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On weighted weak type norm inequalities for one-sided oscillatory singular integrals

by

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Abstract. We consider one-sided weight classes of Muckenhoupt type and study the weighted weak type $(1,1)$ norm inequalities for a class of one-sided oscillatory singular integrals with smooth kernel.

1. Introduction. Oscillatory integrals have been an essential part of harmonic analysis; three chapters are devoted to them in Stein's celebrated book [ST]. Many important operators in harmonic analysis are some versions of oscillatory integrals, such as the Fourier transform, the Bochner–Riesz means, the Radon transform in CT technology and so on. For a more complete account on oscillatory integrals in classical harmonic analysis, we refer the interested reader to [G], [L1], [L2], [LDY], [LZ], [PS] and references therein. Another early impetus for the study of oscillatory integrals came with their application to number theory [B]. In more recent times, the operators fashioned from oscillatory integrals, such as pseudo-differential operators in PDE theory, have become another motivation to study them. Based on the estimates of some kinds of oscillatory integrals, one can establish the well-posedness theory of a class of dispersive equations; for some of this work, we refer to [CM], [KPV1], [KPV2].

This paper is focused on a class of oscillatory singular integrals related to the one defined by Ricci and Stein [RS]

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}} e^{iP(x,y)} K(x-y) f(y)\, dy,$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}\times\mathbb{R}$, and $K \in C^1(\mathbb{R}\setminus\{0\})$

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is a Calderón–Zygmund kernel which satisfies

(1.1) $$|K(x)| \le C/|x|, \qquad |\nabla K(x)| \le C/|x|^2,$$

(1.2) $$\int\limits_{a<|x|<b} K(x)\,dx = 0 \qquad \text{for all } a, b\ (0 < a < b).$$

Obviously, K is an odd function under the condition (1.2).

THEOREM 1.1 ([RS]). *Suppose K satisfies* (1.1), (1.2). *Then for any real polynomial $P(x,y)$, the oscillatory singular integral operator T is of type $(L^p(\mathbb{R}), L^p(\mathbb{R}))$, $1 < p < \infty$, with operator norm bounded by a constant depending on the total degree of P, but not on the coefficients of P in other respects.*

Let A_p $(1 < p < \infty)$ denote the Muckenhoupt classes [CF]. This class consists of positive locally integrable functions (weight functions) w for which

$$\sup_I \bigg(\frac{1}{|I|}\int\limits_I w(x)\,dx\bigg)\bigg(\frac{1}{|I|}\int\limits_I w(x)^{1-p'}\,dx\bigg)^{p-1} < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ and $1/p + 1/p' = 1$.

In 1992, Lu and Zhang [LZ] gave a weighted version of Theorem 1.1.

THEOREM 1.2. *Suppose K satisfies* (1.1), (1.2). *Then for any real polynomial $P(x,y)$, the oscillatory singular integral operator T is of type $(L^p(w), L^p(w))$, where $w \in A_p$, $1 < p < \infty$, and the operator norm is bounded by a constant depending on the total degree of P, but not on the coefficients of P in other respects.*

For the case $p = 1$, Chanillo and Christ [CC] gave a supplement for Theorem 1.1.

THEOREM 1.3. *Under the same assumptions as in Theorem* 1.1, *we have*

$$\|Tf\|_{L^{1,\infty}} \le C\|f\|_{L^1},$$

where $L^{1,\infty}$ denotes the weak L^1 space, and the constant C is independent of P if the total degree of the polynomial is fixed.

Let A_1 be the class of weight functions w satisfying $Mw(x) \le Cw(x)$ a.e., where M denotes the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{h>0} \frac{1}{2h} \int\limits_{x-h}^{x+h} |f(y)|\,dy.$$

We write $w(E) = \int_E w$ for a measurable set E. The third author of this paper gave a weighted version of Theorem 1.3.

Theorem 1.4 ([SA]). *Under the same assumptions as in Theorem* 1.1, *if* $w \in A_1$, *then*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R} : |Tf(x)| > \lambda\}) \le C\|f\|_{L^1(w)}$$

where C *depends on the total degree of* P *and, in other respects, is independent of the coefficients of* P.

We point out that Theorems 1.1–1.4 also hold for dimension $n \ge 2$.

