STUDIA MATHEMATICA 207 (2) (2011)

Geometry of the Banach spaces $C(\beta\mathbb{N} \times K, X)$ for compact metric spaces K

by

DALE E. ALSPACH (Stillwater, OK) and
ELÓI MEDINA GALEGO (São Paulo)

Abstract. A classical result of Cembranos and Freniche states that the $C(K, X)$ space contains a complemented copy of c_0 whenever K is an infinite compact Hausdorff space and X is an infinite-dimensional Banach space. This paper takes this result as a starting point and begins a study of conditions under which the spaces $C(\alpha)$, $\alpha < \omega_1$, are quotients of or complemented in $C(K, X)$.

In contrast to the c_0 result, we prove that if $C(\beta\mathbb{N} \times [1, \omega], X)$ contains a complemented copy of $C(\omega^\omega)$ then X contains a copy of c_0. Moreover, we show that $C(\omega^\omega)$ is not even a quotient of $C(\beta\mathbb{N} \times [1, \omega], \ell_p)$, $1 < p < \infty$.

We then completely determine the separable $C(K)$ spaces which are isomorphic to a complemented subspace or a quotient of a $C(\beta\mathbb{N} \times [1, \alpha], \ell_p)$ space for countable ordinals α and $1 \le p < \infty$. As a consequence, we obtain the isomorphic classification of the $C(\beta\mathbb{N} \times K, \ell_p)$ spaces for infinite compact metric spaces K and $1 \le p < \infty$. Indeed, we establish the following more general cancellation law. Suppose that the Banach space X contains no copy of c_0 and K_1 and K_2 are infinite compact metric spaces, then the following statements are equivalent:

(1) $C(\beta\mathbb{N} \times K_1, X)$ is isomorphic to $C(\beta\mathbb{N} \times K_2, X)$.
(2) $C(K_1)$ is isomorphic to $C(K_2)$.

These results are applied to the isomorphic classification of some spaces of compact operators.

1. Introduction. The isomorphic classification of the separable spaces of continuous functions on a compact Hausdorff space was completed in 1966 when Milyutin [22], [24] showed that there was a single isomorphism class for the continuous functions on uncountable compact metric spaces. For general compact Hausdorff spaces some work has been done in special cases, e.g., [15] or [18], but, unlike the isometric case which is completely determined by the

2010 *Mathematics Subject Classification*: Primary 46B03; Secondary 46B25.
Key words and phrases: isomorphic classifications of $C(K, X)$ spaces, Stone–Čech compactification, compact metric spaces, isomorphic classifications of spaces of compact operators.

DOI: 10.4064/sm207-2-4

Banach–Stone theorem, extended in [2] and [5], the isomorphic classification seems hopeless.

In this paper we consider a special class of compact Hausdorff spaces but allow the range space to be a Banach space instead of $\mathbb{R}$. Thus we study the spaces $C(K, X)$ of continuous functions from K into X where X is a Banach space, K is a compact Hausdorff space and the norm of an element f is $\|f\| = \sup_{k\in K} \|f(k)\|_X$. Usually K will be a compact metric space or a product of a compact metric space and the Stone–Čech compactification of the natural numbers, $\beta\mathbb{N}$. Our interest in $\beta\mathbb{N}$ stems from application of some of the results to the following question, [11, Problem 4.2.2]. From now on $\mathcal{K}(X, Y)$ denotes the space of compact operators from X to another Banach space Y and $[1, \alpha]$ is the compact Hausdorff space of ordinals between 1 and α in the order topology.

Problem 1.1. *Classify, up to an isomorphism, the spaces of compact operators* $\mathcal{K}(\ell_1, C([1, \alpha], \ell_p))$, *where* $\alpha \geq \omega$ *and* $1 \leq p < \infty$.

This problem covers some cases remaining from the development in [11]–[14] and [25] of the isomorphic classification of some spaces of compact operators. We give the solution to the above problem in the case where α is countable. The connection to the spaces $C(K, X)$ comes through the injective tensor product. Notice that that since ℓ_1 has the approximation property, by [7, Proposition 5.3] we know that for every ordinal α and $1 \leq p < \infty$,

$$\mathcal{K}(\ell_1, C([1, \alpha], \ell_p)) \sim C(\beta\mathbb{N} \times [1, \alpha], \ell_p).$$

The notation for the spaces is a bit cumbersome so we will shorten some expressions. When the context clearly requires a compact Hausdorff space we will write α rather than $[1, \alpha]$. In particular, $C(\alpha, X) = C([1, \alpha], X)$. If $X = \mathbb{R}$, we will write $C(K)$ rather than $C(K, \mathbb{R})$. We will also adopt some standard notational conventions from Banach space theory. We write $X \sim Y$ when the Banach spaces X and Y are isomorphic, $Y \hookrightarrow X$ when X contains a copy of Y, that is, a subspace isomorphic to Y, $Y \overset{c}{\hookrightarrow} X$ if X contains a complemented copy of Y, and $X \twoheadrightarrow Y$ when Y is a quotient of X. For other notation and terminology we refer the reader to [16] and [20].

In $C(K, X)$ an obvious part of the difficulty with the isomorphic classification is that structures in X can be used to find an alternate compact Hausdorff space K_1 so that $C(K_1, X)$ is isomorphic to $C(K, X)$. A second difficulty is that structures may arise that are present in neither $C(K)$ nor X. Consider the following result, which was obtained independently by Cembranos [6, Main Theorem] and Freniche [9, Corollary 2.5].

Theorem 1.2. *Let* K *be an infinite compact Hausdorff space and* X *an infinite-dimensional Banach space. Then*

$$c_0 \overset{c}{\hookrightarrow} C(K, X).$$

Consequently, both $C(\beta\mathbb{N}, C(\beta\mathbb{N}))$ and $C(\beta\mathbb{N}, \ell_2)$ contain a complemented subspace isomorphic to c_0 despite the fact that neither $C(\beta\mathbb{N})$ nor ℓ_2 contain complemented copies of c_0.

It is natural to ask if there are other $C(K)$ spaces for which the analogous result holds. The next more complicated $C(K)$ space is $C(\omega^\omega)$. Our first result gives a negative answer in this case. We prove that even when $C(K)$ contains a complemented copy of c_0 and X is an infinite-dimensional Banach space, $C(K, X)$ may not contain a complemented copy of $C(\omega^\omega)$. Indeed, it is easy to see that $C(\beta\mathbb{N} \times \omega)$ contains a complemented copy of c_0. However, in Section 3, we prove the following.

THEOREM 3.2. *Let X be a Banach space. Then*

$$C(\omega^\omega) \overset{c}{\hookrightarrow} C(\beta\mathbb{N} \times \omega, X) \ \Rightarrow\ c_0 \hookrightarrow X.$$

We then extend Theorem 3.2 to larger ordinals by using that result and the structure of the ordinals.

THEOREM 3.5. *Let X be a Banach space containing no copy of c_0, K an infinite compact metric space and $0 \le \alpha < \omega_1$. Then*

$$C(K) \overset{c}{\hookrightarrow} C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X) \Leftrightarrow C(K) \sim C(\omega^{\omega^\xi}) \quad \textit{for some } 0 \le \xi \le \alpha.$$

In Section 4 we turn our attention to spaces of the form $C(\beta\mathbb{N} \times \alpha, X)$ where X satisfies some geometrical properties, (†) and (‡), that are modeled on simple properties of ℓ_p, $1 \le p < \infty$. In particular we show that $C(\omega^\omega)$ is not a quotient of $C(\beta\mathbb{N} \times \omega, \ell_p)$, $1 < p < \infty$.

THEOREM 4.2. *Suppose that X is a Banach space satisfying the daggers. Then $C(\omega^\omega)$ is not a quotient of $C(\beta\mathbb{N} \times \omega, X)$.*

Similar to the way we obtained Theorem 3.5 from Theorem 3.2 in Section 4, we also extend Theorem 4.2 by proving

THEOREM 4.5. *Let K be an infinite compact metric space and $0 \le \alpha < \omega_1$. Then*

$$C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, \ell_p) \twoheadrightarrow C(K) \ \Leftrightarrow\ C(K) \sim C(\omega^{\omega^\xi}) \quad \textit{for some } 0 \le \xi \le \alpha.$$

The next section concerns the isomorphic classification of the spaces $C(\beta\mathbb{N} \times \alpha, \ell_p)$. Our results also provide us with immediate information about the isomorphic classifications of a wider class of Banach spaces, namely, the $C(\beta\mathbb{N} \times K, X)$ spaces, where X contains no copy of c_0 and K is a metrizable compact space, that is, $C(K)$ is a separable space. Indeed, in Section 3 we prove the following cancellation law which is the main application of the results of the paper. The case $X = \ell_p$, $1 \le p < \infty$, gives the solution to Problem 1.1.

Theorem 3.7. *Let X be a Banach space containing no copy of c_0. Then for any infinite compact metric spaces K_1 and K_2 we have*

$$C(\beta\mathbb{N} \times K_1, X) \sim C(\beta\mathbb{N} \times K_2, X) \;\Leftrightarrow\; C(K_1) \sim C(K_2).$$

Moreover, in Section 5, we accomplish the isomorphic classification of the spaces $C(\beta\mathbb{N} \times K, \ell_p)$ by considering also the case where K is finite. In order to do this, we first prove a general result about the spaces of compact operators $\mathcal{K}(\ell_p(X), \ell_q(Y))$ (Theorem 5.3). From that we deduce the following.

Corollary 5.4. *$C(\beta\mathbb{N}, \ell_q)$ is isomorphic to $C(\omega \times \beta\mathbb{N}, \ell_q)$ for $1 \le q < \infty$.*

Finally, in Section 6, we pose some elementary questions which this work raises.

2. Preliminaries. In this section we recall some results that we will use in what follows.

In 1920 Mazurkiewicz and Sierpiński showed that if K is a countable compact metric space then it is homeomorphic to an interval of ordinals $[1, \alpha]$ with $\omega \le \alpha < \omega_1$ [21]. This was used in the isomorphic classification of the $C(\alpha)$ spaces, $\omega \le \alpha < \omega_1$, obtained in 1960 by Bessaga and Pełczyński. They showed that if $\omega \le \alpha \le \beta < \omega_1$ then $C(\alpha)$ is isomorphic to $C(\beta)$ if and only if $\beta < \alpha^\omega$ (see [4] and [24]). In particular this means that the spaces $C(\omega^{\omega^\gamma})$, for $0 \le \gamma < \omega_1$, are a complete set of representatives of the isomorphism classes of $C(K)$ where K is a countably infinite, compact metric space.

Bessaga and Pełczyński actually prove some things for $C(K, X)$, where X is a Banach space.

Proposition 2.1. *Suppose X is a Banach space and α is an infinite ordinal. Then $C(\alpha, X)$ is isomorphic to*

(1) $C_0(\alpha, X) = \{f \in C(\alpha, X) : f(\alpha) = 0\}$

and to

(2) $C(\omega^{\omega^\beta}, X)$ *whenever* $\omega^{\omega^\beta} \le \alpha < \omega^{\omega^{\beta+1}}$.

Thus for some Banach spaces X there may be fewer isomorphism classes for the spaces $C(K, X)$ with K countable, compact metric, than for the case $X = \mathbb{R}$. That is what happens for Banach spaces which are isomorphic to their squares or to the c_0-sum of infinitely many copies of the space. Indeed, for ℓ_p, $1 \le p < \infty$, the finite ordinals all yield the same space; for c_0 all of the spaces $C(\alpha, c_0)$, $\alpha < \omega^\omega$, are isomorphic.

