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# Invertible harmonic mappings beyond the Kneser theorem and quasiconformal harmonic mappings

by

David Kalaj (Podgorica)

**Abstract.** We extend the Rado–Choquet–Kneser theorem to mappings with Lipschitz boundary data and essentially positive Jacobian at the boundary without restriction on the convexity of image domain. The proof is based on a recent extension of the Rado–Choquet–Kneser theorem by Alessandrini and Nesi and it uses an approximation scheme. Some applications to families of quasiconformal harmonic mappings between Jordan domains are given.

**1. Introduction and statement of the main result.** Harmonic mappings in the plane are univalent complex-valued harmonic functions of a complex variable. Conformal mappings are a special case where the real and imaginary parts are conjugate harmonic functions, satisfying the Cauchy–Riemann equations. Harmonic mappings were studied classically by differential geometers because they provide isothermal (or conformal) coordinates for minimal surfaces. More recently they have been actively investigated by complex analysts as generalizations of univalent analytic functions, or conformal mappings. For the background to this theory we refer to the book of Duren [6]. If $w$ is a univalent complex-valued harmonic function, then by Lewy's theorem (see [24]), $w$ has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, $w$ is a diffeomorphism. Moreover, if $w$ is a harmonic mapping of the unit disk $\mathbb{U}$ onto a convex Jordan domain $\Omega$, mapping the boundary $\mathbb{T} = \partial\mathbb{U}$ onto $\partial\Omega$ homeomorphically, then $w$ is a diffeomorphism. This is a celebrated theorem of Rado, Kneser and Choquet ([20]). This theorem has been extended in various directions (see for example [11], [3], [31] and [32]). One of the recent extensions is the following proposition, due to Nesi and Alessandrini, which is one of the main tools in proving our main result.

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Proposition 1.1 ([2]). *Let $F : \mathbb{T} \to \gamma \subset \mathbb{C}$ be an orientation preserving diffeomorphism of class $C^1$ onto a simple closed curve $\gamma$ of the complex plane $\mathbb{C}$. Let $D$ be a bounded domain such that $\partial D = \gamma$. Let $w = P[F] \in C^1(\overline{\mathbb{U}}; \mathbb{C})$, where $P[f]$ is the Poisson extension of $F$. The mapping $w$ is a diffeomorphism of $\overline{\mathbb{U}}$ onto $\overline{D}$ if and only if*

$$J_w(e^{it}) > 0 \quad \textit{everywhere on } \mathbb{T}, \tag{1.1}$$

*where $J_w(e^{it}) := \lim_{r\to 1^-} J_w(re^{it})$, and $J_w(re^{it})$ is the Jacobian of $w$ at $re^{it}$.*

In this paper we generalize the Rado–Kneser–Choquet theorem as follows.

Theorem 1.2 (The main result). *Let $F : \mathbb{T} \to \gamma \subset \mathbb{C}$ be an orientation preserving Lipschitz weak homeomorphism of the unit circle $\mathbb{T}$ onto a $C^{1,\alpha}$ smooth Jordan curve $\gamma$. Let $D$ be a bounded domain such that $\partial D = \gamma$. Then $J_w(e^{it})/|F'(t)|$ exists a.e. in $\mathbb{T}$ and has a continuous extension $T_w(e^{it})$ to $\mathbb{T}$. If*

$$T_w(e^{it}) > 0 \quad \textit{everywhere on } \mathbb{T}, \tag{1.2}$$

*then the mapping $w = P[F]$ is a diffeomorphism of $\mathbb{U}$ onto $D$.*

In order to compare this statement with Kneser's Theorem, it is worth noticing that when $D$ is convex, then by Remark 3.2 the condition (1.2) is automatically satisfied.

It follows from Theorem 1.2 that under its conditions, the Jacobian $J_w$ of $w$ has a continuous extension to the boundary provided that $F \in C^1(\mathbb{T})$ and it should be noticed that this *does not* mean that the partial derivatives of $w$ necessarily have a continuous extension to the boundary (see e.g. [26] for a counterexample).

Note that we do not have any restriction on convexity of the image domain in Theorem 1.2, which is proved in Section 3.

Using this theorem, in Section 4 we characterize all quasiconformal harmonic mappings between the unit disk $\mathbb{U}$ and a smooth Jordan domain $D$ in terms of boundary data (see Theorem 4.1), which could be considered as a variation of Proposition 1.1.

## 2. Preliminaries

**2.1. Arc length parameterization of a Jordan curve.** Suppose that $\gamma$ is a rectifiable Jordan curve in the complex plane $\mathbb{C}$. Denote by $l$ the length of $\gamma$ and let $g : [0, l] \to \gamma$ be the arc length parameterization of $\gamma$, i.e. a parameterization satisfying the condition

$$|g'(s)| = 1 \quad \text{for all } s \in [0, l].$$

We will say that $\gamma$ is of class $C^{1,\alpha}$, $0<\alpha\le 1$, if $g$ is of class $C^1$ and

$$\sup_{t,s}\frac{|g'(t)-g'(s)|}{|t-s|^\alpha}<\infty.$$

Definition 2.1. Let $l=|\gamma|$. We will say that a surjective function $F=g\circ f:\mathbb{T}\to\gamma$ is a *weak homeomorphism* if $f:[0,2\pi]\to[0,l]$ is a nondecreasing surjective function.

Definition 2.2. Let $f:[a,b]\to\mathbb{C}$ be a continuous function. The *modulus of continuity* of $f$ is

$$\omega(t)=\omega_f(t)=\sup_{|x-y|\le t}|f(x)-f(y)|.$$

The function $f$ is called *Dini continuous* if

$$\int_{0+}\frac{\omega_f(t)}{t}\,dt<\infty. \tag{2.1}$$

Here $\int_{0+}:=\int_0^k$ for some positive constant $k$. A smooth Jordan curve $\gamma$ is said to be *Dini smooth* if $g'$ is Dini continuous. Observe that every smooth $C^{1,\alpha}$ Jordan curve is Dini smooth.

Let

$$K(s,t)=\mathrm{Re}[\overline{(g(t)-g(s))}\cdot ig'(s)] \tag{2.2}$$

for $(s,t)\in[0,l]\times[0,l]$. We extend it on $\mathbb{R}\times\mathbb{R}$ by $K(s\pm l,t\pm l)=K(s,t)$. Note that $ig'(s)$ is the inner unit normal vector of $\gamma$ at $g(s)$, and therefore if $\gamma$ is convex then

$$K(s,t)\ge 0\quad\text{for every } s \text{ and } t. \tag{2.3}$$

Suppose now that $F:\mathbb{R}\to\gamma$ is an arbitrary $2\pi$-periodic Lipschitz function such that $F|_{[0,2\pi)}:[0,2\pi)\to\gamma$ is an orientation preserving bijective function. Then there exists an increasing continuous function $f:[0,2\pi]\to[0,l]$ such that

$$F(\tau)=g(f(\tau)). \tag{2.4}$$

In the remainder of this paper we will identify $[0,2\pi)$ with the unit circle $\mathbb{T}$, and $F(s)$ with $F(e^{is})$. In view of the previous convention we have, for a.e. $e^{i\tau}\in\mathbb{T}$,