The study of weights for one-sided operators was motivated not only as the generalization of the theory of both-sided ones but also by their natural appearance in harmonic analysis; for example, they are required when we treat the one-sided Hardy–Littlewood maximal operator [SAW]

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\,dy, \tag{1.3}$$

and

$$M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\,dy, \tag{1.4}$$

arising in the ergodic maximal function. The classical Dunford–Schwartz ergodic theorem can be considered as the first result about weights for (1.3) and (1.4). In [SAW], Sawyer introduced the one-sided A_p classes A_p^+, A_p^-; they are defined by the following conditions:

$$A_p^+: \quad A_p^+(w) := \sup_{a<b<c} \frac{1}{(c-a)^p} \int_a^b w(x)\,dx \Big(\int_b^c w(x)^{1-p'}\,dx\Big)^{p-1} < \infty,$$

$$A_p^-: \quad A_p^-(w) := \sup_{a<b<c} \frac{1}{(c-a)^p} \int_b^c w(x)\,dx \Big(\int_a^b w(x)^{1-p'}\,dx\Big)^{p-1} < \infty,$$

when $1 < p < \infty$; also, for $p = 1$,

$$A_1^+: \quad M^-w \le Cw,$$
$$A_1^-: \quad M^+w \le Cw,$$

for some constant C. The smallest constant C for which the above inequalities are satisfied will be denoted by $A_1^+(w)$ and $A_1^-(w)$. The number $A_p^+(w)$ (resp. $A_p^-(w)$), $p \ge 1$, will be called the A_p^+ (resp. A_p^-) *constant* of w.

Theorem 1.5 ([SAW]). *Let* M^+ *be as in* (1.3).

(i) *Let* $1 \le p < \infty$. *Then there exists* $C > 0$ *such that the inequality*

$$\sup_{\lambda>0} \lambda^p w(\{x \in \mathbb{R} : |M^+f(x)| > \lambda\}) \le C\|f\|^p_{L^p(w)}$$

holds for all f *if and only if* $w \in A_p^+$.

(ii) *Let* $1 < p < \infty$. *Then there exists* $C > 0$ *such that the inequality*

$$\|M^+ f\|_{L^p(w)} \le C\|f\|_{L^p(w)}$$

holds for all $f \in L^p(w)$ *if and only if* $w \in A_p^+$.

Remark. Similar results can be obtained for the left-hand-side operator with the condition A_p^+ replaced by A_p^-.

Together with the characterizations of the weighted inequalities for M^+ and M^-, Sawyer obtained some properties of the classes A_p^+ and A_p^-.

Proposition 1.6 (see also [SAW]).

(i) *If* $w \in A_1^+$, *then* $w^{1+\varepsilon} \in A_1^+$ *for some* $\varepsilon > 0$.
(ii) $w \in A_p^+$ *for* $1 < p < \infty$ *if and only if there exist* $w_1 \in A_1^+$ *and* $w_2 \in A_1^-$ *such that* $w = w_1(w_2)^{1-p}$.
(iii) *If* $1 \le p < \infty$, *then* $A_p = A_p^+ \cap A_p^-$, $A_p \subset A_p^+$, $A_p \subset A_p^-$.
(iv) $A_p^+ \subset A_r^+$, $A_p^- \subset A_r^-$ *if* $1 \le p \le r$.

Perhaps it is worth pointing out that these classes not only control the boundedness of M^+ and M^-, but also they are the right weight classes for one-sided singular integrals [AFM], and they also appear in PDE theory [GS].

We say a Calderón–Zygmund kernel K is a *one-sided Calderón–Zygmund kernel* (OCZK) if K satisfies (1.1) and

$$\Big|\int\limits_{a<|x|<b} K(x)\,dx\Big| \le C, \quad 0 < a < b, \tag{1.5}$$

with support in $\mathbb{R}^- = (-\infty, 0)$ or $\mathbb{R}^+ = (0, +\infty)$. The smallest constant for which (1.1) and (1.5) hold will be denoted by $C(K)$. An example is

$$K(x) = \frac{\sin(\log|x|)}{(x\log|x|)}\chi_{(-\infty,0)}(x),$$

where χ_E denotes the characteristic function of a set E. In [AFM], Aimar, Forzani and Martín-Reyes studied the one-sided Calderón–Zygmund singular integrals defined by

$$\widetilde{T}^+ f(x) = \lim_{\varepsilon\to 0^+} \int\limits_{x+\varepsilon}^{\infty} K(x-y) f(y)\,dy$$

and

$$\widetilde{T}^- f(x) = \lim_{\varepsilon\to 0^+} \int\limits_{-\infty}^{x-\varepsilon} K(x-y) f(y)\,dy$$

where K is a OCZK.

Theorem 1.7 ([AFM]). *Let* K *be a OCZK with support in* $\mathbb{R}^-$. *Then*

(i) $\widetilde{T}^+$ *is bounded on* $L^p(w)$ $(1 < p < \infty)$ *if* $w \in A_p^+$.
(ii) $\widetilde{T}^+$ *maps* $L^1(w)$ *into* $L^{1,\infty}(w)$ *if* $w \in A_1^+$.