Remark 2.2. The order structure on the spaces of ordinals make it easy to find contractively complemented subspaces of $C(\omega^\alpha)$ isometric to $C(\omega^\beta)$ for $\beta < \alpha$. Indeed, if A is a closed subset of $[1, \omega^\alpha]$ and $A^{(1)}$ is the set of non-isolated points of A, we can define a subspace Y of $C(\omega^\alpha)$ isometric to $C(A)$ by

$$Y = \{f \in C(\omega^\alpha) : f(\gamma) = f(\xi) \text{ for all } \gamma \text{ such that} \\ \sup\{\rho < \xi : \rho \in A\} < \gamma < \xi \text{ and } \xi \in A \setminus A^{(1)}\}.$$

We can define a projection onto Y by restricting to A and then extending by the formula in the definition of Y, i.e.,

$$Lg(\gamma) = g(\xi) \text{ for all } \gamma \text{ such that} \\ \sup\{\rho < \xi : \rho \in A\} < \gamma < \xi \text{ and } \xi \in A \setminus A^{(1)}.$$

For $\gamma > \sup A$ let $Lg(\gamma) = 0$.

The spaces c_0 and $C(\beta\mathbb{N})$ play a prominent role in this paper so we now recall some important properties of these spaces. Bessaga and Pełczyński made a study of c_0 in [3] and introduced the notion of a *weakly unconditionally Cauchy* (w.u.c.) sequence. A sequence (x_n) in a Banach space X is said to be w.u.c. if and only if for every $x^* \in X^*$, $\sum_n |x^*(x_n)| < \infty$. A sequence equivalent to the standard basis of c_0 is clearly w.u.c. We will use the following result from their paper.

PROPOSITION 2.3. *Suppose that X is a Banach space which has no subspace isomorphic to c_0. Then every w.u.c. sequence in X is unconditionally convergent. Consequently, if (x_n) is w.u.c. in X, then* $\lim_n \|x_n\| = 0$.

$C(\beta\mathbb{N})$ is isometric to $\ell_\infty = \ell_\infty(\mathbb{N})$, the space of bounded sequences with the supremum norm. For any non-empty index set Γ, $\ell_\infty(\Gamma)$ is *injective*, i.e., it is complemented in any space which contains it. Furthermore, c_0 is *separably injective*, i.e., it is complemented in any separable Banach space that contains it. c_0 is not complemented in ℓ_∞ and in fact the only infinite-dimensional complemented subspaces of c_0 or ℓ_∞ are isomorphs of the whole space [20, pp. 54 and 57]. ℓ_∞ is an example of a *Grothendieck space*, i.e., any weak* convergent sequence in the dual is actually weakly convergent [8, p. 179]. Actually this is the essential property of $C(\beta\mathbb{N})$ that we use. Except in one or two cases, e.g., Theorem 5.5, the results could be rewritten with $C(K)$ which is a Grothendieck space in place of $C(\beta\mathbb{N})$.

While $C(\beta\mathbb{N})$ and ℓ_∞ are isometric, for many infinite-dimensional Banach spaces X, $C(\beta\mathbb{N}, X)$ is not isomorphic to $\ell_\infty(X) = \{(x_n) : x_n \in X$ for all n, $\|(x_n)\| = \sup_n \|x_n\|_X < \infty\}$. This is the case if X does not contain a complemented subspace isomorphic to c_0 since, by [19], $\ell_\infty(X)$ only contains a complemented subspace isomorphic to c_0 if X does.

We identify $C(\beta\mathbb{N}, X)^*$ with the space of X^*-valued regular Borel measures μ on $C(\beta\mathbb{N})$ with $\|\mu\| = \sup \sum_n \|\mu(A_n)\|_{X^*}$ where the supremum is over all partitions of $\beta\mathbb{N}$ into disjoint clopen sets A_n [8, p. 182]. Moreover if

(μ_n) is a weak* convergent sequence of measures with limit μ, then for any clopen set A, $(\mu_n(A))$ converges weak* in X^* to $\mu(A)$.

It will be convenient at times to shift the point of view as to the underlying compact Hausdorff space and the range space. Thus we will use the fact that $C(K_1 \times K_2, X)$ is isomorphic to $C(K_1, C(K_2, X))$ and to $C(K_2, C(K_1, X))$ where K_1 and K_2 are compact Hausdorff spaces. Also because $C(K, X)$ is isometric to the injective tensor product $C(K) \check{\otimes} X$, we may replace K by K_1 if $C(K)$ is isomorphic to $C(K_1)$.

Let $\max_\sigma\{\beta_1, \beta_2\}$ be the largest ordinal $\beta = \gamma_1 + \alpha_1 + \cdots + \gamma_k + \alpha_k$ obtained by writing $\beta_1 = \gamma_1 + \cdots + \gamma_k$ and $\beta_2 = \alpha_1 + \cdots + \alpha_k$, where $\gamma_j \geq 0$ and $\alpha_j \geq 0$ for all j. This can also be obtained by writing the ordinals β_1 and β_2 in terms of prime components and arranging the terms of the sum in decreasing order.

The topological results in [21] are based on the notion of derived set. Recall that $K^{(0)} = K$. For any ordinal α, $K^{(\alpha+1)}$ is the set of non-isolated points in $K^{(\alpha)}$, and for a limit ordinal β, $K^{(\beta)} = \bigcap_{\alpha<\beta} K^{(\alpha)}$. We will only use this with countable compact spaces and will refer to the smallest ordinal α such that $K^{(\alpha)} \neq \emptyset$ and $K^{(\alpha+1)} = \emptyset$ as the *derived order* of K.

LEMMA 2.4. *Let K_1 and K_2 be countable compact metric spaces and β_1, β_2 be countable ordinals such that $K_1^{(\beta_1)}$ and $K_2^{(\beta_2)}$ are finite non-empty sets. Then $(K_1 \times K_2)^{(\max_\sigma\{\beta_1,\beta_2\})}$ is a finite non-empty set.*

Sketch of proof. First we can assume that $K_1^{(\beta_1)}$ and $K_2^{(\beta_2)}$ are singletons, k_1 and k_2, respectively. The proof is by induction on β_2 and for each β_2 on β_1, $0 \leq \beta_1 \leq \beta_2$. The result is clear for $\beta_2 = 0, 1$ and $\beta_1 = 0, 1$ and for all $\beta_2 < \omega_1$ and $\beta_1 = 0$. Assume $1 < \beta_2$, $0 < \beta_1 \leq \beta_2$ and that the result holds for K_2 of derived order $\beta < \beta_2$ and K_1 of derived order $\gamma \leq \beta$ and for $\beta = \beta_2$ and $\gamma < \beta_1$.

To see that

$$(K_1 \times K_2)^{(\max_\sigma\{\beta_1,\beta_2\})} = \{(k_1, k_2)\},$$

notice that by the induction assumption, if $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, $m_1 \in K_1^{(\alpha_1)} \setminus K_1^{(\alpha_1+1)}$ and $m_2 \in K_2^{(\alpha_2)} \setminus K_2^{(\alpha_2+1)}$, then

$$(m_1, m_2) \in (C_1 \times C_2)^{(\max_\sigma\{\alpha_1,\alpha_2\})},$$

where C_1 and C_2, are appropriately chosen clopen subsets of K_1 and K_2, respectively. For $i = 1, 2$ write $K_i \setminus \{k_i\}$ as a disjoint union of clopen sets $C_{1,n}$, of derived order $\beta_i - 1$ for all n or of derived order $\beta_{i,n}$ with $(\beta_{i,n})$ increasing to β_i. Each set $C_{1,n} \times [1, \beta_2]$ and $[1, \beta_1] \times C_{2,n}$ satisfies the induction hypothesis (possibly symmetrized). Notice that

$$\bigcup_n C_{1,n} \times [1, \beta_2] \cup [1, \beta_1] \times C_{2,n} = K_1 \times K_2 \setminus \{(k_1, k_2)\}.$$

There are four cases to check. It is easy to see that

$$\max_{\sigma}\{\beta_1, \beta_2 - 1\} + 1 = \max_{\sigma}\{\beta_1, \beta_2\},$$

in the first case, and $\max_\sigma\{\beta_1, \beta_{2,n}\}$ increases to $\max_\sigma\{\beta_1, \beta_2\}$ in the second case. The other two cases are similar. ∎

The lemma shows that we do not really gain anything from simple manipulations of compact metric spaces. Indeed, for countable ordinals α and γ, we have

$$C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, C(\omega^{\omega^\gamma}, X)) \sim C(\beta\mathbb{N} \times \omega^{\omega^\alpha} \times \omega^{\omega^\gamma}, X).$$

However

$$\omega^{\omega^{\max\{\alpha,\gamma\}}} \le \omega^{\max_\sigma\{\omega^\alpha, \omega^\gamma\}} \le \omega^{\omega^{\max\{\alpha,\gamma\}}2}$$

and thus

$$C(\beta\mathbb{N} \times \omega^{\omega^\alpha} \times \omega^{\omega^\gamma}, X) \sim C(\beta\mathbb{N} \times \omega^{\omega^{\max\{\alpha,\gamma\}}}, X).$$

In a series of papers from the 1970's the first author developed some tools for working with subspaces of $C(K)$ spaces isomorphic to $C(\alpha)$. Some of the proofs in this paper are motivated in part by that work, and versions of some of the technical tools will be needed here. The first is similar to [1, Lemma 2.5].

Lemma 2.5. *Given a positive integer k and $\epsilon > 0$ there is a positive integer n such that if $(x^*_\alpha)_{\alpha \le \omega^n}$ is a sequence in the unit ball of the dual of a Banach space X such that the function $\alpha \mapsto x^*_\alpha$ is an order-to-weak* continuous map, then there is a closed subset B of $[1, \omega^n]$, order isomorphic and homeomorphic via the order isomorphism to $[1, \omega^k]$ such that*

$$\big|\|x^*_\beta\| - \|x^*_{\beta'}\|\big| < \epsilon \quad \text{for all } \beta, \beta' \in B.$$

The next lemma is a vector valued version of a typical construction of a sequence of disjointly supported functions normed by a sequence of measures.

Lemma 2.6. *Suppose that X is a Banach space, C, D are positive constants, K is a compact Hausdorff space, (μ_n) is a sequence of elements of $C(K, X)^*$ represented as X^*-valued measures on K, with $\|\mu_n\| \le C$ for all n, and (g_n) is a sequence of norm at most D elements of $C(K, X)$ such that*

$$\int g_n \, d\mu_n \ge 1 \quad \text{for all } n$$

and $\|g_n(t)\| \to 0$ as $n \to \infty$ for all $t \in K$. Then for any $\epsilon > 0$ there are an infinite subset M of $\mathbb{N}$ and open subsets $(G_m)_{m\in M}$ of K such that $\overline{G_m} \cap \overline{G_j} = \emptyset$ if $m \ne j$, and

$$\int g_m 1_{G_m} \, d\mu_m > 1 - \epsilon \quad \text{for all } m \in M.$$

Proof. Let $\epsilon > 0$, and $\epsilon_k = \epsilon/2^{k+2}$ for $k \in \mathbb{N}$. Choose $\rho > 0$ such that $\rho < \epsilon/(4C)$. Then

$$\int g_n 1_{\{t:\, \|g_n(t)\|\ge\rho\}}\, d\mu_n > 1-\epsilon/4, \quad \forall n,\ 1\le n<\omega.$$

Because $\|g_n(t)\|$ converges to 0, $\int \|g_n(t)\|\, d|\mu_i|(t)$ converges to 0 for each i.