$$F'(\tau)=g'(f(\tau))\cdot f'(\tau),$$

and therefore

$$|F'(\tau)|=|g'(f(\tau))|\cdot|f'(\tau)|=f'(\tau).$$

Along with the function $K$ we will also consider the function $K_F$ defined by

$$K_F(t,\tau)=\mathrm{Re}[\overline{(F(t)-F(\tau))}\cdot iF'(\tau)].$$

It is easy to see that

$$K_F(t,\tau) = f'(\tau)K(f(t), f(\tau)). \tag{2.5}$$

LEMMA 2.3. *If $\gamma$ is Dini smooth, and $\omega$ is the modulus of continuity of $g'$, where $g$ denotes the arc-length parameterization of $\gamma$, then*

$$|K(s,t)| \le \int_0^{\min\{|s-t|,l-|s-t|\}} \omega(\tau)\,d\tau. \tag{2.6}$$

*Proof.* Note that

$$\begin{aligned} K(s,t) &= \operatorname{Re}[\overline{(g(t)-g(s))}\cdot ig'(s)] \\ &= \operatorname{Re}\Big[\overline{(g(t)-g(s))}\cdot i\Big(g'(s) - \frac{g(t)-g(s)}{t-s}\Big)\Big], \end{aligned}$$

and

$$g'(s) - \frac{g(t)-g(s)}{t-s} = \int_s^t \frac{g'(s)-g'(\tau)}{t-s}\,d\tau.$$

Therefore

$$\begin{aligned} \left|g'(s) - \frac{g(t)-g(s)}{t-s}\right| &\le \int_s^t \frac{|g'(s)-g'(\tau)|}{t-s}\,d\tau \le \int_s^t \frac{\omega(\tau-s)}{t-s}\,d\tau \\ &= \frac{1}{t-s}\int_0^{t-s} \omega(\tau)\,d\tau. \end{aligned}$$

On the other hand

$$|\overline{g(t)-g(s)}| \le \sup_{s\le x\le t} |g'(x)|(t-s) = t-s, \qquad s \le t.$$

It follows that

$$|K(s,t)| \le \int_0^{|s-t|} \omega(\tau)\,d\tau. \tag{2.7}$$

Since $K(s\pm l, t\pm l) = K(s,t)$, from (2.7) we obtain (2.6). ∎

LEMMA 2.4. *If $\omega : [0,l] \to [0,\infty)$, $\omega(0) = 0$, is a bounded function satisfying $\int_{0^+} \omega(x)\,dx/x < \infty$, then $\int_{0^+} \omega(ax)\,dx/x < \infty$ for every constant $a$. Moreover, for every $0 < y \le l$,*

$$\int_{0^+}^y \frac{1}{x^2}\int_0^x \omega(at)\,dt\,dx = \int_{0^+}^y \Big(\frac{\omega(ax)}{x} - \frac{\omega(ax)}{y}\Big)\,dx. \tag{2.8}$$

*Proof.* The first statement is immediate. Making the substitutions $u = \int_0^x \omega(at)\,dt$ and $dv = x^{-2}dx$, and using the fact that

$$\lim_{\alpha\to 0^+} \frac{\int_0^\alpha \omega(at)\,dt}{\alpha} = \lim_{\alpha\to 0^+} \omega(\alpha a) = \omega(0) = 0,$$

which follows from l'Hôpital's rule, we obtain

$$\begin{aligned}\int_{0^+}^{y} \frac{1}{x^2}\int_0^x \omega(at)\,dt\,dx &= \lim_{\alpha\to 0^+}\int_\alpha^y \frac{1}{x^2}\int_0^x \omega(at)\,dt\,dx\\ &= -\lim_{\alpha\to 0^+}\frac{\int_0^x \omega(at)\,dt}{x}\bigg|_\alpha^y + \lim_{\alpha\to 0^+}\int_\alpha^y \frac{\omega(ax)}{x}\,dx\\ &= \int_{0^+}^{y}\bigg(\frac{\omega(ax)}{x}-\frac{\omega(ax)}{y}\bigg)\,dx. \ \blacksquare\end{aligned}$$

A function $\varphi : A \to B$ is called *$\mathcal{L}$-bi-Lipschitz*, where $0 < \mathcal{L} < \infty$, if $\mathcal{L}^{-1}|x-y| \le |\varphi(x)-\varphi(y)| \le \mathcal{L}|x-y|$ for $x, y \in A$.

Lemma 2.5. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is an $\mathcal{L}$-bi-Lipschitz mapping (or an $\mathcal{L}$-Lipschitz weak homeomorphism) such that $\varphi(x+a) = \varphi(x)+b$ for some $a$ and $b$ and every $x$, then there exists a sequence of $\mathcal{L}$-bi-Lipschitz diffeomorphisms (respectively of diffeomorphisms) $\varphi_n : \mathbb{R}\to\mathbb{R}$ such that $\varphi_n$ converges uniformly to $\varphi$, and $\varphi_n(x+a) = \varphi_n(x)+b$.*

*Proof.* We introduce appropriate mollifiers: fix a smooth function $\rho : \mathbb{R} \to [0,1]$ which is compactly supported in $(-1,1)$ and satisfies $\int_{\mathbb{R}}\rho = 1$. For $\varepsilon = 1/n$ consider the mollifier

$$\rho_\varepsilon(t) := \frac{1}{\varepsilon}\rho\bigg(\frac{t}{\varepsilon}\bigg). \tag{2.9}$$

It is compactly supported in $(-\varepsilon,\varepsilon)$ and $\int_{\mathbb{R}}\rho_\varepsilon = 1$. Define

$$\varphi_\varepsilon(x) = \varphi * \rho_\varepsilon = \int_{\mathbb{R}} \varphi(y)\frac{1}{\varepsilon}\rho\bigg(\frac{x-y}{\varepsilon}\bigg)\,dy = \int_{\mathbb{R}}\varphi(x-\varepsilon z)\rho(z)\,dz.$$

Then

$$\varphi_\varepsilon'(x) = \int_{\mathbb{R}}\varphi'(x-\varepsilon z)\rho(z)\,dz.$$

It follows that

$$\mathcal{L}^{-1}\int_{\mathbb{R}}\rho(z)\,dz = \mathcal{L}^{-1} \le |\varphi_\varepsilon'(x)| \le \mathcal{L}\int_{\mathbb{R}}\rho(z)\,dz = \mathcal{L}.$$

The fact that $\varphi_\varepsilon(x)$ converges uniformly to $\varphi$ follows from the Arzelà–Ascoli theorem.

In the case when $\varphi$ is an $\mathcal{L}$-Lipschitz weak homeomorphism, we make use of the following simple fact. Since $\varphi$ is $\mathcal{L}$-Lipschitz, the function

$$\varphi_m(x) = \frac{mb}{mb+a}(\varphi(x)+x/m)$$

is $\mathcal{L}_m$-bi-Lipschitz for some $\mathcal{L}_m > 0$, with $\varphi_m(x+a) = \varphi_m(x)+b$, and $\varphi_m$ converges uniformly to $\varphi$. By the previous case, we can choose a diffeomorphism

$$\psi_m = \varphi_m * \rho_{\varepsilon_m} = \frac{mb}{mb+a}\left(\varphi * \rho_{\varepsilon_m} + \frac{x}{m}\right) \tag{2.10}$$

such that $\|\psi_m - \varphi_m\|_\infty \le 1/m$. Thus

$$\lim_{n\to\infty} \|\psi_n - \varphi\|_\infty = 0. \ \blacksquare$$

**2.2. Harmonic functions and Poisson integral.** The function

$$P(r,t) = \frac{1-r^2}{2\pi(1-2r\cos t + r^2)}, \qquad 0 \le r < 1,\ t \in [0,2\pi],$$

is called the *Poisson kernel*. The *Poisson integral* of a complex function $F \in L^1(\mathbb{T})$ is the complex harmonic function given by

$$w(z) = u(z) + iv(z) = P[F](z) = \int_0^{2\pi} P(r, t-\tau) F(e^{it})\, dt, \tag{2.11}$$

where $z = re^{i\tau} \in \mathbb{U}$. The following claim holds:

Claim 1 (see e.g. [4, Theorem 3.13 b), $p = \infty$]). *If $w$ is a bounded harmonic function, then there exists a function $F \in L^\infty(\mathbb{T})$ such that $w(z) = P[F](z)$.*

We refer to the book of Axler, Bourdon and Ramey [4] for a good account of harmonic functions.