Also, a converse of Theorem 1.7 is given in [AFM]. Inspired by [CC], [SA] and [SAW], we will study the one-sided version of Theorem 1.4 by using induction, Calderón–Zygmund decomposition, estimates for oscillatory integrals in the unweighted case and interpolation of operators with change of measures. In the following, the letter C will stand for a positive constant which may vary from line to line.

2. Main results. We first give the definition of one-sided oscillatory singular integral operators T^+, T^-:

$$T^+f(x) = \lim_{\varepsilon\to 0^+} \int_{x+\varepsilon}^{\infty} e^{iP(x,y)}K(x-y)f(y)\,dy = \text{p.v.}\int_{x}^{\infty} e^{iP(x,y)}K(x-y)f(y)\,dy$$

and

$$T^-f(x) = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{x-\varepsilon} e^{iP(x,y)}K(x-y)f(y)\,dy = \text{p.v.}\int_{-\infty}^{x} e^{iP(x,y)}K(x-y)f(y)\,dy,$$

where $P(x,y)$ is a real polynomial defined on $\mathbb{R}\times\mathbb{R}$, and K is a OCZK with support in $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively. Now, we formulate our result:

Main Theorem 2.1. *If $w \in A_1^+$, then there exists a constant C depending on the total degree of P, $C(K)$ and $A_1^+(w)$ such that*

$$\sup_{\lambda>0} \lambda w(\{x\in\mathbb{R} : |T^+f(x)| > \lambda\}) \le C\|f\|_{L^1(w)} \tag{2.1}$$

for $f\in\mathcal{S}(\mathbb{R})$ (the Schwartz class).

We shall prove Theorem 2.1 by induction, as in [LZ], [RS] and [SA]. Suppose $P(x,y)$ is a real polynomial in x and y. First, we assume that the conclusion of Theorem 2.1 is valid for all polynomials which are sums of monomials of degree less than k in x and of any degree in y, and of monomials which are of degree k in x and of degree less than l in y. Let

$$P(x,y) = a_{kl}x^k y^l + R(x,y)$$

with

$$R(x,y) = \sum_{\alpha<k,\beta} a_{\alpha\beta}x^\alpha y^\beta + \sum_{\beta<l} a_{k\beta}x^k y^\beta$$

satisfying the above induction assumption.

Let us now prove that (2.1) holds for $P(x,y)$. Arguing as in [RS, p. 188], with the aid of weighted theory of one-sided Calderón–Zygmund operators, without loss of generality, we may assume $k>0$, $l>0$ and $|a_{kl}|\ne 0$ (for if $|a_{kl}|=0$, then (2.1) holds by the induction assumption). By dilation invariance of the operators and weights, we only need to consider the case $|a_{kl}|=1$.

We split the kernel K as

$$K(x-y) = K(x-y)\chi_{\{|x-y|\le 1\}}(y) + K(x-y)\chi_{\{|x-y|>1\}}(y) = K_0 + K_\infty,$$

and consider the corresponding splitting $T^+ = T_0^+ + T_\infty^+$:

$$T_0^+ f(x) = \text{p.v.} \int_x^\infty e^{iP(x,y)} K_0(x-y) f(y)\, dy,$$

$$T_\infty^+ f(x) = \int_x^\infty e^{iP(x,y)} K_\infty(x-y) f(y)\, dy.$$

In Section 4, we will prove the following proposition under the induction assumption.

Proposition 2.2. *If $w \in A_1^+$, then there exists a constant C depending on the total degree of P, $C(K)$ and $A_1^+(w)$ such that*

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R} : |T_0^+ f(x)| > \lambda\}) \le C\|f\|_{L^1(w)} \tag{2.2}$$

and

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R} : |T_\infty^+ f(x)| > \lambda\}) \le C\|f\|_{L^1(w)}. \tag{2.3}$$

Obviously, this will complete the proof of Theorem 2.1.

The rest of this paper is devoted to the argument for Proposition 2.2. Section 3 contains some preliminaries which are essential to our proof. In Section 4, we prove Proposition 2.2; this part is partially motivated by [LZ] and [SA].

3. Preliminaries. Let $w \in A_1^+$ and $f \in \mathcal{S}(\mathbb{R})$. We perform the following Calderón–Zygmund decomposition at height $\lambda > 0$.