Thus for $i = 1,\dots,j_1$ with $j_1 > C/\epsilon_1$, there are infinite subsets $\mathbb{N} \supset N_1 \supset \cdots \supset N_{j_1}$ such that, setting $n_1 = 1$ and $n_{i+1} = \min N_i$,

$$\sum_{n\in N_i} |\mu_{n_i}|(\{t : \|g_n(t)\| > \rho/2\}) \le \sum_{n\in N_i} (2/\rho)\int \|g_n(t)\|\, d|\mu_{n_i}|(t) < \epsilon/(4C).$$

For $i = 1,\dots,j_1$, let

$$A_i = \{t : \|g_{n_i}(t)\| \ge \rho\} \setminus \bigcup_{n\in N_i} \{t : \|g_n(t)\| > \rho/2\}.$$

Find disjoint open sets $H_1,\dots,H_{j_1}$ such that

$$\{t : \|g_{n_i}(t)\| > \rho/2\} \supset H_i \supset A_i$$

for each i. Because $\|\mu_m\| \le C$ for all m and the sets H_i are disjoint, for some infinite subset M_1 of N_{j_1} and some i_1, $1 \le i_1 \le j_1$,

$$|\mu_m|(H_{i_1}) < \epsilon/8 = \epsilon_1$$

for all $m \in M_1$. Let $m_1 = n_{i_1}$ and G_{m_1} be an open set containing A_{i_1} such that $\overline{G_{m_1}} \subset H_{i_1}$. Then m_1 is the first element of M, and μ_{m_1} and G_{m_1} are the corresponding measure and open set.

Let $K_1 = K \setminus H_{i_1}$. Now notice that if we consider $(\mu_m|_{K_1})_{m\in M_1}$ and $(g_m|_{K_1})_{m\in M_1}$ we have the original situation with 1 replaced by $1-\epsilon/8$ as the lower bound on

$$\int_{K_1} g_m\, d\mu_m.$$

Thus repeating the argument above with ϵ_2 but choosing open sets as open subsets of $K\setminus\overline{G_{m_1}}$ rather than K_1 (and hence open in K), we get μ_{m_2} and G_{m_2} with $G_{m_1} \cap G_{m_2} = \emptyset$.

Continuing in this way we can construct the required indices and open sets. ∎

3. Complemented separable $C(K)$ subspaces of $C(\beta\mathbb{N}\times\alpha, X)$. It is clear that for any Banach space X, $C(\beta\mathbb{N}\times\omega, X)$ contains a complemented copy of c_0. This section is devoted to proving that c_0 is, up to isomorphism, the only separable $C(K)$ space which is complemented in $C(\beta\mathbb{N}\times\omega, X)$ whenever X contains no copy of c_0. This is a direct consequence of Theorem 3.2 below.

The next lemma is a technical analog of a result of Bessaga and Pełczyński [3, Theorem 4].

LEMMA 3.1. *Suppose that X and Y are Banach spaces and that T is an operator from $C(\beta\mathbb{N}, X)$ into Y. If there exist an element f of $C(\beta\mathbb{N}, X)$, $\delta > 0$, a sequence (G_n) of disjoint non-empty clopen subsets of $\beta\mathbb{N}$ and a sequence (μ_n) of X^*-valued measures contained in $T^*(B_{Y^*})$ such that*

$$\Big|\int f1_{G_n}\,d\mu_n\Big| > \delta \quad \text{for all } n,$$

then there is a subspace Z of $C(\beta\mathbb{N}, X)$ such that Z is isomorphic to ℓ_∞ and $T|_Z$ is an isomorphism into Y.

Proof. Let $|\mu_n|$ denote the real-valued total variation measure induced by μ_n. Observe that the sequence of pairs $(|\mu_n|, G_n)$ satisfy the hypotheses of Rosenthal's disjointness lemma [8, Lemma 1, p. 18]. Therefore there exists a subsequence $(|\mu_n|, G_n)_{n\in M}$ such that for all $n \in M$,

$$\sum_{j\in M,\, j\neq n} |\mu_n|(G_j) < \delta/(8\|f\|).$$

We also need to have

$$|\mu_n|\Big(\overline{\bigcup_{j\in M} G_j} \setminus \bigcup_{j\in M} G_j\Big) < \delta/(8\|f\|) \quad \text{for all } n \in M.$$

If for some $n \in M$,

$$|\mu_n|\Big(\overline{\bigcup_{j\in M} G_j} \setminus \bigcup_{j\in M} G_j\Big) \geq \delta/(8\|f\|),$$

we can argue as follows. Partition M into an infinite number of infinite sets M_k. If for some k, for all $n \in M_k$,

$$|\mu_n|\Big(\overline{\bigcup_{j\in M_k} G_j} \setminus \bigcup_{j\in M_k} G_j\Big) < \delta/(8\|f\|),$$

we can continue with M_k in place of M. If not, for each k choose $n_k \in M_k$ such that

$$|\mu_{n_k}|\Big(\overline{\bigcup_{j\in M_k} G_j} \setminus \bigcup_{j\in M_k} G_j\Big) \geq \delta/(8\|f\|).$$

Let $M^1 = \{n_k : k \in \mathbb{N}\}$. Observe that for all k,

$$\Big(\overline{\bigcup_{j\in M_k} G_j} \setminus \bigcup_{j\in M_k} G_j\Big) \cap \overline{\bigcup_{j\in M^1} G_j} = \emptyset.$$

Now if for all $n \in M^1$,

$$|\mu_n|\Big(\overline{\bigcup_{j\in M^1} G_j} \setminus \bigcup_{j\in M^1} G_j\Big) < \delta/(8\|f\|),$$

we can use M^1 in place of M. If not, notice that for all $n \in M^1$,

$$\|\mu_n|_{\overline{\bigcup_{j\in M^1} G_j}}\| \leq \|\mu_n\| - \delta/(8\|f\|).$$

We can split M^1 into infinitely many infinite sets and repeat the previous argument. Each time this process reduces the norm of the part of μ_n under consideration by $\delta/(8\|f\|)$. Thus in at most $\|f\|\,\|T\|8/\delta$ repetitions of the argument we will find the required infinite set M such that for all $n \in M$,

$$|\mu_n|\Big(\overline{\bigcup_{j\in M} G_j} \setminus \bigcup_{j\in M} G_j\Big) < \delta/8,$$

and

$$\sum_{j\in M,\, j\neq n} |\mu_n|(G_j) < \delta/(8\|f\|).$$

Let Z be given by

$$\Big\{g \in C(\beta\mathbb{N}, X) : g(t) = 0 \ \ \forall t \notin \overline{\bigcup_{n\in M} G_n},\ g1_{G_n} = c_n f1_{G_n},\ c_n \in \mathbb{R} \ \ \forall n \in M\Big\}.$$

Because the range of f is compact, for any bounded sequence $(c_n)_{n\in M}$ of real numbers, the function h defined on $\mathbb{N}$ by

$$h(k) = \begin{cases} 0 & \text{if } k \notin \bigcup_{m\in M} G_m, \\ c_n f(k) & \text{if } k \in G_n \text{ and } n \in M, \end{cases}$$

is in $\ell_\infty(X)$ with relatively compact range and hence extends continuously to some function H on $\beta\mathbb{N}$ with values in the symmetric radial hull of $\|(c_n)\|_\infty$ times the range of f. Moreover because $\mathbb{N}$ is dense, the extension is unique and must agree with $c_n f1_{G_n}$ on G_n for all $n \in M$, and be 0 on the closure of

$$\Big\{k \in \mathbb{N} : k \notin \bigcup_{n\in M} G_n\Big\}.$$

Therefore Z is isomorphic to ℓ_∞, and for (c_n) and H as above,

$$\begin{aligned}(\delta/\|T\|)\|(c_n)_{n\in M}\|_\infty &\le \inf_{n\in M} \|f1_{G_n}\|\,\|(c_n)_{n\in M}\|_\infty \\ &\le \|H\| \le \|f\|\,\|(c_n)_{n\in M}\|_\infty.\end{aligned}$$

Continuing with the same notation, we can get a lower bound on $\|TH\|$ as follows. Observe that for each $n \in M$,

$$\begin{aligned}&\Big|\int H\,d\mu_n\Big| \ge \\ &\Big|\int H1_{G_n}\,d\mu_n\Big| - \sum_{j\in M, j\neq n} |\mu_n|(G_j)\|f\||c_j| - |\mu_n|\Big(\overline{\bigcup_{j\in M} G_j} \setminus \bigcup_{j\in M} G_j\Big)\|f\|\,\|(c_j)\|_\infty \\ &\qquad \ge |c_n|\Big|\int f1_{G_n}\,d\mu_n\Big| - \|(c_j)_{j\in M}\|_\infty \delta/4 \ge \delta(|c_n| - \|(c_j)_{j\in M}\|_\infty/4).\end{aligned}$$

Taking the supremum over n and noting that $\mu_n \in T(B_{Y^*})$ completes the proof. ∎

THEOREM 3.2. *Let X be a Banach space. Then*

$$C(\omega^\omega) \overset{c}{\hookrightarrow} C(\beta\mathbb{N} \times \omega, X) \;\Rightarrow\; c_0 \hookrightarrow X.$$

Proof. Assume that X does not contain a subspace isomorphic to c_0. We will show that the existence of a complemented subspace isomorphic to $C(\omega^\omega)$ produces the situation in the hypothesis of the previous lemma. First we will reduce to a simplified situation. By Proposition 2.1(1), $C(\beta\mathbb{N}\times\omega, X)$ is isomorphic to $C_0(\omega \times \beta\mathbb{N}, X)$, i.e., the c_0-sum of $C(\beta\mathbb{N}, X)$. Assume now that T is a projection from $C_0(\omega \times \beta\mathbb{N}, X)$ onto a subspace Y isomorphic to $C(\omega^\omega)$. Let $S : Y \to C(\omega^\omega)$ be the isomorphism and suppose, without loss of generality, that $\|S\| \le 1$. Then T^*S^* is an isomorphism with some lower bound $\epsilon > 0$, i.e.,

$$\|T^*S^*z\| \ge \epsilon\|z\| \quad \text{for all } z \in C(\omega^\omega)^*.$$

Choose N by Lemma 2.5 so that for $n > 8\|T\|$, there exists a subfamily $\{\mu_\beta : \beta \le \omega^n\}$ of $\{T^*S^*\delta_\gamma : \gamma \le \omega^N\}$ such that $\beta \mapsto \mu_\beta$ is a (order-to-weak*) homeomorphism, $\beta \mapsto \gamma(\beta)$ defined by

$$\mu_\beta = T^*S^*\delta_{\gamma(\beta)}$$

is an order isomorphism and homeomorphism, $n \ge 8\|T\|/\epsilon$, and

$$\big|\|\mu_\beta\| - \|\mu_{\beta'}\|\big| < \epsilon/(32\|T\|) \quad \text{for all } \beta, \beta' \le \omega^n.$$

The family of measures $\{\delta_{\gamma(\beta)} : \beta \le \omega^n\}$ is a natural basis of the dual of a 1-complemented subspace Z of $C(\omega^\omega)$ isometric to $C(\omega^n)$. Indeed, according to Remark 2.2 it suffices to take for Z the subspace of $C(\omega^\omega)$ of all functions constant on order intervals $(\gamma(\beta), \gamma(\beta+1)]$ for $\beta < \omega^n$. Further because

$$\lim_K \|\mu_{\omega^n}|_{[K,\omega)\times\beta\mathbb{N}}\| = 0,$$

and the restriction to $[1, K] \times \beta\mathbb{N}$ is weak* continuous, we can assume that the support of μ_γ is contained in $[1, K] \times \beta\mathbb{N}$ for all $\gamma \le \omega^n$. Notice that $[1, K] \times \beta\mathbb{N}$ is homeomorphic to $\beta\mathbb{N}$ so we may replace $[1, K] \times \beta\mathbb{N}$ by $\beta\mathbb{N}$.