The *Hilbert transformation* of a function $\chi \in L^1(\mathbb{T})$ is defined by the formula

$$\tilde{\chi}(\tau) = H(\chi)(\tau) = -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{\chi(\tau+t) - \chi(\tau-t)}{2\tan(t/2)}\, dt.$$

Here $\int_{0^+}^{\pi} \Phi(t)\, dt := \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\pi} \Phi(t)\, dt$. This integral is improper and converges for a.e. $\tau \in [0,2\pi]$; this and other facts concerning $H$ can be found in Zygmund's book [35, Chapter VII]. If $f$ is a harmonic function then a harmonic function $\tilde{f}$ is called the *harmonic conjugate* of $f$ if $f + i\tilde{f}$ is an analytic function. Let $\chi, \tilde{\chi} \in L^1(\mathbb{T})$. Then

$$P[\tilde{\chi}] = \widetilde{P[\chi]}, \tag{2.12}$$

where $\tilde{k}(z)$ is the harmonic conjugate of $k(z)$ (see e.g. [30, Theorem 6.1.3]).

Assume that $z = x + iy = re^{i\tau} \in \mathbb{U}$. The complex derivatives of a differentiable mapping $w : \mathbb{U} \to \mathbb{C}$ are defined as follows:

$$w_z = \frac{1}{2}\left(w_x + \frac{1}{i} w_y\right), \qquad w_{\bar{z}} = \frac{1}{2}\left(w_x - \frac{1}{i} w_y\right).$$

The derivatives of $w$ in polar coordinates can be expressed as

$$w_\tau(z) := \frac{\partial w(z)}{\partial \tau} = i(z w_z - \bar{z} w_{\bar{z}}), \qquad w_r(z) := \frac{\partial w(z)}{\partial r} = e^{i\tau} w_z + e^{-i\tau} w_{\bar{z}}.$$

The Jacobian determinant of $w$ is expressed in polar coordinates as

$$J_w(z) = |w_z|^2 - |w_{\bar z}|^2 = \frac{1}{r}\operatorname{Im}(w_\tau \overline{w}_r) = \frac{1}{r}\operatorname{Re}(i w_r \overline{w}_\tau). \tag{2.13}$$

Assume that $w = P[F](z)$ is a harmonic function defined on $\mathbb{U}$. Then there exist two analytic functions $h$ and $k$ defined on $\mathbb{U}$ such that $w = h + \overline{k}$. Moreover $w_\tau = i(zh'(z) - \overline{\bar z k'(z)})$ is a harmonic function and $rw_r = zh'(z) + \overline{\bar z k'(z)}$ is its harmonic conjugate.

Assume now that $F$ is Lipschitz continuous. Then $F' \in L^1(\mathbb{T})$ and by (2.11), using integration by parts, it follows that $w_\tau$ equals the Poisson integral of $F'$:

$$\begin{aligned} w_\tau(re^{i\tau}) &= \int_0^{2\pi} \partial_\tau P(r,\tau - t)F(t)\,dt = -\int_0^{2\pi} \partial_t P(r,\tau - t)F(t)\,dt \\ &= -P(r,\tau - t)F(t)\Big|_{t=0}^{2\pi} + \int_0^{2\pi} P(r,\tau - t)F'(t)\,dt \\ &= \int_0^{2\pi} P(r,\tau - t)F'(t)\,dt. \end{aligned}$$

Let $0 < \alpha < \pi/2$ and define

$$\Gamma_\alpha = \{z : \arg z \in [\pi - \alpha, \pi + \alpha]\}, \qquad \Gamma_\alpha(s) = \mathbb{U} \cap e^{is}(\Gamma_\alpha + 1).$$

That is, $\Gamma_\alpha(s)$ is the wedge inside the unit disk with angle $2\alpha$, whose axis passes between $e^{is}$ and zero. We say that a function $f\colon \mathbb{U} \to \mathbb{C}$ has a *nontangential limit* at $e^{is}$ if for $0 < \alpha < \pi/2$ the limit

$$g(s) = \lim_{\Gamma_\alpha(s) \ni z \to e^{is}} f(z)$$

exists and does not depend on $\alpha$.

We now recall Fatou's theorem [4, Theorem 6.39]:

Claim 2. *If $G \in L^1(\mathbb{T})$, then the Poisson extension $W(z) = P[G](z)$ has nontangential limit at almost every $\zeta \in \mathbb{T}$.*

By using Fatou's theorem we find that the radial limits of $w_\tau$ exist a.e. and

$$\lim_{r\to 1^-} w_\tau(re^{i\tau}) = F'(\tau) \quad \text{a.e.} \tag{2.14}$$

If $F$ is Lipschitz continuous, then $\Phi = F' \in L^\infty(\mathbb{T})$, and by Marcel Riesz's famous theorem (see e.g. [8, Theorem 2.3]), for $1 < p < \infty$ there is a constant $A_p$ such that

$$\|H(F')\|_{L^p(\mathbb{T})} \le A_p \|F'\|_{L^p(\mathbb{T})}.$$

It follows that $\tilde{\Phi} = H(F') \in L^1$. Since $rw_r$ is the harmonic conjugate of $w_\tau$, according to (2.12), we have $rw_r = P[H(F')]$, and by Fatou's theorem again,

$$\lim_{r\to 1^-} w_r(re^{i\tau}) = H(F')(\tau) \quad \text{a.e.} \tag{2.15}$$

**3. The proof of the main theorem.** The aim of this section is to prove Theorem 1.2. We will construct a suitable sequence $w_n$ of univalent harmonic mappings, converging almost uniformly to $w = P[F]$. To do so, we will mollify the boundary function $F$ by a sequence of diffeomorphisms $F_n$ and take the Poisson extension $w_n = P[F_n]$. We will show that under the assumption of Theorem 1.2, for large $n$, $w_n$ satisfies the conditions of the theorem of Alessandrini and Nesi. By a result of Hengartner and Schober [9], the limit function $w$ of a locally uniformly convergent sequence of univalent harmonic mappings $w_n$ is univalent, implying that $F$ is a surjective mapping.

We begin by the following lemma.