Lemma 3.1. *We have a collection $\{I\}$ of non-overlapping closed intervals in $\mathbb{R}$ and functions g, b on $\mathbb{R}$ such that*

$$f = g + b, \tag{3.1}$$

$$\lambda \le |I|^{-1} \int_I |f| \le C\lambda, \tag{3.2}$$

$$w\Big(\bigcup I\Big) \le C\lambda^{-1}\|f\|_{L^1(w)}, \tag{3.3}$$

$$\|g\|_{L^1(w)} \le C\|f\|_{L^1(w)}, \tag{3.4}$$

$$\|g\|_\infty \le C\lambda, \tag{3.5}$$

$$b = \sum_I b_I, \quad \text{supp}(b_I) \subset I, \quad \int b_I = 0, \quad \|b_I\|_{L^1} \le C\lambda|I|. \tag{3.6}$$

Proof. Let

$$\{x \in \mathbb{R} : M^+ f(x) > \lambda\} = \bigcup I'$$

be the component decomposition. Let I be the closure of I'. By Lemma 2.1 of [SAW] we see that $|I|^{-1}\int_I |f| \geq \lambda$, which proves (3.2). Define $b_I = (f - |I|^{-1}\int_I f)\chi_I$, $b = \sum_I b_I$ and $g = f\chi_F + \sum_I |I|^{-1}(\int_I f)\chi_I$, where $F = \mathbb{R} \setminus \bigcup I$. Then we only need to prove (3.3) and (3.4) because (3.1), (3.5) and (3.6) are straightforward.

Let I be one of the intervals obtained above. By Lemma 1 of [MOT] and Lemma 2.1 of [SAW], for any positive increasing function U_I on I we have

$$\int_I U_I \leq \lambda^{-1} \int_I U_I |f|. \tag{3.7}$$

Also, since $w \in A_1^+$, by Lemma 2 of [MOT] there exists a positive increasing function $V_{w,I}$ on I such that

$$V_{w,I} \leq Cw \quad \text{a.e. on } I, \quad \int_I w \leq \int_I V_{w,I}, \tag{3.8}$$

where C is independent of I. By (3.7) and (3.8) with $V_{w,I}$ in place of U_I, we can prove (3.3) as follows (see [MOT, p. 520]):

$$\begin{aligned} w\Big(\bigcup I\Big) &\leq \sum \int_I w \leq \sum \int_I V_{w,I} \\ &\leq \lambda^{-1} \sum \int_I V_{w,I}|f| \leq C\lambda^{-1} \sum \int_I |f|w \leq C\lambda^{-1}\|f\|_{L^1(w)}. \end{aligned}$$

The estimate (3.4) can be proved similarly:

$$\begin{aligned} \|g\|_{L^1(w)} &\leq \int_F |f|w + \sum |I|^{-1}\Big|\int_I f\Big| \int_I w \leq \int_F |f|w + C\lambda \sum \int_I V_{w,I} \\ &\leq \int_F |f|w + C\sum \int_I V_{w,I}|f| \leq \int_F |f|w + C\sum \int_I |f|w \leq C\|f\|_{L^1(w)}. \ \blacksquare \end{aligned}$$

We decompose $K_\infty(x,y) = e^{iP(x,y)}K_\infty(x-y) = \sum_{j=0}^\infty K_j(x,y)$, where

$$K_j(x,y) = \varphi(2^{-j}(x-y))K_\infty(x,y),$$

and $\varphi \in C_0^\infty(\mathbb{R})$ is such that $\mathrm{supp}(\varphi) \subset \{1/2 \leq |x| \leq 2\}$ and $\sum_{j=0}^\infty \varphi(2^{-j}x) = 1$ if $|x| \geq 1$. For $j \geq 0$, we define

$$W_j^+(f)(x) = \int K_j(x,y)f(y)\,dy. \tag{3.9}$$

Let

$$W^+(f)(x) = \sum_{j=1}^\infty W_j^+(f)(x).$$

Then $T_\infty = W_0^+ + W^+$. We set

$$\mathcal{B}_i = \sum_{2^{i-1}<|I|\le 2^i} b_I \quad (i\ge 1), \qquad \mathcal{B}_0 = \sum_{|I|\le 1} b_I$$

and put $\mathcal{E} = \bigcup \tilde{I}$, where $\tilde{I}$ denotes the interval with the same right end point as I and with length 100 times that of I. When $x \in \mathbb{R}\setminus\mathcal{E}$, we have

$$\begin{aligned} W^+(b)(x) = W^+\Big(\sum_{i\ge 0}\mathcal{B}_i\Big)(x) &= \sum_{i\ge 0}\sum_{j\ge 1}\int K_j(x,y)\mathcal{B}_i(y)\,dy \\ &= \sum_{s\ge 1}\sum_{j\ge s} W_j^+(\mathcal{B}_{j-s})(x). \end{aligned}$$