In order to simplify notation we can now assume that we have a projection T from $C(\beta\mathbb{N}, X)$ onto a subspace Y isomorphic by an operator S to $C(\omega^n)$ such that

$$\|T^*S^*z\| \ge \epsilon\|z\| \quad \text{for all } z \in C(\omega^n)^*$$

and

$$\big|\|T^*S^*\delta_\beta\| - \|T^*S^*\delta_{\beta'}\|\big| < \epsilon/(8\|T\|) \quad \text{for all } \beta, \beta' \le \omega^n.$$

Let

$$g_{\omega^n} = S^{-1}(1_{(0,\omega^n]}) \;\text{ and }\; g_{\omega^{n-1}k} = S^{-1}(1_{(\omega^{n-1}(k-1),\omega^{n-1}k]}) \quad \text{for all } k \in \mathbb{N}.$$

Then $(g_{\omega^{n-1}k})$ is w.u.c. Because X does not contain c_0, for each t, $(g_{\omega^{n-1}k}(t))$ is unconditionally convergent and thus converges in norm to 0 for all $t \in \beta\mathbb{N}$.

Because $\|g_{\omega^{n-1}k}(\cdot)\| \le \|S^{-1}\|$ for all k, $(\|g_{\omega^{n-1}k}(\cdot)\|)$ converges to 0 weakly in $C(\beta\mathbb{N})$. Because

$$\int g_{\omega^{n-1}k}\,d\mu_{\omega^{n-1}k} = 1,$$

by Lemma 2.6 there exists a subsequence $(\mu_{\omega^{n-1}k})_{k\in M_1}$ and a sequence of disjoint clopen sets $(G_{\omega^{n-1}k})_{k\in M_1}$ such that

$$\int g_{\omega^{n-1}k}1_{G_{\omega^{n-1}k}}\,d\mu_{\omega^{n-1}k} \ge 7/8 \quad \text{for all } k\in M_1.$$

If there is an infinite subset K of M_1 and $\delta>0$ such that

$$\Big|\int g_{\omega^n}1_{G_{\omega^{n-1}k}}\,d\mu_{\omega^{n-1}k}\Big| \ge \delta \quad \text{for all } k\in K, \tag{3.1}$$

then Lemma 3.1 would imply that $C(\omega^\omega)$ is non-separable. Notice that the same contradiction would result if for each k, we replace $G_{\omega^{n-1}k}$ in (3.1) by any of its clopen subsets.

We also have

$$\int g_{\omega^n}\,d\mu_{\omega^{n-1}k} = 1 \quad \text{for all } k,$$

thus, by replacing M_1 by an infinite subset, for each $k\in M_1$ there are disjoint clopen sets $G^0_{\omega^{n-1}k}$ and $G^1_{\omega^{n-1}k}$ such that

- $\int g_{\omega^{n-1}k}1_{G^1_{\omega^{n-1}k}}\,d\mu_{\omega^{n-1}k} > 3/4$,
- $\int g_{\omega^n}1_G\,d\mu_{\omega^{n-1}k} < 1/8$ for all clopen $G\subset G^1_{\omega^{n-1}k}$,
- $\int g_{\omega^n}1_{G^0_{\omega^{n-1}k}}\,d\mu_{\omega^{n-1}k} > 3/4$.

This is the first step of an at most n-step induction argument.

Fix $k_1\in M_1$. Consider the sequence $(\omega^{n-1}(k_1-1)+\omega^{n-2}k)$. For sufficiently large k,

$$\int g_{\omega^{n-1}k_1}1_{G^1_{\omega^{n-1}k_1}}\,d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 3/4,$$

and

$$\int g_{\omega^n}1_{G^0_{\omega^{n-1}k_1}}\,d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 3/4.$$

Because

$$g_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} = S^{-1}(1_{(\omega^{n-1}(k_1-1)+\omega^{n-2}(k-1),\omega^{n-1}(k_1-1)+\omega^{n-2}k]})$$

converges weakly to 0, by applying Lemma 2.6, there exist a subsequence

$$(\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k})_{k\in M_2}$$

and disjoint clopen sets $(G_k)_{k\in M_2}$ such that

$$\int g_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}1_{G_k}\,d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 7/8 \quad \text{for all } k\in M_2.$$

For every $\delta>0$ and clopen $H_k\subset G_k$ for $k\in M_2$, Lemma 3.1 tells us that there are only finitely many k for which

$$\Big|\int g_{\omega^{n-1}k_1}1_{H_k}\,d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}\Big| > \delta,$$

or

$$\left|\int g_{\omega^n} 1_{H_k}\, d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}\right| > \delta.$$

Thus taking $\delta = 1/8$, for sufficiently large $k \in M_2$ we can find disjoint clopen sets

$$G^j_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}, \qquad j = 0, 1, 2,$$

such that

- $\int g_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} 1_{G^2_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}}\, d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 5/8$,
- $\int g_{\omega^{n-1}k_1} 1_{G^1_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}}\, d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 5/8$,
- $\int g_{\omega^n} 1_{G^0_{\omega^{n-1}(k_1-1)+\omega^{n-2}k}}\, d\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}k} > 5/8$.

An induction argument shows that we can choose $k_1, \ldots, k_n$ and disjoint clopen sets

$$G^j_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\cdots+k_n}, \qquad j = 0, 1, \ldots, n-1,$$

such that

$$\int g_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\cdots+\omega^{n-j}k_j} 1_{G^j_{\omega^{n-1}(k_1-1)+\cdots+k_n}}\, d\mu_{\omega^{n-1}(k_1-1)+\cdots+k_n}$$

is strictly greater than $1/2 + 1/2^n$. This implies that

$$\|\mu_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\cdots+k_n}\| > n/2 > \|T\|\,\|\delta_{\omega^{n-1}(k_1-1)+\omega^{n-2}(k_2-1)+\cdots+k_n}\|.$$

This contradiction shows that no such projection T exists. ∎

Remark 3.3. The conclusion of this proposition is equivalent to the statement that $C(\beta\mathbb{N} \times \omega, X)$ does not contain $C(\omega^n)$ uniformly complemented. Obviously if X contains $C(\omega^n)$ uniformly complemented then it follows that $C(\beta\mathbb{N} \times \omega, X)$ contains $C(\omega^n)$ uniformly complemented. It is possible that the hypothesis on X could be weakened to something like X does not contain $C(\omega^n)$ uniformly or uniformly complemented. We do not know whether $C(\beta\mathbb{N}\times\omega, C(\beta\mathbb{N}))$ contains a complemented subspace isomorphic to $C(\omega^\omega)$. If there is a counterexample then assuming additionally that X is separable may provide a strong enough hypothesis.

The next result generalizes the previous one to larger ordinals.

Theorem 3.4. *Let X be a Banach space and $0 \le \alpha < \beta < \omega_1$. Then*

$$C(\omega^{\omega^\beta}) \overset{c}{\hookrightarrow} C(\beta\mathbb{N} \times \omega^{\omega^\alpha}, X) \;\Rightarrow\; c_0 \hookrightarrow X.$$

Proof. Let α be a countable ordinal and X a Banach space containing no copy of c_0. We will show by induction that α is the smallest ordinal γ such that $C(\beta\mathbb{N}\times\omega^{\omega^\gamma}, X)$ contains a complemented subspace isomorphic to $C(\omega^{\omega^\alpha})$. Theorem 3.2 shows this for $\alpha = 1$. Assume that the result holds for ordinals less than α, $\alpha > 1$, and that $\gamma < \alpha$ is the smallest ordinal such that

$C(\beta\mathbb{N}\times\omega^{\omega^\gamma},X)$ contains a complemented subspace isomorphic to $C(\omega^{\omega^\alpha})$. We will show that this leads to a contradiction.

In place of $C(\beta\mathbb{N}\times\omega^{\omega^\gamma},X)$, we will use the isomorphic space, $C_0(\omega^{\omega^\gamma}\times\beta\mathbb{N},X)$. Now assume that T is a projection defined on the latter space with range isomorphic to $C(\omega^{\omega^\alpha})$. Let $\alpha_k\uparrow\omega^\alpha$ and $\beta_k\uparrow\omega^\gamma$, where $\alpha_k=\omega^{\alpha'}k$ if $\alpha=\alpha'+1$ for some α', or $\alpha_k=\omega^{\xi_k}$ if α is a limit ordinal and $\xi_k\uparrow\alpha$, and $\beta_k=\omega^{\gamma'}k$ if $\gamma=\gamma'+1$ for some γ', or $\beta_k=\omega^{\gamma_k}$ if γ is a limit ordinal and $\gamma_k\uparrow\gamma$. Choose k_0 such that $\alpha_k\ge\omega^\gamma$ for all $k\ge k_0$. By Lemma 2.5 for each $k>k_0$, there is a k' such that $\{T^*\delta_\beta:\beta\le\omega^{\alpha_{k'}}\}$ contains a subfamily $\{\mu_\rho:\rho\le\omega^{\alpha_k}\}$ such that $\rho\mapsto\mu_\rho$ is a homeomorphism, $\rho\mapsto\beta(\rho)$ is an order homeomorphism, where $\mu_\rho=T^*\delta_{\beta(\rho)}$, and

$$\big|\|\mu_\rho\|-\|\mu_{\rho'}\|\big|<1/(4\|T\|)\quad\text{for all }\rho,\rho'\le\omega^{\alpha_k}.$$

For each m let P_m be the canonical projection from $C_0(\omega^{\omega^\gamma}\times\beta\mathbb{N},X)$ onto $C(\omega^{\beta_m}\times\beta\mathbb{N},X)$. There exists an m such that

$$\|(I-P_m^*)(\mu_{\omega^{\alpha_k}})\|<1/(8\|T\|).$$

It follows by passing to a suitable neighborhood of ω^{α_k} that we may assume that

$$\|(I-P_m^*)(\mu_\rho)\|<3/(8\|T\|)\quad\text{for all }\rho\le\omega^{\alpha_k}.$$

According to Remark 2.2 we can find a 1-complemented subspace Z of $C(\omega^{\omega^\alpha})$ which is isometric to $C(\omega^{\alpha_k})$ and has natural basis of its dual $\{\mu_\rho:\rho\le\omega^{\alpha_k}\}$. This implies that $P_m(Z)$ is a complemented subspace of $C(\beta\mathbb{N}\times\omega^{\beta_m},X)$ isomorphic to $C(\omega^{\alpha_k})$. Because $\beta_m<\omega^\gamma$ and $\alpha_k\ge\omega^\gamma$, $C(\beta\mathbb{N}\times\omega^{\beta_m},X)$ cannot contain a complemented copy of $C(\omega^{\omega^\gamma})$ by the inductive hypothesis. Thus we have a contradiction and the theorem is proved. ∎

Now we can prove

THEOREM 3.5. *Let X be a Banach space containing no copy of c_0, K an infinite compact metric space and $0\le\alpha<\omega_1$. Then*

$$C(K)\overset{c}{\hookrightarrow}C(\beta\mathbb{N}\times\omega^{\omega^\alpha},X)\ \Leftrightarrow\ C(K)\sim C(\omega^{\omega^\xi})\ \textit{for some}\ 0\le\xi\le\alpha.$$

Proof. Since $C([0,1])$ contains complemented copies of every $C(\omega^{\omega^\alpha})$, $0\le\alpha<\omega_1$, it follows directly from Milyutin's theorem and Theorem 3.4 that K must be countable if $C(K)\overset{c}{\hookrightarrow}C(\beta\mathbb{N}\times\omega^{\omega^\alpha},X)$. If K is countable, then $C(K)$ is isomorphic to $C(\omega^{\omega^\xi})$ for some countable ordinal ξ and Theorem 3.4 determines the possible values of ξ. The converse is obvious. ∎

REMARK 3.6. Because $C(\beta\mathbb{N},\ell_1)$ is isomorphic to its c_0-sum (see Theorem 5.4), the above result in the case $X=\ell_1$ does not mimic that for the scalar case where there is an additional isomorphism class. Indeed, since c_0

is not isomorphic to a complemented subspace of $C(\beta\mathbb{N})$, the spaces $C(\beta\mathbb{N})$ and $C(\beta\mathbb{N}\times\omega)$ are not isomorphic.