LEMMA 3.1. *Let $\gamma$ be a Dini smooth Jordan curve, denote by $g$ its arc-length parameterization and assume that $F(t) = g(f(t))$ is a Lipschitz weak homeomorphism from the unit circle onto $\gamma$. If $w(z) = u(z) + iv(z) = P[F](z)$ is the Poisson extension of $F$, then for almost every $\tau \in [0, 2\pi]$ the limit*

$$J_w(e^{i\tau}) := \lim_{r\to 1^-} J_w(re^{i\tau})$$

*exists and*

$$J_w(e^{i\tau}) = f'(\tau) \int_0^{2\pi} \frac{\operatorname{Re}\left[\overline{(g(f(t)) - g(f(\tau))) \cdot ig'(f(\tau))}\right]}{2\sin^2 \frac{t-\tau}{2}}\, dt. \tag{3.1}$$

*Proof.* Let $z = re^{i\tau}$. Since $F$ is Lipschitz it is absolutely continuous and by (2.14) and (2.15) the radial derivatives of $w_\tau$ and $w_r$ exist for a.e. $e^{i\tau} \in \mathbb{T}$. Let $w(e^{it}) := F(t)$, $u(e^{it}) := \operatorname{Re}(F(t))$ and $v(e^{it}) := \operatorname{Im}(F(t))$. Now, for a.e. $\tau \in [0, 2\pi]$, by Lagrange's theorem,

$$\frac{u(e^{i\tau}) - u(re^{i\tau})}{1-r} = u_r(pe^{i\tau}), \quad r < p < 1,$$
$$\frac{v(e^{i\tau}) - v(re^{i\tau})}{1-r} = v_r(qe^{i\tau}), \quad r < q < 1.$$

It follows that for a.e. $\tau \in [0, 2\pi]$,

$$\lim_{r\to 1^-} \frac{u(e^{i\tau}) - u(re^{i\tau})}{1-r} = \lim_{r\to 1^-} u_r(re^{i\tau}), \tag{3.2}$$

$$\lim_{r\to 1^-} \frac{v(e^{i\tau}) - v(re^{i\tau})}{1-r} = \lim_{r\to 1^-} v_r(re^{i\tau}), \tag{3.3}$$

and consequently for a.e. $\tau \in [0, 2\pi]$

$$\lim_{r\to 1^-} \frac{w(e^{i\tau}) - w(re^{i\tau})}{1-r} = \lim_{r\to 1^-} w_r(re^{i\tau}). \tag{3.4}$$

Furthermore

$$w(e^{i\tau}) - w(re^{i\tau}) = \int_0^{2\pi} [F(\tau) - F(t)] P(r, \tau - t)\, dt$$

and therefore, for a.e. $\tau \in [0, 2\pi]$,

$$\begin{aligned} \lim_{r\to 1^-} w_r(re^{i\tau}) &= \lim_{r\to 1^-} \frac{w(e^{i\tau}) - w(re^{i\tau})}{1-r} \\ &= \lim_{r\to 1^-} \int_0^{2\pi} [F(\tau) - F(t)] \frac{P(r, \tau - t)}{1-r}\, dt. \end{aligned} \tag{3.5}$$

By using the previous facts and the formulae (2.13) and (2.14), since

$$\lim_{r\to 1^-} A(r)B(r) = \lim_{r\to 1^-} A(r) \lim_{r\to 1^-} B(r)$$

provided the limits on the right-hand side exist, we obtain

$$\begin{aligned} \lim_{r\to 1^-} J_w(re^{i\tau}) &= \lim_{r\to 1^-} \frac{\operatorname{Re}[iw_r(re^{i\tau})\overline{w_\tau(re^{i\tau})}]}{r} \\ &= \lim_{r\to 1^-} \frac{\operatorname{Re}[i(w(e^{i\tau}) - w(re^{i\tau}))\overline{F'(\tau)}]}{(1-r)r} \\ &= \lim_{r\to 1^-} \int_0^{2\pi} \frac{P(r, \tau - t)}{1-r} \operatorname{Re}[i(F(\tau) - F(t))\overline{F'(\tau)}]\, dt \\ &= \lim_{r\to 1^-} \int_{-\pi}^{\pi} K_F(t+\tau, \tau) \frac{P(r,t)}{1-r}\, dt \qquad \text{a.e.}, \end{aligned} \tag{3.6}$$

where

$$K_F(t, \tau) = f'(\tau) \operatorname{Re}[\overline{(g(f(t)) - g(f(\tau))) \cdot ig'(f(\tau))}]. \tag{3.7}$$

We refer to [22, (5.6)] for a similar approach, but for some other purpose.

To continue, observe first that

$$\frac{P(r,t)}{1-r} = \frac{1+r}{2\pi(1 + r^2 - 2r\cos t)} \le \frac{1}{\pi((1-r)^2 + 4r\sin^2 t/2)} \le \frac{\pi}{4rt^2}$$

for $0 < r < 1$ and $t \in [-\pi, \pi]$ because $|\sin(t/2)| \ge t/\pi$. On the other hand, by (2.6) and (3.7), for

$$\sigma = \min\{|f(t+\tau) - f(\tau)|, l - |f(t+\tau) - f(\tau)|\}$$

we obtain

$$|K_F(t+\tau,\tau)| \le \|F'\|_\infty \int_0^\sigma \omega(u)\,du,$$

where $\omega$ is the modulus of continuity of $g'$. Therefore for $r \ge 1/2$,

$$\begin{aligned}(3.8)\quad \left|K_F(t+\tau,\tau)\frac{P(r,t)}{1-r}\right| &\le \frac{\|F'\|_\infty \pi}{4rt^2}\int_0^\sigma \omega(u)\,du \le \frac{\sigma}{t}\,\frac{\|F'\|_\infty\pi}{4rt^2}\int_0^t \omega\left(\frac{\sigma}{t}u\right)du \\ &\le \frac{\pi\|F'\|_\infty^2}{2}\,\frac{1}{t^2}\int_0^t \omega(\|F'\|_\infty u)\,du := Q(t).\end{aligned}$$

Having in mind (2.8), we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} |Q(t)|\,dt &\le \frac{2\pi\|F'\|_\infty^2}{2}\int_0^\pi \frac{1}{t^2}\int_0^t \omega(\|F'\|_\infty u)\,du \\ &= \pi\|F'\|_\infty^2 \int_0^\pi \left(\frac{\omega(\|F'\|_\infty u)}{u} - \frac{\omega(\|F'\|_\infty u)}{\pi}\right)du < M < \infty.\end{aligned}$$

According to the Lebesgue Dominated Convergence Theorem, taking the limit under the integral sign in the last integral in (3.6), from

$$\lim_{r\to 1^-}\frac{P(r,t)}{1-r} = \frac{1}{2\pi}\lim_{r\to 1^-}\frac{1+r}{1+r^2-2r\cos t} = \frac{1}{4\pi\sin^2\frac{t}{2}}$$

we obtain (3.1). ∎

For a Lipschitz nondecreasing function $f$ and an arc-length parameterization $g$ of the Dini smooth curve $\gamma$ we define an operator $T$ as follows:

$$(3.9)\qquad T[f](\tau) = \int_0^{2\pi} \frac{\mathrm{Re}[\overline{(g(f(t)) - g(f(\tau)))}\cdot ig'(f(\tau))]}{2\sin^2\frac{t-\tau}{2}}\,\frac{dt}{2\pi}, \qquad \tau\in[0,2\pi].$$

According to Lemma 3.1, this integral converges. Notice that if $\gamma$ is a convex Jordan curve then $\mathrm{Re}[\overline{(g(f(t)) - g(f(\tau)))}\cdot ig'(f(\tau))] \ge 0$, and therefore $T[f]>0$. In the next proof, we will show that under the condition $T[f]>0$, the harmonic extension of a bi-Lipschitz mapping is a diffeomorphism regardless of the condition of convexity.