Lemma 3.2. *Suppose that $w\in A_1^+$ and s is a positive integer. For $\alpha>0$, put*

$$E_\alpha^s = \Big\{x\in\mathbb{R} : \Big|\sum_{j\ge s} W_j^+(\mathcal{B}_{j-s})(x)\Big| > \alpha\Big\}.$$

Then there exists $\varepsilon>0$ such that

$$w(E_\lambda^s) \le C\lambda^{-1}2^{-\varepsilon s}\int |f(x)|w(x)\,dx.$$

Lemma 3.2 will be proved by applying a variant of the interpolation argument of [V] (see [FS1, FS2]). We first give some lemmas which are essential to our analysis. Some of them are almost the same as their counterparts in [CC], [FS1], [FS2] and [SA]. Our results differ from the previous ones only in that we set them up based on one-sided singular integrals and the weight $w\in A_1^+$. We use some results and notation of [SA]. Let $\lambda>0$ and $\{\mathcal{G}_j\}_{j\ge 0}$ be a family of measurable functions such that

$$\int_I |\mathcal{G}_j| \le \lambda |I|$$

for all intervals I in $\mathbb{R}$ with length $|I| = 2^j$.

Lemma 3.3 (see also [SA]). *Suppose $\sum_{j\ge 0}\|\mathcal{G}_j\|_{L^1}<\infty$. Then, for any positive integer s, we have*

$$\Big\|\sum_{j\ge s} W_j^+(\mathcal{G}_{j-s})\Big\|_{L^2}^2 \le C\lambda 2^{-s}\sum_{j\ge 0}\|\mathcal{G}_j\|_{L^1}.$$

For each $j\ge 0$, let $\mathcal{I}_j$ be a family of non-overlapping closed intervals I such that $|I|\le 2^j$. We assume I and J are non-overlapping if $I\in\mathcal{I}_i$, $J\in\mathcal{I}_j$ for $i\ne j$ and $\sum_{j\ge 0}\sum_{I\in\mathcal{I}_j}|I|<\infty$. Put $\mathcal{I}=\bigcup_{j\ge 0}\mathcal{I}_j$. Let $\lambda>0$. With each $I\in\mathcal{I}$, we associate $f_I\in L^1$ such that $\int|f_I|\le\lambda|I|$ and $\operatorname{supp}(f_I)\subset I$. Define

$$\mathcal{F}_i = \sum_{I\in\mathcal{I}_i} f_I.$$

LEMMA 3.4. *Let $w \in A_1^+$ and s be a positive integer. Then*
$$\Big\|\sum_{j\ge s} W_j^+(\mathcal{F}_{j-s})\Big\|_{L^1(w)} \le C_w\lambda \sum_{J\in\mathcal{I}} |J| \inf_J w,$$
where $\inf_J f = \inf_{x\in J} f(x)$.

Proof. By the triangle inequality we have
$$\Big\|\sum_{j\ge s} W_j^+(\mathcal{F}_{j-s})\Big\|_{L^1(w)} \le \sum_j \sum_{I\in\mathcal{I}_{j-s}} \int |f_I(y)|\Big(\int |K_j(x,y)|w(x)\,dx\Big)\,dy.$$
We note that $K_j(x,y)$ is supported in the interval $[y-2^{j+1}, y-2^{j-1}]$ as a function of x, for each fixed y, and
$$\sup[y-2^{j+1}, y-2^{j-1}] \le \inf I \qquad \text{for all } y\in I\in\mathcal{I}_{j-s}.$$
Also, $|K_j|\le C2^{-j}$. Thus we have
$$\int |f_I(y)|\Big(\int |K_j(x,y)|w(x)\,dx\Big)\,dy \le C\int |f_I(y)| \inf_I M^-(w)\,dy \le C\lambda|I| \inf_I w,$$
where M^- is as in (1.4). Combining the results, we get the conclusion. ∎

Let $\mathcal{J}$ denote the family of intervals arising from the Calderón–Zygmund decomposition of Lemma 3.1.

