in the last equality we have used Lemma 4. Summing up, we have shown

$$K(\varepsilon) \geq 1 + c \sum_{k=1}^{\infty} \frac{1}{c^k} \|f_j^k\|.$$

Letting $j \to \infty$ we get $K(\varepsilon) \geq 1 + c \sum_{k=1}^{\infty} 1/c^k = (2c-1)/(c-1)$. From this (11) follows. ■

Remark 5. In [1] (page 283, last paragraph), one more question besides the one alluded to in the introduction is asked. Namely, for a unitary commutative semi-simple Banach algebra A, does $r_n(A) < 1$ for some $n > 2$ imply $r_2(A) < 1$? Recently, in [2] (Remarque 5.7), O. El-Fallah has, for given $m \geq 2$, constructed a weighted analytic Beurling algebra A_ω^+ with $r_1(A_\omega^+) = r_2(A_\omega^+) = \ldots = r_m(A_\omega^+) = 1$ and $r_n(A_\omega^+) = 0$ for $n > m$.

Remark 6. Let A be a commutative semisimple Banach algebra with unit element. Regarding the quantity $\lim_{n\to\infty} r_n(A)$ there is an amount of slack between the upper bound in Corollary 1 and the examples in Theorem 3. The right upper bound for the quantity $\lim_{n\to\infty} r_n(A)$ remains to be found.

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