*Proof of Theorem 1.2.* Assume for simplicity that $|\gamma| = 2\pi$. The general case follows by normalization. Let $g : [0,2\pi] \to \gamma$ be an arc-length parameterization of $\gamma$. Then $F(e^{it}) = g(f(t))$, where $f : \mathbb{R}\to\mathbb{R}$ is a Lipschitz weak homeomorphism such that $f(t+2\pi) = f(t) + 2\pi$. From (3.9) we have

$$T[f](\tau) = \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\pi} \frac{\mathrm{Re}[\overline{(g(f(t+\tau)) - g(f(\tau)))} \cdot ig'(f(\tau))]}{2\sin^2\frac{t}{2}} \frac{dt}{2\pi} + \lim_{\epsilon\to 0^+} \int_{-\pi}^{-\epsilon} \frac{\mathrm{Re}[\overline{(g(f(t+\tau)) - g(f(\tau)))} \cdot ig'(f(\tau))]}{2\sin^2\frac{t}{2}} \frac{dt}{2\pi}.$$

Assume that $\beta : [0, 2\pi] \to \mathbb{R}$ is a continuous function such that

$$g'(s) = e^{i\beta(s)}, \qquad \beta(0) = \beta(2\pi). \tag{3.10}$$

Then

$$|g'(s) - g'(t)| = 2\left|\sin\frac{\beta(t)-\beta(s)}{2}\right|. \tag{3.11}$$

Let $\omega_\beta$ be the modulus of continuity of $g'$. Then

$$\omega_\beta(\rho) = \max_{|t-s|\le\rho} 2\left|\sin\frac{\beta(t)-\beta(s)}{2}\right|. \tag{3.12}$$

Since $\gamma \in C^{1,\alpha}$,

$$\omega_\beta(\rho) \le c(\gamma)\rho^\alpha. \tag{3.13}$$

Further from (3.10), we have

$$\begin{aligned}\frac{\mathrm{Re}[\overline{(g(f(t+\tau)) - g(f(\tau)))} \cdot ig'(f(\tau))]}{2\sin^2\frac{t}{2}} &= \frac{\mathrm{Re}\left[\overline{\int_{f(\tau)}^{f(t+\tau)} g'(s)\,ds} \cdot ig'(f(\tau))\right]}{2\sin^2\frac{t}{2}}\\ &= \frac{\mathrm{Re}\left[\overline{\int_{f(\tau)}^{f(t+\tau)} e^{i\beta(s)}\,ds} \cdot ie^{i\beta(f(\tau))}\right]}{2\sin^2\frac{t}{2}}\\ &= \frac{-\mathrm{Im}\left[\overline{\int_{f(\tau)}^{f(t+\tau)} e^{i\beta(s)-\beta(f(\tau))}\,ds}\right]}{2\sin^2\frac{t}{2}}\\ &= \frac{\int_{f(\tau)}^{f(t+\tau)} \sin[\beta(s)-\beta(f(\tau))]\,ds}{2\sin^2\frac{t}{2}}.\end{aligned}$$

Taking

$$dU = \frac{1}{2\sin^2\frac{t}{2}}\,dt \quad \text{and} \quad V = \int_{f(\tau)}^{f(t+\tau)} \sin[\beta(s)-\beta(f(\tau))]\,ds,$$

we obtain

$$U = -\cot\frac{t}{2} \quad \text{and} \quad dV = f'(t+\tau)\sin[\beta(f(t+\tau)) - \beta(f(\tau))]\,dt.$$

To continue recall that $f$ is Lipschitz with a Lipschitz constant $L$. Thus

$$\begin{aligned}
|\lim_{\epsilon\to 0^+} U(t)V(t)|_\epsilon^\pi| &= \left|\lim_{\epsilon\to 0^+}\cot\frac{\epsilon}{2}\int\limits_{f(\tau)}^{f(\epsilon+\tau)} \sin[\beta(s)-\beta(f(\tau))]\,ds\right| \\
&\le \lim_{\epsilon\to 0^+}\cot\frac{\epsilon}{2}\cdot|\sin[\beta(\epsilon+\tau)-\beta(f(\tau))]|\,|f(\epsilon+\tau)-f(\tau)| \\
&\le \lim_{\epsilon\to 0^+} L\epsilon\cot\frac{\epsilon}{2}\cdot\omega_\beta(\epsilon)=0.
\end{aligned}$$

Similarly we have

$$\lim_{\epsilon\to 0^+} U(t)V(t)|_{-\pi}^{-\epsilon}=0.$$

By integration by parts we obtain

$$\begin{aligned}
T[f](\tau) &= \lim_{\epsilon\to 0^+}\left(UV|_\epsilon^\pi+\int\limits_\epsilon^\pi f'(t+\tau)\cdot\sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\,\frac{dt}{2\pi}\right) \\
&\quad+\lim_{\epsilon\to 0^+}\left(UV|_{-\pi}^{-\epsilon}+\int\limits_{-\pi}^{-\epsilon} f'(t+\tau)\cdot\sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\,\frac{dt}{2\pi}\right) \\
&= \int\limits_{-\pi}^{\pi} f'(t+\tau)\cdot\sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\,\frac{dt}{2\pi}.
\end{aligned}$$

Hence

$$T[f](\tau)=\int\limits_{-\pi}^{\pi} f'(t+\tau)\cdot\sin[\beta(f(t+\tau))-\beta(f(\tau))]\cot\frac{t}{2}\,\frac{dt}{2\pi}.$$

By using Lemma 2.5, we can choose a family of diffeomorphisms $f_n$ converging uniformly to $f$. Then

$$T[f_n](\tau)=\int\limits_{-\pi}^{\pi} f_n'(t+\tau)\cdot\sin[\beta(f_n(t+\tau))-\beta(f_n(\tau))]\cot\frac{t}{2}\,\frac{dt}{2\pi}.$$

We are going to show that $T[f_n]$ converges uniformly to $T[f]$. In order to do this, we apply the Arzelà–Ascoli theorem.

First of all

$$\begin{aligned}
|T[f_n](\tau)| &\le \frac{1}{\pi}\|f_n'\|_\infty\int\limits_0^\pi \omega_\beta(\|f_n'\|_\infty t)\cot\frac{t}{2}\,dt \\
&\le \frac{1}{\pi}\|f'\|_\infty\int\limits_0^\pi \omega_\beta(\|f'\|_\infty t)\cot\frac{t}{2}\,dt = C(f,\gamma)<\infty.
\end{aligned}$$

We prove now that $T[f_n]$ is an equicontinuous family of functions. We have to estimate the quantity

$$|T[f_n](\tau)-T[f_n](\tau_0)|.$$

Assume without loss of generality that $\tau_0 = 0$. Then

$$|T[f_n](\tau) - T[f_n](0)| = \bigg| \int_{-\pi}^{\pi} f_n'(t+\tau) \cdot \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))] \cot\frac{t}{2}\,\frac{dt}{2\pi} - \int_{-\pi}^{\pi} f_n'(t) \cdot \sin[\beta(f_n(t)) - \beta(f_n(0))] \cot\frac{t}{2}\,\frac{dt}{2\pi} \bigg| \le A + B,$$

where

$$A = \bigg| \int_{-\pi}^{\pi} (f_n'(t+\tau) - f_n'(t)) \cdot \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))] \cot\frac{t}{2}\,\frac{dt}{2\pi} \bigg|,$$

$$B = \bigg| \int_{-\pi}^{\pi} f_n'(t) \cdot \{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\} \times \cot\frac{t}{2}\,\frac{dt}{2\pi} \bigg|.$$

Take $r \ge 1$, $p > 1$, $q > 1$ such that $1/p + 1/q = 1$, and $\delta \in (0,1)$.