LEMMA 3.5. *Let $t>0$, $w\in A_1^+$ and s be a positive integer. Let $\mathcal{B}_j, E_\alpha^s$ be as above. Then*
$$\int_{E_\lambda^s} \min(w(x),t)\,dx \le C\sum_{J\in\mathcal{J}} |J| \min\Big(t2^{-s}, \inf_J w\Big). \tag{3.10}$$

Proof. Let
$$\mathcal{J}_t = \Big\{J\in\mathcal{J} : \inf_J w(x) < t2^{-s}\Big\}$$
and $\mathcal{J}_t^c = \mathcal{J}\setminus\mathcal{J}_t$. For $j>0$, put
$$\mathcal{B}_j' = \sum_{2^{j-1}<|J|\le 2^j,\, J\in\mathcal{J}_t} b_J, \qquad \mathcal{B}_j'' = \sum_{2^{j-1}<|J|\le 2^j,\, J\in\mathcal{J}_t^c} b_J,$$
and
$$\mathcal{B}_0' = \sum_{|J|\le 1,\, J\in\mathcal{J}_t} b_J, \qquad \mathcal{B}_0'' = \sum_{|J|\le 1,\, J\in\mathcal{J}_t^c} b_J.$$
Then $\mathcal{B}_j = \mathcal{B}_j' + \mathcal{B}_j''$ for $j\ge 0$. Define
$$E_\alpha' = \Big\{x\in\mathbb{R} : \Big|\sum_{j\ge s} W_j^+(\mathcal{B}_{j-s}')(x)\Big| > \alpha\Big\},$$
$$E_\alpha'' = \Big\{x\in\mathbb{R} : \Big|\sum_{j\ge s} W_j^+(\mathcal{B}_{j-s}'')(x)\Big| > \alpha\Big\},$$

for $\alpha > 0$. Then $E_\lambda^s \subset E'_{\lambda/2} \cup E''_{\lambda/2}$, and hence

$$\int_{E_\lambda^s} \min(w(x), t)\, dx \le \int_{E'_{\lambda/2}} \min(w(x), t)\, dx + \int_{E''_{\lambda/2}} \min(w(x), t)\, dx$$
$$\le \int_{E'_{\lambda/2}} w(x)\, dx + \int_{E''_{\lambda/2}} t\, dx.$$

By Lemmas 3.3 and 3.4, with $\mathcal{G}_j = C_1\mathcal{B}''_j$ and $\mathcal{F}_j = C_2\mathcal{B}'_j$, via Chebyshev's inequality, we have

$$\int_{E'_{\lambda/2}} w(x)\, dx \le C \sum_{J\in\mathcal{J}_t} |J| \inf_J w = C \sum_{J\in\mathcal{J}_t} |J| \min\Big(t2^{-s}, \inf_J w\Big),$$
$$\int_{E''_{\lambda/2}} t\, dx \le Ct2^{-s} \sum_{J\in\mathcal{J}_t^c} |J| = C \sum_{J\in\mathcal{J}_t^c} |J| \min\Big(t2^{-s}, \inf_J w\Big).$$

Combining these estimates, we conclude the proof of Lemma 3.5. ■

Now, we prove Lemma 3.2. Since

$$\int_0^\infty \min(N, t)t^{-1+\theta}\, dt/t = C_\theta N^\theta,$$

for $0 < \theta < 1$, $C_\theta, N > 0$, multiplying both sides of (3.10) by $t^{-1+\theta}$ $(0 < \theta < 1)$, then integrating them on $(0,\infty)$ with respect to the measure dt/t, we get

$$\int_{E_\lambda^s} w(x)^\theta\, dx \le C \sum_{J\in\mathcal{J}} |J| 2^{-(1-\theta)s} \inf_J w^\theta \le C\lambda^{-1} 2^{-(1-\theta)s} \sum_{J\in\mathcal{J}} \inf_J w^\theta \int_J |f(x)|\, dx$$
$$\le C\lambda^{-1}2^{-(1-\theta)s} \int |f(x)| w(x)^\theta\, dx.$$

By Proposition 1.6, if $w \in A_1^+$, then $w^{1+\delta} \in A_1^+$ for some $\delta > 0$. Therefore, we complete the proof of Lemma 3.2 by substituting $w^{1+\delta}$ for w and putting $\theta = 1/(1+\delta)$ in the above inequalities.

Lemma 3.6. *Let W_j^+ be as in (3.9). Suppose $w \in A_1^+$. There exist $C, \delta > 0$ such that*

$$\|W_j^+\|_{L^2(w)} \le C2^{-j\delta}$$

for all $j \ge 1$, where $\|\cdot\|_{L^2(w)}$ denotes the operator norm on $L^2(w)$.

Before proving Lemma 3.6, we first give a lemma obtained by Ricci–Stein.