We can now prove the main result of this section.

THEOREM 3.7. *Let X be a Banach space containing no copy of c_0. Then for any infinite compact metric spaces K_1 and K_2 we have*

$$C(\beta\mathbb{N}\times K_1, X)\sim C(\beta\mathbb{N}\times K_2, X) \;\Leftrightarrow\; C(K_1)\sim C(K_2).$$

Proof. Let us show the non-trivial implication. Suppose that

$$C(\beta\mathbb{N}\times K_1, X)\sim C(\beta\mathbb{N}\times K_2, X).$$

It is convenient to consider two subcases:

CASE 1: *K_1 and K_2 are countable.* Hence there are countable ordinals ξ and η such that $C(K_1)\sim C(\omega^{\omega^{\xi}})$ and $C(K_2)\sim C(\omega^{\omega^{\eta}})$. Then according to our hypothesis,

$$C(\omega^{\omega^{\eta}})\overset{c}{\hookrightarrow} C(\omega^{\omega^{\eta}}\times\beta\mathbb{N}, X)\sim C(\omega^{\omega^{\xi}}\times\beta\mathbb{N}, X).$$

Therefore by Theorem 3.5 we deduce that $\omega^{\omega^{\eta}}\le\omega^{\omega^{\xi}}$. Similarly, we show that $\omega^{\omega^{\xi}}\le\omega^{\omega^{\eta}}$. Hence $C(K_1)\sim C(K_2)$.

CASE 2: *K_1 or K_2 is uncountable.* Without loss of generality we suppose that K_2 is uncountable. To prove that $C(K_1)\sim C(K_2)$, it is enough by Milyutin's theorem to show that K_1 is also uncountable. Assume the contrary. Then there exists an ordinal ξ such that $C(K_1)\sim C(\omega^{\omega^{\xi}})$. Since $C(K_2)\sim C([0,1])$, by our hypothesis we have

$$C(\omega^{\omega^{\xi+1}})\overset{c}{\hookrightarrow} C(\beta\mathbb{N}\times[0,1], X)\overset{c}{\hookrightarrow} C(\omega^{\omega^{\xi}}\times\beta\mathbb{N}, X),$$

a contradiction of Theorem 3.5. This completes the proof. ∎

COROLLARY 3.8. *Let X be a Banach space containing no copy of c_0. Then for any infinite compact metric spaces K_1 and K_2 we have*

$$C(K_1, X)\sim C(K_2, X) \;\Leftrightarrow\; C(K_1)\sim C(K_2).$$

Proof. One direction is immediate. If $C(K_1,X)\sim C(K_2,X)$ then $C(\beta\mathbb{N}\times K_1, X)\sim C(\beta\mathbb{N}\times K_2, X)$, so this follows from the previous result. ∎

The next result can be considered as an extension of the Cembranos–Freniche result although the proof does not yield a proof of that result. To include the original result we would need to use the Josefson–Nissenzweig Theorem [17], [23].

PROPOSITION 3.9. *Suppose that $0\le\alpha<\omega_1$, $0\le\gamma<\omega_1$, and either (γ_n) is (ω^{β_n}) where (β_n) increases to γ, or (γ_n) is $(\omega^{\beta}n)$ and $\gamma=\beta+1$ for some ordinal β. Let X be a Banach space such that with constants independent of n, $C(\omega^{\gamma_n})$ is isomorphic to a complemented subspace of X. Then for* *any infinite compact Hausdorff space K, $C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma})$ is isomorphic to a complemented subspace of $C(K \times \omega^{\omega^\alpha}, X)$.*

If X is also separable then $C(\omega^{\omega^\gamma})$ is isomorphic to a complemented subspace of $C(K, X)$.

Proof. Clearly $C(\omega^{\omega^\alpha})$ is isomorphic to a complemented subspace of $C(K \times \omega^{\omega^\alpha}, X)$. We also know by Lemma 2.4 that

$$C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma}) \sim C(\omega^{\omega^{\max\{\alpha,\gamma\}}}).$$

So we need only show that $C(\omega^{\omega^\gamma})$ is isomorphic to a complemented subspace of $C(K \times \omega^{\omega^\alpha}, X)$. The case $\gamma = 0$ is the Cembranos–Freniche result but also is immediate from the fact that $\alpha \geq 0$. Now assume $\gamma \geq 1$.

Notice that $C(K \times \omega^{\omega^\alpha}, X)$ is isomorphic to $C_0(\omega \times \omega^{\omega^\alpha} \times K, X)$. This in turn is isomorphic to

$$\Big(\sum_{j\in\mathbb{N}} C(\omega^{\omega^\alpha} \times K, X)\Big)_{c_0}.$$

For each $n \in \mathbb{N}$ let X_n be a complemented subspace of X which is isomorphic to $C(\omega^{\gamma_n})$ and let P_n be a projection from X onto X_n. By the hypothesis we can assume that the norms of the isomorphisms and the projections are bounded independently of n. Choose any point $a \in \omega^{\omega^\alpha} \times K$. If $(f_j) \in (\sum_j C(\omega^{\omega^\alpha} \times K, X))_{c_0}$, then

$$P((f_j)) = (P_j(f_j(a))1_{\omega^{\omega^\alpha} \times K})$$

defines a projection onto a space isometric to the c_0-sum of X_j. Because $(\sum_{j\in\mathbb{N}} C(\omega^{\gamma_j}))_{c_0}$ is isomorphic to $C(\omega^{\omega^\gamma})$, the c_0-sum of X_j is isomorphic to $C(\omega^{\omega^\gamma})$.

If X is separable, then with X_n as before let Y_n be the subspace of X_n which is the image of $C_0(\omega^{\gamma_n})$ under the isomorphism from $C(\omega^{\gamma_n})$, and Q_n be the projection from X onto Y_n. Because X is separable there is a decreasing sequence of weak* open sets G_j which is a base for the neighborhoods of 0 in the ball of X^*, B_{X^*}. For each n there is a sequence of complemented subspaces $Y_{n,k}$ of Y_n with projections $Q_{n,k}$, $Y_{n,k} \supset Y_{n,k+1}$ and $Y_{n,k}$ isomorphic to $C_0(\omega^{\gamma_n})$ for all k, such that for each j and n there is a K such that for all $k \geq K$,

$$Q^*_{n,k}(X^*) \cap B_{X^*} \subset G_j.$$

Indeed if $\rho_k \nearrow \omega^{\gamma_n}$, then $f \mapsto f1_{(\rho_k,\omega^{\gamma_n}]}$ is a projection onto a subspace of $C_0(\omega^{\gamma_n})$ isomorphic to $C_0(\omega^{\gamma_n})$ and $Y_{n,k}$ can be taken to be the image of this subspace in Y_n.

Let (g_n) be a sequence of disjointly supported non-negative norm one elements in $C(K)$ such that for each n there is an open set

$$H_n \supset \overline{\{t : g_n(t) > 0\}},$$

with $H_n\cap H_m=\emptyset$ for all $m\neq n$, and $t_n\in K$ such that $g_n(t_n)=1$ for all n. Let

$$D=\sup_{s,k}\|Q_{s,k}\|,$$

and choose k_n such that

$$D^{-1}Q^*_{n,k_n}(B_{X^*})\subset G_n \quad \text{for all } n,$$

and let

$$Z=[g_ny_n : y_n\in Y_{n,k_n},\ n\in\mathbb{N}].$$

Clearly Z is isomorphic to $(\sum_n C_0(\omega^{\gamma_n}))_{c_0}$, which is isomorphic to $C(\omega^{\omega^\gamma})$. Define an operator T from $C(K,X)$ into Z by

$$Tf(t)=g_n(t)Q_{n,k_n}(f(t_n)) \quad \text{for all } t\in\operatorname{supp} g_n$$

and $Tf(t)=0$ if $t\notin\bigcup_n \operatorname{supp} g_n$. Because each g_n is continuous, Tf is continuous on H_n for all n. If $\epsilon>0$ and $t'_n\in\operatorname{supp} g_n$, and if t is an accumulation point of $\{t'_n\}$, then by the continuity of f, $\|f(t)-f(t'_n)\|<\epsilon$ for all $t'_n\in H$ where H is some neighborhood of t. Because t cannot be in any H_n, $T(f(t))=0$. By the choice of k_n we have

$$\lim_n \sup_{x^*\in B_{X^*}} |(Q^*_{n,k_n}x^*)(f(t))|=0$$

and for $t'_n\in H$,

$$\epsilon\sup_{s,k\in\mathbb{N}}\|Q_{s,k}\|>\sup_{x^*\in B_{X^*}}|(Q^*_{n,k_n}x^*)(f(t'_n))-(Q^*_{n,k_n}x^*)(f(t))|.$$

Thus

$$\|(Tf)(t)\|=0=\lim_{n\in\mathcal{N}}\sup_{x^*\in B_{X^*}}|(Q^*_{n,k_n}x^*)(f(t_n))g(t'_n)|=\lim_{n\in\mathcal{N}}\|(Tf)(t'_n)\|,$$

where the limit is over some net $(t'_n)_{n\in\mathcal{N}}$ such that $\lim_{n\in\mathcal{N}} t'_n=t$.

It is easy to see that

$$\|T\|\le\sup_{s,k\in\mathbb{N}}\|Q_{s,k}\|$$

and, because $g_n(t_n)=1$ and each Q_{n,k_n} is a projection, T is a projection. ∎

REMARK 3.10. We do not know whether the separability condition in the second part is necessary but the argument fails for the natural choices of X_n if $X=\{(x_n):\ \forall n\in\mathbb{N}, x_n\in C(\omega^n), \|(x_n)\|=\sup_n\|x_n\|<\infty\}$.

In the next section we will prove some results about quotients of $C(K,X)$ isomorphic to $C(\omega^{\omega^\alpha})$. If we consider quotients in the previous proposition instead of complemented subspaces, the analogous results hold. The proof is similar to the previous one except that the argument is now entirely in the dual.