In what follows, for a function $g \in L^a(\mathbb{T})$, $a > 0$, we consider the following $a$-norm:

$$\|g\|_a = \bigg( \int_0^{2\pi} |g(e^{it})|^a \,\frac{dt}{2\pi} \bigg)^{1/a}.$$

Define $f_\tau(x) := f(x+\tau)$. By (2.10) we have

$$f_n = \frac{n}{n+1}\bigg( f * \rho_{\varepsilon_n} + \frac{x}{n} \bigg).$$

Thus

$$|f_{n,\tau}' - f_n'| = \frac{n}{n+1} |(f_\tau' - f') * \rho_{\varepsilon_n}|. \tag{3.14}$$

According to Young's inequality for convolution ([34, pp. 54–55], [8, Theorem 20.18]), we obtain

$$\|(f_\tau' - f') * \rho_{\varepsilon_n}\|_r \le \|f_\tau' - f'\|_r.$$

In view of (3.13) and (3.14), for $1 < q < \frac{1}{1-\alpha}$, by the Hölder inequality we have

$$\begin{aligned} A &\le \|f_n'(t+\tau) - f_n'(t)\|_p \cdot \bigg\| \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))] \cot\frac{t}{2} \bigg\|_q \\ &\le \|f'(t+\tau) - f'(t)\|_p \cdot \bigg\| \omega_\beta(|f_n|_\infty t) \cot\frac{t}{2} \bigg\|_q \\ &\le C_1(\gamma) \|f'\|_\infty \|f'(t+\tau) - f'(t)\|_p. \end{aligned} \tag{3.15}$$

Let now estimate $B$. First of all

(3.16)
$$B \le \|f'\|_\infty \left\| \{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\} \cot \frac{t}{2} \right\|_1 .$$

On the other hand, using again the Hölder inequality we have

$$\begin{aligned}
&\left\| \{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\} \cot \frac{t}{2} \right\|_1 \\
&\quad \le \|\{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}^\delta\|_p \\
&\qquad \times \left\| \{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}^{1-\delta} \cot \frac{t}{2} \right\|_q .
\end{aligned}$$

Further

$$\begin{aligned}
&\|\{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}^\delta\|_p \\
&\qquad \le \left\| \left\{ \left| 2\sin \frac{\beta(f_n(t)) - \beta(f_n(0)) - \beta(f_n(t+\tau)) + \beta(f_n(\tau))}{2} \right| \right\}^\delta \right\|_p \\
&\qquad \le \left\| \left\{ \left| 2\sin \frac{\beta(f_n(t+\tau)) - \beta(f_n(t))}{2} \right| \right\}^\delta \right\|_p \\
&\qquad\quad + \left\| \left\{ \left| 2\sin \frac{\beta(f_n(\tau)) - \beta(f_n(0))}{2} \right| \right\}^\delta \right\|_p \\
&\qquad \le \omega_\beta(|f_n'|_\infty \tau)^\delta + \omega_\beta(|f_n'|_\infty \tau)^\delta = 2\omega_\beta(|f_n'|_\infty \tau)^\delta \le 2\omega_\beta(|f'|_\infty \tau)^\delta,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \{\sin[\beta(f_n(t)) - \beta(f_n(0))] - \sin[\beta(f_n(t+\tau)) - \beta(f_n(\tau))]\}^{1-\delta} \cot \frac{t}{2} \right\|_q \\
&\qquad\qquad \le \left\| 2\omega_\beta(|f_n'|_\infty t)^{1-\delta} \cot \frac{t}{2} \right\|_q .
\end{aligned}$$

Choose $q$ and $\delta$ such that

$$(\alpha - \alpha\delta - 1)q > -1.$$

Then the integral

$$\left\| 2\omega_\beta(|f_n'|_\infty t)^{1-\delta} \cot \frac{t}{2} \right\|_q$$

converges and it is less than or equal to

$$C(\gamma)\|f_n'\|_\infty^{1-\delta} \le C(\gamma)\|f'\|_\infty^{1-\delta}.$$

Therefore

$$B \le 2\|f'\|_\infty C(\gamma)\|f'\|_\infty^{1-\delta} \omega_\beta(\|f'\|_\infty \tau)^\delta. \tag{3.17}$$

Since translation is continuous (see [33, Theorem 9.5]), (3.15) and (3.17) imply that the family $\{T[f_n]\}$ is equicontinuous. By the Arzelà–Ascoli theorem it follows that

$$\lim_{n\to\infty} \|T[f_n]-T[f]\|_\infty = 0. \tag{3.18}$$

Thus $T[f]$ is continuous. Moreover for sufficiently large $n$, for

$$\delta = \min\{T[f](s) : 0 \le s \le 2\pi\} > 0,$$

from (3.18), we obtain

$$T[f_n](s) \ge T[f](s) - \delta/2 \ge \delta/2 > 0, \qquad s \in [0, 2\pi].$$

Since $f_n$ is a diffeomorphism, $f_n'(\tau) > 0$. Thus for sufficiently large $n$,

$$J_{w_n}(e^{i\tau}) = f_n'(\tau)T[f_n](e^{i\tau}) > 0, \qquad e^{i\tau} \in \mathbb{T}.$$

Since $f_n \in C^\infty$, it follows that

$$w_n = P[F_n] \in C^1(\overline{\mathbb{U}}).$$

Therefore all the conditions of Proposition 1.1 are satisfied. This means that $w_n$ is a harmonic diffeomorphism of the unit disk onto the domain $D$.

Since, by a result of Hengartner and Schober [9], the limit function $w$ of a locally uniformly convergent sequence of univalent harmonic mappings $w_n$ on $\mathbb{U}$ is either univalent on $\mathbb{U}$, a constant, or its image lies on a straight line, we deduce that $w = P[F]$ is univalent. ∎

Remark 3.2. If $\gamma$ is a $C^{1,\alpha}$ convex curve, then

$$\mathrm{Re}[\overline{(g(f(t)) - g(f(\tau)))} \cdot ig'(f(\tau))] \ge 0$$

and therefore $T[f](\tau) > 0$. By the proof of Theorem 1.2, $\tau \mapsto T[f](\tau)$ is continuous. Therefore $\min_{\tau\in[0,2\pi]} T[f](\tau) = \delta > 0$.