Lemma 3.7 ([RS]). *For $j \ge 1$, if $k \ne l$, we have*

$$\|W_j^+\|_{L^2} \le C_{k,l} 2^{-j/2-\min(l/k,k/l)j/2}$$

and if $k = l$,

$$\|W_j^+\|_{L^2} \le C_k 2^{-j} j^{1/2}.$$

To prove Lemma 3.6, we apply interpolation with change of measures [SW]. For $j \ge 1$, since

$$|W_j^+(f)| \le C \int_{2^{j-1}+x}^{2^{j+1}+x} \frac{|f(y)|}{|x-y|}\, dy \le CM^+(f)(x),$$

Theorem 1.5 and Proposition 1.6 imply that $\|W_j^+\|_{L^2(w)} \le C$ for $w \in A_1^+$. Consequently,

$$\|W_j^+\|_{L^2(w^{1+\varepsilon})} \le C \tag{3.11}$$

for some $\varepsilon > 0$ for which $w^{1+\varepsilon} \in A_1^+$ (see Proposition 1.6). So, Lemma 3.6 follows from Lemma 3.7 and (3.11) by interpolation with change of measures.

Lemmas 3.2 and 3.6 are essential to the proof of Proposition 2.2.

4. Proof of Proposition 2.2. We first prove (2.2). Take any $h \in \mathbb{R}$, and write

$$P(x,y) = a_{kl}(x-h)^k(y-h)^l + R(x,y,h),$$

where the polynomial $R(x,y,h)$ satisfies the induction assumption for Theorem 2.1, and the coefficients of $R(x,y,h)$ depend on h. Write

$$T_0^+ f(x) = T_{01}^+ f(x) + T_{02}^+ f(x),$$

where

$$T_{01}^+ f(x) = \text{p.v.} \int_x^{1+x} e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})} K(x-y) f(y)\, dy,$$

$$T_{02}^+ f(x) = \text{p.v.} \int_x^{1+x} \{e^{iP(x,y)} - e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})}\} K(x-y) f(y)\, dy.$$

Now we split f into three parts as follows:

$$f = f\chi_{\{|y-h|<1/2\}}(y) + f\chi_{\{1/2\le|y-h|<5/4\}}(y) + f\chi_{\{|y-h|\ge 5/4\}}(y) = f_1 + f_2 + f_3.$$

It is easy to see that $|x-h| < 1/4$ and $|y-h| < 1/2$ imply $|y-x| < 1$, and hence

$$T_{01}^+ f_1(x) = \text{p.v.} \int e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})} K(x-y) f_1(y)\, dy.$$

Thus, from the induction assumption, it follows that

$$w(\{x \in I(h,1/4) : |T_{01}^+ f_1(x)| > \lambda\}) \le \frac{C}{\lambda} \int_{|y-h|<1/2} |f(y)| w(y)\, dy, \tag{4.1}$$

where C is independent of h and of the coefficients of $P(x,y)$. Here and below, $I(x,r)$ denotes the interval $(x-r,x+r)$.

Notice that if $|x-h|<1/4$ and $1/2\le|y-h|<5/4$, then $|y-x|>1/4$. Thus

$$|T_{01}^{+}f_2(x)|\le\int_{x+1/4}^{x+1}|K(x-y)f_2(y)|\,dy\le CM^{+}(f_2)(x).$$

So we have

$$w(\{x\in I(h,1/4):|T_{01}^{+}f_2(x)|>\lambda\})\le\frac{C}{\lambda}\int_{|y-h|<5/4}|f(y)|w(y)\,dy \tag{4.2}$$

for some constant C independent of h and of the coefficients of $P(x,y)$.

Finally, if $|x-h|<1/4$ and $|y-h|\ge 5/4$, then $|y-x|>1$, thus

$$T_{01}^{+}f_3(x)=0. \tag{4.3}$$

From (4.1)–(4.3), it follows that

$$w(\{x\in I(h,1/4):|T_{01}^{+}f(x)|>\lambda\})\le\frac{C}{\lambda}\int_{|y-h|<5/4}|f(y)|w(y)\,dy, \tag{4.4}$$

where C is independent of h and of the coefficients of $P(x,y)$.

Evidently, if $|x-h|<1/4$ and $0<y-x<1$, then

$$|e^{iP(x,y)}-e^{i(R(x,y,h)+a_{kl}(y-h)^{k+l})}|\le C|a_{kl}|\,|x-y|=C(y-x).$$

Therefore, when $|x-h|<1/4$, we have

$$|T_{02}^{+}f(x)|\le C\int_{x}^{x+1}|f(y)|\,dy\le CM^{+}(f(\cdot)\chi_{B(h,5/4)}(\cdot))(x).$$