Proposition 3.11. *Suppose that $0 \le \alpha < \omega_1$, $0 \le \gamma < \omega_1$, and (γ_n) is either (ω^{β_n}) where (β_n) increases to γ, or $(\omega^{\beta} n)$ and $\gamma = \beta + 1$. Let X be a Banach space such that with constants independent of n, $C(\omega^{\gamma_n})$ is isomorphic to a quotient of X. Then for any infinite compact Hausdorff space K, $C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma})$ is isomorphic to a quotient of $C(K \times \omega^{\omega^\alpha}, X)$.*

If X is also separable then $C(\omega^{\omega^\gamma})$ is isomorphic to a quotient of $C(K, X)$.

Proof. Clearly $C(\omega^{\omega^\alpha})$ is isomorphic to a quotient of $C(K \times \omega^{\omega^\alpha}, X)$. We also know by Lemma 2.4 that

$$C(\omega^{\omega^\alpha} \times \omega^{\omega^\gamma}) \sim C(\omega^{\omega^{\max\{\alpha,\gamma\}}}).$$

Thus as before we need only show that $C(\omega^{\omega^\gamma})$ is isomorphic to a quotient of $C(K \times \omega^{\omega^\alpha}, X)$. The case $\gamma = 0$ is the Cembranos–Freniche result but is also immediate from the fact that $\alpha \ge 0$. Assume $\gamma \ge 1$.

Notice that $C(K \times \omega^{\omega^\alpha}, X)$ is isomorphic to $C_0(\omega \times \omega^{\omega^\alpha} \times K, X)$, and this is isomorphic to

$$\Big(\sum_j C(\omega^{\omega^\alpha} \times K, X)\Big)_{c_0}.$$

For each $n \in \mathbb{N}$ let X_n be a quotient of X which is isomorphic to $C(\omega^{\gamma_n})$, and P_n be the quotient map from X onto X_n. By hypothesis we can assume that the norms of the isomorphisms and the quotient maps are bounded independently of n. Thus $P_n^*(X_n^*)$ is weak* isomorphic to $C(\omega^{\gamma_n})^*$, and the subspace

$$Z = \{(z_n) : z_n \in P_n^*(X_n^*) \text{ for all } n \in \mathbb{N}\}$$

of $(\sum_j C(\omega^{\omega^\alpha} \times K, X))_{c_0})^*$ is weak* isomorphic to $((\sum_j C(\omega^{\gamma_n}))_{c_0})^*$. This space is weak* isomorphic to $C(\omega^\gamma)^*$, giving us the required quotient.

If X is separable, then with X_n as before let Y_n be the complemented subspace of X_n which is the image of $C_0(\omega^{\gamma_n})$ under the isomorphism, and Q_n be the quotient map from X onto Y_n. Because X is separable there is a decreasing sequence of weak* open sets G_j which is a base for the neighborhoods of 0 in B_{X^*}. As in the proof of Proposition 3.9 for each n there is a sequence of complemented subspaces $Y_{n,k}$ of Y_n with projections $Q_{n,k}$, $Y_{n,k} \supset Y_{n,k+1}$ and $Y_{n,k}$ isomorphic to $C_0(\omega^{\gamma_n})$ for all k, such that for each j and n there is a K such that for all $k \ge K$,

$$Q_{n,k}^*(Y_n^*) \cap B_{X^*} \subset G_j.$$

Let (t_n) be a sequence of points in K such that for each n there is an open set H_n containing t_n with $H_n \cap H_m = \emptyset$ for all $m \ne n$. Let

$$D = \sup_{s,k} \|Q_{s,k}\|,$$

and choose k_n such that

$$D^{-1} Q_{n,k_n}^*(B_{Y_n^*}) \subset G_n \quad \text{for all } n,$$

and let

$$Z = [z_n\delta_{t_n} : z_n \in Q^*_{n,k_n}(Y^*_n) \text{ for all } n \in \mathbb{N}].$$

Clearly Z is isomorphic to $(\sum_n C_0(\omega^{\gamma_n})^*)_{\ell_1}$. We need to show that Z is weak* isomorphic to $((\sum_n C_0(\omega^{\gamma_n}))_{c_0})^*$. This however follows immediately from the choice of (k_n) and the fact that no point t_j is an accumulation point of $\{t_n : n \in \mathbb{N}\}$ (as was shown in the proof of Proposition 3.9). ∎

4. Separable $C(K)$ quotients of $C(\beta\mathbb{N}\times\alpha, X)$. By Theorem 1.2 we know that $C(\beta\mathbb{N}, \ell_p)$, $1 < p < \infty$, contains a complemented copy of c_0. The main aim of this section is to show that $C(\omega^\omega)$ is not even a quotient of this space (Proposition 4.2). Of course, this implies that c_0 is, up to isomorphism, the only separable $C(K)$ space which is a quotient of $C(\beta\mathbb{N}, \ell_p)$, $1 < p < \infty$.

In this section we will work with Banach spaces X that satisfy the following properties:

(†) X^* has a monotone weak* FDD (X^*_m).
(‡) For every constant C, $0 < C < 1$, there is a constant C' such that for all $x^* \in X^*$ and $j \in \mathbb{N}$,

$$\|(I - P_j)x^*\| \le C\|x^*\| + C'(\|x^*\| - \|P_jx^*\|),$$

where (P_j) is the sequence of FDD projections, i.e., $P_j(X^*) = [X^*_m : m \le j]$.

We will refer to such spaces as *satisfying the daggers.* Before proceeding to the main results we will verify that ℓ_p, for $1 < p < \infty$, satisfies the daggers.

Remark 4.1. Suppose that P is an operator on X^* such that for some $x^* \in X^*$,

$$\|(I - P)x^*\|^p + \|Px^*\|^p \le \|x^*\|^p$$

and $0 < C < 1$. We claim

$$\|(I - P)x^*\| \le C\|x^*\| + C^{1-p}(\|x^*\| - \|Px^*\|).$$

Indeed, by Hölder's inequality with $q = p/(p - 1)$,

$$\begin{aligned}\|(I - P)x^*\| + C^{1-p}\|Px^*\| &\le (1 + C^{-p})^{(p-1)/p}(\|(I - P)x^*\|^p + \|Px^*\|^p)^{1/p}\\ &\le (1 + C^{-p})(1 + C^{-p})^{-1/p}\|x^*\|\\ &\le (1 + C^{-p})C\|x^*\|.\end{aligned}$$

Thus any space with a weak* FDD satisfying the p-concavity inequality

$$\|(I - P_j)x^*\|^p + \|P_jx^*\|^p \le \|x^*\|^p$$

for all j and $x^* \in X^*$ for some $1 < p < \infty$ will satisfy (‡). In particular ℓ_q, $1 < q < \infty$, satisfies the daggers.

The next result is the initial case of the main theorem of this section.

Theorem 4.2. *Suppose that X is a Banach space satisfying the daggers. Then $C(\omega^\omega)$ is not a quotient of $C(\beta\mathbb{N}\times\omega, X)$.*

Proof. First we can replace $C(\beta\mathbb{N}\times\omega, X) = C(\omega\times\beta\mathbb{N}, X)$ by the isomorphic space $C_0(\omega\times\beta\mathbb{N}, X)$. Suppose that there exists a bounded linear operator T from $C_0(\omega\times\beta\mathbb{N}, X)$ onto $C(\omega^\omega)$. We will show that this leads to a contradiction. We may assume that $\|T\| = 1$. Because T^* is a weak* isomorphism from $C(\omega^\omega)^*$ into $C_0(\omega\times\beta\mathbb{N}, X)^*$, there is a constant K such that for every $n\in\mathbb{N}$, $\{\delta_\alpha : \alpha\le\omega^n\}$ is mapped to $\{x^*_\alpha : \alpha\le\omega^n\}\subset C_0(\omega\times\beta\mathbb{N}, X)^*$ and $\{x^*_\alpha : \alpha\le\omega^n\}$ is K-equivalent to the usual unit vector basis of ℓ_1, i.e.,

$$\Big\|\sum_\alpha c_\alpha x^*_\alpha\Big\| \ge \sum_\alpha |c_\alpha|/K$$

for all sequences (c_α) of scalars.

Let $C = (8K)^{-1}$ in (‡), and choose ρ, $1/2 > \rho > 0$, such that

$$(1-\rho)^{-1}\left(1-\frac{(1-\rho)^3}{1+\rho}\right)C' < (4K)^{-1}.$$

By Lemma 2.5 for $k=1$ and n sufficiently large, we can find $(x^*_{\alpha(\gamma)})_{\gamma\le\omega}$ with

$$\big|\|x^*_{\alpha(\gamma)}\| - \|x^*_{\alpha(\gamma')}\|\big| < \rho/K \quad \text{for all } \gamma,\gamma'\le\omega.$$

Moreover, as in the proof of Theorem 3.2, we may assume that the measures are all supported in $[1,K]\times\beta\mathbb{N}$ and thus reduce to measures supported on $\beta\mathbb{N}$. By our identification of $C(\beta\mathbb{N}, X)^*$ with a space of X^*-valued measures and switching to a more suggestive notation we have (μ_n), a weak* convergent sequence of X^*-valued measures with limit μ, with

$$\|\mu\|(1+\rho) > \|\mu_n\| > \|\mu\|(1-\rho) \quad \text{for all } n.$$

Choose a finite partition $\{B_j\}$ of $\beta\mathbb{N}$ into clopen sets such that

$$\sum_j \|\mu(B_j)\|_{X^*} > \|\mu\|(1-\rho).$$

As in (‡) let P_m denote the FDD projection of X^* onto the span of the first m subspaces. With a slight abuse of notation we will also use P_m for the operator on the X^*-valued measures defined by

$$(P_m\mu)(A) = P_m(\mu(A)) \quad \text{for all measurable } A.$$

Choose N such that

$$\|P_N(\mu(B_j))\| > (1-\rho)\|\mu(B_j)\| \quad \text{for all } j.$$

By passing to a subsequence we may assume that

$$\|P_N(\mu_n(B_j))\| > (1-\rho)\|\mu(B_j)\| \quad \text{for all } j \text{ and } n.$$

Hence

$$\|P_N\mu_n\| \ge \sum_j \|P_N\mu_n(B_j)\| \ge (1-\rho)\sum_j \|\mu(B_j)\|$$
$$\ge (1-\rho)^2\|\mu\| \ge \frac{(1-\rho)^2}{1+\rho}\|\mu_n\|.$$

Fix n and choose a partition $\{A_k\}$ that refines $\{B_j\}$ such that

- $\sum_k \|\mu_n(A_k)\| \ge (1-\rho)\|\mu_n\|$,
- $\sum_k \|P_N\mu_n(A_k)\| \ge (1-\rho)\|P_N\mu_n\|$,
- $\sum_k \|(I-P_N)\mu_n(A_k)\| \ge (1-\rho)\|(I-P_N)\mu_n\|$.