**4. Quasiconformal harmonic mappings.** An injective harmonic mapping $w = u + iv$ is called $K$-*quasiconformal* ($K$-q.c), $K \ge 1$, if

$$|w_{\bar z}| \le k|w_z| \tag{4.1}$$

on $D$ where $k = (K-1)/(K+1)$. Notice that, since

$$|\nabla w(z)| := \max\{|\nabla w(z)h| : |h| = 1\} = |w_z| + |w_{\bar z}|$$

and

$$l(\nabla w(z)) := \min\{|\nabla w(z)h| : |h| = 1\} = \big||w_z| - |w_{\bar z}|\big|,$$

the condition (4.1) is equivalent to

$$|\nabla w(z)| \le Kl(\nabla w(z)). \tag{4.2}$$

For a general definition of quasiregular mappings and quasiconformal mappings we refer to the book of Ahlfors [1]. In this section we apply Theorem 1.2 to the class of q.c. harmonic mappings. The area of quasiconformal harmonic mappings is a very active area of research. For background on this theory we refer [10], [18]–[25], [26], [28], [29], [5]. In this section we obtain some new results concerning a characterization of this class. We will restrict ourselves to the class of q.c. harmonic mappings $w$ between the unit disk $\mathbb{U}$ and a Jordan domain $D$. The unit disk is taken because of simplicity. Namely, if $w : \Omega \to D$ is q.c. harmonic, and $a : \mathbb{U} \to \Omega$ is conformal, then $w \circ a$ is also q.c. harmonic. However the image domain $D$ cannot be replaced by the unit disk.

The case when $D$ is a convex domain is treated in detail by the author and others in the above cited papers. In this section we will use our main result to yield a characterization of quasiconformal harmonic mappings of the unit disk onto a Jordan domain that is not necessarily convex in terms of boundary data.

To state the main result of this section, we make use of Hilbert transform formalism. It provides a necessary and a sufficient condition for the harmonic extension of a homeomorphism from the unit circle to a $C^2$ Jordan curve $\gamma$ to be a q.c mapping. It is an extension of the corresponding result [12, Theorem 3.1] relating to convex Jordan domains.

THEOREM 4.1. *Let $F : \mathbb{T} \to \gamma$ be a sense preserving homeomorphism of the unit circle onto the Jordan curve $\gamma = \partial D \in C^2$. Then $w = P[F]$ is a quasiconformal mapping of the unit disk onto $D$ if and only if $F$ is absolutely continuous and*

$$0 < l(F) := \operatorname{ess\,inf} l(\nabla w(e^{i\tau})), \tag{4.3}$$

$$\|F'\|_\infty := \operatorname{ess\,sup} |F'(\tau)| < \infty, \tag{4.4}$$

$$\|H(F')\|_\infty := \operatorname{ess\,sup} |H(F')(\tau)| < \infty. \tag{4.5}$$

*If $F$ satisfies* (4.3)–(4.5), *then $w = P[F]$ is $K$-quasiconformal, where*

$$K := \frac{\sqrt{\|F'\|_\infty^2 + \|H(F')\|_\infty^2 - l(F)^2}}{l(F)}. \tag{4.6}$$

*The constant $K$ is approximately sharp for small values of $K$: if $w$ is the identity or if it is a mapping close to the identity, then $K = 1$ or $K$ is close to* 1 (*respectively*).

*Proof of necessity.* Suppose that $w = P[F] = g + \overline{h}$ is a $K$-q.c. harmonic mapping that satisfies the conditions of the theorem. By [12, Theorem 2.1], we see that $w$ is Lipschitz continuous,

$$L := \|F'\|_\infty < \infty \tag{4.7}$$

and

$$|\nabla w(z)| \le KL. \tag{4.8}$$

By [16, Theorem 1.4] we have, for $b = w(0)$,

$$|\partial w(z)| - |\bar{\partial} w(z)| \geq C(\Omega, K, b) > 0, \quad z \in \mathbb{U}. \tag{4.9}$$

Because of (4.8), the analytic functions $\partial w(z)$ and $\bar{\partial} w(z)$ are bounded, and thus there exist functions $F_1, F_2 \in L^\infty(\mathbb{T})$ such that $\partial w(z) = P[F_1](z)$ and $\bar{\partial} w(z) = P[F_2](z)$ (see Claim 1 in Subsection 2.2). Therefore by Fatou's theorem,

$$\lim_{r \to 1^-} (|\partial w(re^{i\tau})| - |\bar{\partial} w(re^{i\tau})|) = |\partial w(e^{i\tau})| - |\bar{\partial} w(e^{i\tau})| \quad \text{a.e.} \tag{4.10}$$

Combining (4.7), (4.10) and (4.9), we get (4.3) and (4.4).

Next we prove (4.5). Observe first that $w_r = e^{i\tau} w_z + e^{-i\tau} w_{\bar{z}}$. Thus

$$|w_r| \leq |\nabla w| \leq KL. \tag{4.11}$$

Therefore $r w_r = P[H(F')]$ is a bounded harmonic function, which implies that $H(F') \in L^\infty(\mathbb{T})$. Therefore (4.5) holds and the necessity is proved.

*Proof of sufficiency.* We have to prove that under the conditions (4.3)–(4.5), $w$ is quasiconformal. From

$$0 < l(F) = \operatorname{ess\,inf} l(\nabla w(e^{i\tau}))$$

we obtain

$$J_w(e^{i\tau}) = (|w_z| + |w_{\bar{z}}|) l(\nabla w(e^{i\tau})) \geq l(F)^2 \quad \text{a.e.}$$

Since $F$ is absolutely continuous with $\|F'\|_\infty < \infty$, it follows that $F' \in L^\infty(\mathbb{T})$. From (2.14) and (2.15) we have

$$\lim_{r \to 1^-} w_r(re^{i\tau}) = H(F')(\tau) \quad \text{and} \quad \lim_{r \to 1^-} w_\tau(re^{i\tau}) = F'(\tau) \quad \text{a.e.} \tag{4.12}$$

As

$$|w_z|^2 + |w_{\bar{z}}|^2 = \frac{1}{2}\left(|w_r|^2 + \frac{|w_\tau|^2}{r^2}\right),$$

it follows that for a.e. $\tau \in [0, 2\pi)$,

$$\begin{aligned} \lim_{r \to 1^-} (|w_z(re^{i\tau})|^2 + |w_{\bar{z}}(re^{i\tau})|^2) &= |w_z(e^{i\tau})|^2 + |w_{\bar{z}}(e^{i\tau})|^2 \\ &\leq \frac{1}{2}(\|F'\|_\infty^2 + \|H(F')\|_\infty^2). \end{aligned} \tag{4.13}$$

To continue we make use of (4.3). From (4.13), (4.3) and (4.2), for a.e. $\tau \in [0, 2\pi)$,

$$\frac{|w_z(e^{i\tau})|^2 + |w_{\bar{z}}(e^{i\tau})|^2}{(|w_z(e^{i\tau})| - |w_{\bar{z}}(e^{i\tau})|)^2} \leq \frac{\|F'\|_\infty^2 + \|H(F')\|_\infty^2}{2l(F)^2}. \tag{4.14}$$

Hence

$$|w_z(e^{i\tau})|^2 + |w_{\bar{z}}(e^{i\tau})|^2 \leq S(|w_z(e^{i\tau})| - |w_{\bar{z}}(e^{i\tau})|)^2 \quad \text{a.e.,} \tag{4.15}$$

where

$$S := \frac{\|F'\|_\infty^2 + \|H(F')\|_\infty^2}{2l(F)^2}. \tag{4.16}$$

According to (4.14), $S \ge 1$. Let

$$\mu(e^{i\tau}) := \left|\frac{w_{\bar z}(e^{i\tau})}{w_z(e^{i\tau})}\right|.$$

Since every $C^2$ curve is $C^{1,\alpha}$, Theorem 1.2 shows that $w = g + \overline{k}$ is univalent and according to Lewy's theorem, $J_w(z) = |g'(z)|^2 - |h'(z)|^2 > 0$. Thus $a(z) = \overline{w_{\bar z}}/w_z = h'/g'$ is an analytic function bounded by 1. As $\mu(e^{i\tau}) = |a(e^{i\tau})|$, we have $\mu(e^{i\tau}) \le 1$. Then (4.15) can be written as

$$1 + \mu^2(e^{i\tau}) \le S(1 - \mu(e^{i\tau}))^2,$$

i.e. if $S = 1$, then $\mu(e^{i\tau}) = 0$ a.e. and if $S > 1$, then

$$\mu^2(S-1) - 2\mu S + S - 1 = (S-1)(\mu - \mu_1)(\mu - \mu_2) \ge 0, \tag{4.17}$$

where

$$\mu_1 = \frac{S + \sqrt{2S-1}}{S-1}, \qquad \mu_2 = \frac{S-1}{S + \sqrt{2S-1}}.$$