It follows that

$$w(\{x\in I(h,1/4):|T_{02}^{+}f(x)|>\lambda\})\le\frac{C}{\lambda}\int_{|y-h|<5/4}|f(y)|w(y)\,dy \tag{4.5}$$

for some constant C independent of h and of the coefficients of $P(x,y)$. From (4.4) and (4.5), it follows that the inequality

$$w(\{x\in I(h,1/4):|T_{0}^{+}f(x)|>\lambda\})\le\frac{C}{\lambda}\int_{|y-h|<5/4}|f(y)|w(y)\,dy$$

holds uniformly in $h\in\mathbb{R}$, which implies

$$w(\{x\in\mathbb{R}:|T_{0}^{+}f(x)|>\lambda\})\le\frac{C}{\lambda}\|f\|_{L^1(w)}$$

by integration with respect to h, where C is independent of the coefficients of $P(x,y)$. This completes the proof of (2.2).

Now, we turn to the proof of (2.3). Recall that $T_\infty^+ = W_0^+ + W^+$. It is easy to see that

$$\|W_0^+(f)\|_{L^1(w)} \le C\|f\|_{L^1(w)} \tag{4.6}$$

for $w \in A_1^+$, since

$$\begin{aligned}\int |W_0^+(f)(x)|w(x)\,dx &\le \iint |K_0(x-y)|w(x)\,dx\,|f(y)|\,dy\\ &\le C\int M^- w(y)|f(y)|\,dy \le C\int w(y)|f(y)|\,dy.\end{aligned}$$

So, in the following, we only consider W^+.

Now, we recall the decomposition $f = g + b$ and the set $\mathcal{E} = \bigcup \tilde{I}$ of Section 3, and we see that

$$\begin{aligned}&w(\{x\in\mathbb{R}\setminus\mathcal{E} : |W^+(f)(x)| > \lambda\})\\ &\le w(\{x\in\mathbb{R}\setminus\mathcal{E} : |W^+(g)(x)| > \lambda/2\}) + w(\{x\in\mathbb{R}\setminus\mathcal{E} : |W^+(b)(x)| > \lambda/2\})\\ &\le C\lambda^{-2}\|W^+(g)\|_{L^2(w)}^2 + w\Big(\Big\{x\in\mathbb{R}^n : \Big|\sum_{s\ge1}\sum_{j\ge s} W_j^+(\mathcal{B}_{j-s})(x)\Big| > \lambda/2\Big\}\Big).\end{aligned}$$

From Lemma 3.6 we easily see that W^+ is bounded on $L^2(w)$. It follows that $\lambda^{-2}\|W^+(g)\|_{L^2(w)}^2$ is bounded by $C\lambda^{-1}\|f\|_{L^1(w)}$ via (3.4) and (3.5). Checking the constants appearing in the proof of Lemma 3.2 and replacing K by $c2^{\delta s}K$, we have

$$w(E^s_{c_\delta 2^{-\delta s}\lambda}) \le c\lambda^{-1}2^{-\tau s}\|f\|_{L^1(w)},$$

where δ and τ are positive constants depending on w, and c_δ is a constant satisfying $\sum_{s\ge1} c_\delta 2^{-\delta s} = 1/2$. Thus, we have

$$\begin{aligned}&w\Big(\Big\{x\in\mathbb{R}^n : \Big|\sum_{s\ge1}\sum_{j\ge s} W_j^+(\mathcal{B}_{j-s})(x)\Big| > \lambda/2\Big\}\Big)\\ &\qquad\le \sum_{s\ge1} w(E^s_{c_\delta 2^{-\delta s}\lambda}) \le C\lambda^{-1}\|f\|_{L^1(w)}.\end{aligned}$$

Therefore,

$$w(\{x\in\mathbb{R}\setminus\mathcal{E} : |W^+(f)(x)| > \lambda\}) \le C\lambda^{-1}\|f\|_{L^1(w)}. \tag{4.7}$$

On the other hand, by (3.3) and the estimate $w(\tilde{I}) \le Cw(I)$, which is easily proved for $w\in A_1^+$, we see that

$$w(\mathcal{E}) \le C\lambda^{-1}\|f\|_{L^1(w)}. \tag{4.8}$$

By (4.7) and (4.8) for $w \in A_1^+$, we get

$$w(\{x\in\mathbb{R} : |W^+(f)(x)| > \lambda\}) \le C\lambda^{-1}\|f\|_{L^1(w)}. \tag{4.9}$$

Now (4.6) and (4.9) imply

$$w(\{x\in\mathbb{R} : |T_\infty^+(f)(x)| > \lambda\}) \le C\lambda^{-1}\|f\|_{L^1(w)}$$

for $w \in A_1^+$ with a constant C independent of the coefficients of $P(x,y)$, which completes the proof of (2.3).

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