For each k let λ_k satisfy

$$\lambda_k\|\mu_n(A_k)\| = \|P_N\mu_n(A_k)\|.$$

Then

$$\sum_k \lambda_k\|\mu_n(A_k)\| \ge \frac{(1-\rho)^3}{1+\rho}\|\mu_n\| \ge \frac{(1-\rho)^3}{1+\rho}\sum_k \|\mu_n(A_k)\|.$$

Equivalently,

$$\left(1-\frac{(1-\rho)^3}{1+\rho}\right)\sum_k \|\mu_n(A_k)\| \ge \sum_k (1-\lambda_k)\|\mu_n(A_k)\|.$$

We need to estimate $\|(I-P_N)\mu_n\|$. Let

$$M = \{k : \|(I-P_k)\mu_n(A_k)\| \le C\|\mu_n(A_k)\|\}.$$

Then by the choice of C,

$$\sum_k \|(I-P_N)\mu_n(A_k)\| \le \sum_{k\in M} C\|\mu_n(A_k)\| + \sum_{k\notin M}(C+C'(1-\lambda_k))\|\mu_n(A_k)\|$$
$$\le \sum_k C\|\mu_n(A_k)\| + C'\sum_k (1-\lambda_k)\|\mu_n(A_k)\|$$
$$\le C\|\mu_n\| + C'\left(1-\frac{(1-\rho)^3}{1+\rho}\right)\|\mu_n\|.$$

Because $\|T\|=1$, $\|\mu_n\|\le 1$. Therefore

$$\|(I-P_N)\mu_n\| \le (1-\rho)^{-1}C + (1-\rho)^{-1}C'\left(1-\frac{(1-\rho)^3}{1+\rho}\right) \le (2K)^{-1}.$$

Notice that $(P_N\mu_n)$ converges to $P_N\mu$ in the weak* topology. Because $C(\beta\mathbb{N})$ is a Grothendieck space, so is $C(\beta\mathbb{N}, [X_m : m\le N])$. Thus $(P_N\mu_n)$ converges to $P_N\mu$ in the weak topology. Because (μ_n) is K-equivalent to the usual unit vector basis of ℓ_1, the estimate on $\|(I-P_N)\mu_n\|$ implies that $(P_N\mu_n)$ is also equivalent to that basis. This is a contradiction because the unit vector basis of ℓ_1 has no weak Cauchy subsequence. ∎

Remark 4.3. The proof of the theorem shows that $C(\omega^n)$ is not a quotient of $C(\omega \times \beta\mathbb{N}, X)$ uniformly in n. Because $C(\omega^\omega)$ is isomorphic to $(\sum_n C(\omega^n))_{c_0}$, this is equivalent to $C(\omega^\omega)$ not being a quotient. It is conceivable that the conclusion of Theorem 4.2 could be proved under the hypothesis that X does not have $C(\omega^n)$ as a quotient uniformly in n.

The following theorem generalizes Theorem 4.2 to higher ordinals.

Theorem 4.4. *Suppose that X is a Banach space such that $C(\omega^\omega)$ is not isomorphic to a quotient of $C(\omega, X)$. Let $\alpha \geq 1$. Then $C(\omega^{\omega^\alpha})$ is isomorphic to a quotient of $C(\beta\mathbb{N} \times \xi, X)$ if and only if $\xi \geq \omega^{\omega^\alpha}$.*

Proof. It is easy to see that $C(\gamma)$ is isomorphic to a complemented subspace of $C(\beta\mathbb{N} \times \gamma, X)$ for all $\gamma \geq 1$. Thus the sufficiency is clear.

On the other hand, by Proposition 2.1,

$$C(\beta\mathbb{N} \times \omega^n, X) = C(\omega^n, C(\beta\mathbb{N}, X)) \sim C(\omega, C(\beta\mathbb{N}, X)) = C(\beta\mathbb{N} \times \omega, X)$$

for all positive integers n. Thus the necessity is true for $\alpha = 1$. Because $C(\omega^\omega)$ is not isomorphic to a quotient of $C(\omega, X)$ or of $C(\beta\mathbb{N} \times \omega, X)$, it is sufficient to prove the result for $C(\xi, X)$ rather than $C(\beta\mathbb{N} \times \xi, X)$.

Now suppose that $\alpha > 1$ and for all $\gamma < \alpha$, $C(\omega^{\omega^\gamma})$ is not isomorphic to a quotient of $C(\xi, X)$ if $\xi < \omega^{\omega^\gamma}$. We will show that if $\beta < \omega^{\omega^\alpha}$ and there is a bounded operator T from $C(\beta, X)$ onto $C(\omega^{\omega^\alpha})$ then this leads to a contradiction. Without loss of generality we assume that $\|T\| = 1$.

If α is a limit ordinal and $\alpha_n \uparrow \alpha$, then $\omega^{\omega^{\alpha_n}} > \beta$ for some n. Now $C(\omega^{\omega^{\alpha_n}})$ is isomorphic to a quotient of $C(\omega^{\omega^\alpha})$ and by assumption also to a quotient of $C(\beta, X)$, which contradicts the induction hypothesis.

Now suppose that $\alpha = \gamma + 1$ for some ordinal γ. By Proposition 2.1, $C(\beta, X)$ is isomorphic to $C_0(\beta, X)$. Hence we may assume that T goes from $C_0(\beta, X)$ onto $C(\omega^{\omega^\alpha})$. Also by Proposition 2.1 we may assume that $\beta = \omega^{\omega^\zeta}$ for some $1 \leq \zeta < \alpha$. Let $\zeta_k = \omega^{\zeta - 1}k$ if ζ is not a limit ordinal and $\zeta_k \uparrow \omega^\zeta$ otherwise. Let $\{x^*_\rho : \rho \leq \omega^{\omega^\alpha}\}$ be the corresponding images of the Dirac measures $\{\delta_\rho : \rho \leq \omega^{\omega^\alpha}\}$ under T^*. Then $\{x^*_\rho : \rho \leq \omega^{\omega^\alpha}\}$ is, for some $R > 1$, R-equivalent to the usual unit vector basis of ℓ_1. By Lemma 2.5 there is an N sufficiently large such that if $\{y^*_\rho : \rho \leq \omega^{\omega^\gamma N}\}$ is contained in the unit ball of $C_0(\beta, X)^*$ and the mapping $\rho \mapsto y^*_\rho$ is weak* continuous, then there is a continuous map ϕ from $[1, \omega^{\omega^\gamma}]$ into $[1, \omega^{\omega^\gamma N}]$ such that

$$\big| \|y^*_{\phi(\rho)}\| - \|y^*_{\phi(\rho')}\| \big| < \frac{1}{4R} \qquad \text{for all } \rho, \rho' \leq \omega^{\omega^\gamma}.$$

Applying this to $\{x^*_\rho : \rho \leq \omega^{\omega^\gamma N}\}$, we get a map ϕ as above. Because $C_0(\beta, X)^*$ is weak* isomorphic and norm-isometric to $(\sum C(\omega^{\zeta_k}, X)^*)_{c_0^*}$, there exists K such that

$$\|(I - P_K) x^*_{\phi(\omega^{\omega^\gamma})}\| < \frac{1}{4R},$$

where P_K is the weak* continuous projection (truncation) from

$$\Big(\sum_{k=1}^{\infty} C(\omega^{\zeta_k}, X)^*\Big)_{c_0^*} \quad \text{onto} \quad \Big(\sum_{k\le K} C(\omega^{\zeta_k}, X)^*\Big)_{c_0^*}.$$

By passing to a neighborhood of ω^{ω^γ} we may assume that

$$\|P_K x^*_{\phi(\rho)}\| > \|P_K x^*_{\phi(\omega^{\omega^\gamma})}\| - \frac{1}{4R} > \|x^*_{\phi(\rho)}\| - \frac{3}{4R} \quad \text{for all } \rho \le \omega^{\omega^\gamma}.$$

Thus $C(\omega^{\omega^\gamma})^*$ is weak* isomorphic to a subspace of

$$\Big(\sum_{k\le K} C(\omega^{\zeta_k}, X)^*\Big)_{c_0^*},$$

and consequently $C(\omega^{\omega^\gamma})$ is isomorphic to a quotient of

$$C(\omega^{\zeta_1} + \cdots + \omega^{\zeta_K}, X).$$

But $\omega^{\zeta_1} + \cdots + \omega^{\zeta_K} < \omega^{\omega^\gamma}$, contradicting the inductive hypothesis. No such β exists. ∎

The purpose of this section is to prove Theorem 4.5 below. This result now follows from Theorem 4.4 and Milyutin's theorem by an argument similar to the deduction of Theorem 3.5 from Theorem 3.2. We leave the details to the reader.

THEOREM 4.5. *Let K be an infinite compact metric space and $0\le\alpha<\omega_1$, and let X satisfy the daggers. Then*

$$C(\beta\mathbb{N}\times\omega^{\omega^\alpha}, X) \twoheadrightarrow C(K) \iff C(K)\sim C(\omega^{\omega^\xi}) \text{ for some } 0\le\xi\le\alpha.$$

COROLLARY 4.6. *Let K be an infinite compact metric space and $0 \le \alpha < \omega_1$. Then*

$$C(\beta\mathbb{N}\times\omega^{\omega^\alpha}, \ell_p) \twoheadrightarrow C(K) \iff C(K)\sim C(\omega^{\omega^\xi}) \text{ for some } 0\le\xi\le\alpha.$$

5. The isomorphism of $C(\beta\mathbb{N}\times\omega, \ell_p)$ and $C(\beta\mathbb{N}, \ell_p)$, $1\le p<\infty$. As an immediate consequence of Theorem 3.2, for every $1 \le p < \infty$ and $\alpha \ge \omega^\omega$, we have

$$C(\beta\mathbb{N}\times\alpha, \ell_p) \nsim C(\beta\mathbb{N}, \ell_p).$$

For α finite $\beta\mathbb{N}\times\alpha$ is homeomorphic to $\beta\mathbb{N}$, so the case $\omega\le\alpha<\omega^\omega$ of Theorem 5.4 remains. This result is the case $p = 1$ of Theorem 5.3 below. To prove this theorem we need the following lemma.

LEMMA 5.1. *Let X and Y be Banach spaces and let $1 \le p < \infty$ and $p \le q \le \infty$ with $(p, q) \ne (1,\infty)$. Then $c_0(\omega\times\mathcal{K}(X, Y))$ is isomorphic to a complemented subspace of $\mathcal{K}(\ell_p(X), \ell_q(Y))$.*

Proof. Let $T \in \mathcal{K}(\ell_p(X), \ell_p(Y))$. Represent T as a matrix with entries in $\mathcal{K}(X, Y)$. For the moment assume that $q < \infty$. By [20, Proposition 1.c.8

and following Remarks] the operator given by the diagonal of this matrix is a bounded linear operator with norm no larger than $\|T\|$. Therefore the mapping from $\mathcal{K}(\ell_p(X), \ell_q(Y))$ into the diagonal operators with respect to this representation is a contraction. If $p > 1$ and $q = \infty$, then sign change operators are contractive and the argument from [20] shows that the map from $\mathcal{K}(\ell_p(X), \ell_\infty(Y))$ into the diagonal operators is contractive. Also for compact operators that are diagonal the map is the identity.

The norm of a diagonal operator is the supremum of the norms of the operators on the diagonal. For the case $q = \infty$ this is clear. If $q < \infty$, let D_j be the jth block of the diagonal operator D and $(x_j) \in \ell_p(X)$. Then

$$\|(D_j x_j)\|_{\ell_q(Y)} = \Big(\sum \|D_j x_j\|_Y^q\Big)^{1/q} \le \Big(\sum \|D_j\|^q \|x_j\|_X^q\Big)^{1/q} \le (\sup_j \|D_j\|)\|(x_j)\|_{\ell_p(X)},$$

since $q \ge p$. Clearly $\|D\| \ge \sup \|D_j\|$.