If $S > 1$, then from (4.17) it follows that $\mu(e^{i\tau}) \le \mu_2$ or $\mu(e^{i\tau}) \ge \mu_1$. But $\mu(e^{i\tau}) \le 1$ and therefore

$$\mu(e^{i\tau}) \le \frac{S-1}{S+\sqrt{2S-1}} \quad \text{a.e.} \tag{4.18}$$

If $S = 1$, then (4.18) clearly holds. Define $\mu(z) = |a(z)|$. Since $a$ is a bounded analytic function, by the maximum principle it follows that

$$\mu(z) \le k := \mu_2$$

for $z \in \mathbb{U}$. This yields

$$K(z) \le K := \frac{1+k}{1-k} = \frac{2S - 1 + \sqrt{2S-1}}{\sqrt{2S-1}+1} = \sqrt{2S-1},$$

i.e.

$$K(z) \le \frac{\sqrt{\|F'\|_\infty^2 + \|H(F')\|_\infty^2 - l(F)^2}}{l(F)}$$

which means that $w$ is $K = \frac{\sqrt{\|F'\|_\infty^2 + \|H(F')\|_\infty^2 - l(F)^2}}{l(F)}$-quasiconfomal. The result is asymptotically sharp because $K = 1$ for $w$ being the identity. This finishes the proof of Theorem 4.6. ∎

Conjecture. Let $F : \mathbb{T} \to \gamma \subset \mathbb{C}$ be a homeomorphism of bounded variation, where $\gamma$ is Dini smooth. Let $D$ be the bounded domain such that $\partial D = \gamma$. The mapping $w = P[F]$ is a diffeomorphism of $\mathbb{U}$ onto $D$ if and only if

$$\operatorname{ess\,inf}\{J_w(e^{it}) : t \in [0, 2\pi]\} \geq 0. \tag{4.19}$$

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## References

[1] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand Math. Stud. 10, Van Nostrand, 1966.

[2] G. Alessandrini and V. Nesi, *Invertible harmonic mappings, beyond Kneser*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) 8 (2009), 451–468.

[3] —, —, *Univalent $\sigma$-harmonic mappings*, Arch. Ration. Mech. Anal. 158 (2001), 155–171.

[4] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer, New York, 1992.

[5] X. Chen and A. Fang, *A Schwarz–Pick inequality for harmonic quasiconformal mappings and its applications*, J. Math. Anal. Appl. 369 (2010), 22–28.

[6] P. Duren, *Harmonic Mappings in the Plane*, Cambridge Univ. Press, 2004.

[7] P. Duren and W. Hengartner, *Harmonic mappings of multiply connected domains*, Pacific J. Math. 180 (1997), 201–220.

[8] J. Garnett, *Bounded Analytic Functions*, Pure Appl. Math. 96, Academic Press, New York, 1981.

[9] W. Hengartner and G. Schober, *Univalent harmonic mappings*, Trans. Amer. Math. Soc. 299 (1987), 1–31.

[10] —, —, *Harmonic mappings with given dilatation*, J. London Math. Soc. (2) 33 (1986), 473–483.

[11] J. Jost, *Univalency of harmonic mappings between surfaces*, J. Reine Angew. Math. 324 (1981), 141–153.

[12] D. Kalaj, *Quasiconformal harmonic mapping between Jordan domains*, Math. Z. 260 (2008), 237–252.

[13] —, *Harmonic quasiconformal mappings and Lipschitz spaces*, Ann. Acad. Sci. Fenn. Math. 34 (2009), 475–485.

[14] —, *On quasiregular mappings between smooth Jordan domains*, J. Math. Anal. Appl. 362 (2010), 58–63.

[15] —, *On boundary correspondence of q.c. harmonic mappings between smooth Jordan domains*, Math. Nachr., DOI 10.1002/mana.200910053 (arXiv:0910.4950v1).

[16] —, *Harmonic mappings and distance function*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) 10 (2011), 669–681.

[17] D. Kalaj and M. Mateljević, *Inner estimate and quasiconformal harmonic maps between smooth domains*, J. Anal. Math. 100 (2006), 117–132.

[18] D. Kalaj and M. Pavlović, *On quasiconformal self-mappings of the unit disk satisfying the Poisson differential equation*, Trans. Amer. Math. Soc. 363 (2011), 4043–4061.

[19] —, —, *Boundary correspondence under harmonic quasiconformal homeomorphisms of a half-plane*, Ann. Acad. Sci. Fenn. Math. 30 (2005), 159–165.

[20] H. Kneser, *Lösung der Aufgabe 41*, Jahresber. Deutsch. Math.-Verein. 35 (1926), 123–124.

[21] M. Knežević and M. Mateljević, *On the quasi-isometries of harmonic quasiconformal mappings*, J. Math. Anal. Appl. 334 (2007), 404–413.

[22] R. S. Laugesen, *Planar harmonic maps with inner and Blaschke dilatations*, J. London Math. Soc. 56 (1997), 37–48.

[23] F. D. Lesley and S. E. Warschawski, *Boundary behavior of the Riemann mapping function of asymptotically conformal curves*, Math. Z. 179 (1982), 299–323.

[24] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. 42 (1936), 689–692.

[25] V. Manojlović, *Bi-Lipschicity of quasiconformal harmonic mappings in the plane*, Filomat 23 (2009), 85–89.

[26] O. Martio, *On harmonic quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 425 (1968), 3–10.

[27] M. Mateljević and M. Vuorinen, *On harmonic quasiconformal quasi-isometries*, J. Inequal. Appl. 2010, art. ID 178732, 19 pp.

[28] D. Partyka and K. Sakan, *On bi-Lipschitz type inequalities for quasiconformal harmonic mappings*, Ann. Acad. Sci. Fenn. Math. 32 (2007), 579–594.

[29] M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disc*, Ann. Acad. Sci. Fenn. 27 (2002), 365–372.

[30] —, *Introduction to Function Spaces on the Disk*, Mat. Inst. SANU, Belgrade, 2004.

[31] R. Schoen and S.-T. Yau, *On univalent harmonic maps between surfaces*, Invent. Math. 44 (1978), 265–278.

[32] F. Schulz, *Univalent solutions of elliptic systems of Heinz–Lewy type*, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), 347–361.

[33] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[34] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Act. Sci Indust. 869 & 1145, Hermann, 1941 & 1951.

[35] A. Zygmund, *Trigonometric Series I*, Cambridge Univ. Press, 1958.

David Kalaj
Faculty of Natural Sciences and Mathematics
University of Montenegro
Džordža Vasingtona b.b.
81000 Podgorica, Montenegro
E-mail: davidk@t-com.me

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