Let E_j be the natural inclusion map from X into the elements of $\ell_p(X)$ which are zero except in the jth coordinate and P_j be the projection from $\ell_q(Y)$ onto Y given by choosing the jth coordinate. For all j, let $x_j \in B_X$ be such that

$$\|P_j T E_j x_j\| = \|D_j x_j\| \ge \|D_j\|/2.$$

Because we began with a compact operator, $\{TE_j x_j\}$ is relatively compact in $\ell_q(Y)$. If $1 \le q < \infty$, $\|P_j T E_j x_j\|$ converges to 0 because $T(B_{\ell_p(X)})$ is relatively compact and thus $\sum \|P_j y\|_Y^q$ converges uniformly for $y \in T(B_{\ell_p(X)})$. If $q = \infty$, then $p > 1$ and $(E_j x_j)$ converges weakly to 0. Because T is compact, $(TE_j x_j)$ converges in norm to 0. Therefore the limit of the norms of the operators D_j must be 0. Conversely, any sequence (T_i) of operators with $T_i \in \mathcal{K}(X, Y)$ for all i and $\lim \|T_i\| = 0$ induces a diagonal operator in $\mathcal{K}(\ell_p(X), \ell_q(Y))$ by $D(x_j) = (T_j x_j)$. Clearly each truncation $D^{(n)}$ of D,

$$D^{(n)}(x_j) = (T_1 x_1, \ldots, T_n x_n, 0, 0, \ldots),$$

is compact and $D^{(n)}$ converges to D in norm. ∎

Remark 5.2. The conditions on p and q are not necessary for the proof that the space of diagonals of the compact operators is the range of a contractive map. In fact ℓ_p and ℓ_q can be replaced by spaces with unconditional basis. The computation of the bound on the norm of the diagonal operator requires that the norm on the domain dominate the norm on the range. In addition, to prove compactness of the diagonal we used the fact that the norm of the tail of an element in $\ell_q(Y) \cap T(B_{\ell_p(X)})$ goes to zero. If $p = 1$ and $q = \infty$, this may fail and the diagonal of a compact operator may not be compact. An example of this is the one-dimensional operator $T : \ell_1 \to \ell_\infty$

defined by $T(a_j)=(\sum a_j)1_{\mathbb{N}}$. The corresponding diagonal operator is the inclusion map, $J(a_j)=\sum a_j 1_{\{j\}}$.

THEOREM 5.3. *Let X and Y be Banach spaces and $1\le p<\infty$ and $p\le q\le\infty$ with $(p,q)\ne(1,\infty)$. Then*

$$\mathcal{K}(\ell_p(X),\ell_q(Y))\sim C(\omega,\mathcal{K}(\ell_p(X),\ell_q(Y))).$$

Proof. By Lemma 5.1 with $X_1=\ell_p(X)$ and $Y_1=\ell_q(Y)$ in place of X and Y, we see that

$$\begin{aligned}C_0(\omega,\mathcal{K}(\ell_p(X),\ell_q(Y)))&=C_0(\omega,\mathcal{K}(X_1,Y_1))\overset{c}{\hookrightarrow}\mathcal{K}(X_1,Y_1)\\&=\mathcal{K}(\ell_p(X),\ell_q(Y)).\end{aligned}$$

Therefore by the Pełczyński decomposition method [20, p. 54] and 2.1 we infer

$$\mathcal{K}(\ell_p(X),\ell_q(Y))\sim C_0(\omega,\mathcal{K}(\ell_p(X),\ell_q(Y)))\sim C(\omega,\mathcal{K}(\ell_p(X),\ell_q(Y))).\ \blacksquare$$

COROLLARY 5.4. *$C(\beta\mathbb{N},\ell_q)$ is isomorphic to $C(\omega\times\beta\mathbb{N},\ell_q)$ for $1\le q<\infty$.*

Proof. Let $p=1$, $X=\ell_1$ and $Y=\ell_q$ in the previous result. Recall that $\mathcal{K}(\ell_1,\ell_q)$ is isomorphic to $C(\beta\mathbb{N},\ell_q)$. Thus

$$C(\beta\mathbb{N},\ell_q)\sim C(\omega,C(\beta\mathbb{N},\ell_q))\sim C(\omega\times\beta\mathbb{N},\ell_q).\ \blacksquare$$

An analysis of the proof for the special case $p=1$ shows that we can prove a version of the Cembranos–Freniche result for the case $K=\beta\mathbb{N}$.

PROPOSITION 5.5. *Suppose that X is a Banach space, $K<\infty$, and (P_j) is a sequence of projections defined on X with range X_j and $\|P_j\|\le K$ for all j such that $\lim_j\|P_jx\|=0$ for all $x\in X$. Then $(\sum_j C(\beta\mathbb{N},X_j))_{c_0}$ is complemented in $C(\beta\mathbb{N},X)$.*

Proof. Let $\{N_j\}$ be a partition of $\mathbb{N}$ into countably many disjoint infinite sets. Define an operator P on $C(\beta\mathbb{N},X)$ by $(Pf)(n)=P_j(f(n))$ for all $f\in C(\beta\mathbb{N},X)$, for all $n\in N_j$, $j=1,2,\ldots$. We will show that Pf has norm relatively compact range and thus extends to a continuous function on $\beta\mathbb{N}$. The bound K on the norms of the projections shows that the range of Pf is bounded in X. To see that the range is totally bounded, let $\epsilon>0$ and $\{x_m:m\in M\}$ be a finite $\epsilon/(4K)$-net in $f(\beta\mathbb{N})$. For each $m\in M$, there is an integer J_m such that $\|P_jx_m\|<\epsilon/4$ for all $j\ge J_m$, Let $J=\max_m J_m$. If $j\ge J$ and $x\in f(\beta\mathbb{N})$ then

$$\|P_jx\|\le\min_m(\|P_jx_m\|+\|P_j(x_m-x)\|)<\epsilon/2.$$

Thus the range of Pf is contained in

$$\bigcup_{j\le J}P_j(f(\beta\mathbb{N}))\cup\frac{\epsilon}{2}B_X,$$

and an $\epsilon/2$-net in the compact set $\bigcup_{j\le J}P_j(f(\beta\mathbb{N}))$ will yield an ϵ-net.

Clearly P will be linear, bounded and the identity on $C(\beta N_j, X_j)$ for all j. Moreover the argument above shows that

$$Pf \in \Big(\sum_j C(\beta N_j, X_j)\Big)_{c_0}. \ \blacksquare$$

REMARK 5.6. The results in Sections 3–5 allow us to point out some limitations of our approach if one considers more general classification problems for the spaces $C(K, X)$. Notice that $C(\beta\mathbb{N}, \ell_2 \oplus c_0)$ is isomorphic to

$$C(\beta\mathbb{N}, \ell_2) \oplus C(\beta\mathbb{N}, c_0) \sim C(\omega \times \beta\mathbb{N}, \ell_2) \oplus C(\omega \times \beta\mathbb{N}) \sim C(\beta\mathbb{N}, \ell_2).$$

If we instead use $C(\omega^\omega)$, we get a different outcome:

$$C(\beta\mathbb{N}, \ell_2 \oplus C(\omega^\omega)) \sim C(\beta\mathbb{N}, \ell_2) \oplus C(\beta\mathbb{N}, C(\omega^\omega)).$$

This space is not isomorphic to $C(\beta\mathbb{N}, \ell_2)$. It is complemented in $C(\beta\mathbb{N} \times \omega^\omega, \ell_2)$ but it does not seem likely that it contains $C(\omega^\omega, \ell_2)$ as a complemented subspace. It also seems doubtful that there is any compact Hausdorff space K such that $C(\beta\mathbb{N}, \ell_2 \oplus C(\omega^\omega))$ is isomorphic to $C(K, \ell_2)$.

6. Open questions. We end this paper by stating some questions which it raises. We do not know whether the statement of our main result (Theorem 3.7) remains true in the case where $X = \ell_\infty$, that is,

PROBLEM 6.1. *Let K_1 and K_2 be infinite compact metric spaces. Does it follow that*

$$C(\beta\mathbb{N} \times \beta\mathbb{N} \times K_1) \sim C(\beta\mathbb{N} \times \beta\mathbb{N} \times K_2) \ \Rightarrow\ C(K_1) \sim C(K_2)?$$

Notice that with $X = \mathbb{R}$ in Theorem 3.7 we have (see also [10, Theorem 5.7])

$$C(\beta\mathbb{N} \times K_1) \sim C(\beta\mathbb{N} \times K_2) \ \Rightarrow\ C(K_1) \sim C(K_2)$$

for any infinite compact metric spaces K_1 and K_2.

The case $K_1 = \omega$ and $K_2 = \{1\}$ of Problem 6.1 is the statement of Corollary 5.4 when $q = \infty$, that is,

PROBLEM 6.2. *Is it true that*

$$C(\omega \times \beta\mathbb{N} \times \beta\mathbb{N}) \sim C(\beta\mathbb{N} \times \beta\mathbb{N})?$$

This is a special case of the following question for which Corollary 5.4 gives some answers:

PROBLEM 6.3. *For which infinite-dimensional Banach spaces X is*

$$C(\omega \times \beta\mathbb{N}, X) \sim C(\beta\mathbb{N}, X)?$$

Of course if X is a finite-dimensional space, this is false. If X is isomorphic to $C(\omega, Y)$, then

$$C(\omega \times \beta\mathbb{N}, X) \sim C(\beta\mathbb{N}, C(\omega \times \omega, Y)) \sim C(\beta\mathbb{N}, C(\omega, Y)),$$

and thus such X are examples.

One way to approach Problem 6.2 is to study the isomorphic classification of the complemented subspaces of $C(\beta\mathbb{N}\times\beta\mathbb{N})$. Thus, it would be interesting to solve the following intriguing problem which is a particular case of the well known complemented subspace problem for $C(K)$ spaces (see for instance [24, Section 5]).

Problem 6.4. *Let X be a complemented subspace of $C(\beta\mathbb{N}\times\beta\mathbb{N})$. Suppose that X is an infinite-dimensional separable space. Is X isomorphic to c_0?*

In particular, the other separable $C(K)$ spaces would be eliminated if the answer to the following is no.

Problem 6.5. *Is it true that $C(\omega^\omega)\overset{c}{\hookrightarrow}C(\beta\mathbb{N}\times\beta\mathbb{N})$?*

Finally, observe that Theorem 3.2 leads naturally to the following problem which is connected with the Cembranos and Freniche theorem (Theorem 1.2):

Problem 6.6. *Suppose that X is a Banach space and K is an infinite compact space. Is it true that*

(1) $C(\omega^\omega)\overset{c}{\hookrightarrow}C(K,X)\ \Rightarrow\ c_0\overset{c}{\hookrightarrow}C(K)$ *or* $c_0\hookrightarrow X$?
(2) $C(\omega^\omega)\overset{c}{\hookrightarrow}C(\beta\mathbb{N},X)\ \Rightarrow\ c_0\overset{c}{\hookrightarrow}X$?

If X is separable, then c_0 is always complemented so the latter is true by Theorem 3.2. As noted after the proof of that theorem, variations of this problem with either $C(\omega^\omega)$ or uniformly complemented copies of (ω^n) could also be considered.

It is possible that the proper context for this line of investigation is actually injective tensor products.

Problem 6.7. *Suppose that X and Y are Banach spaces, $\alpha>0$ and $\alpha_n\uparrow\omega^\alpha$ are ordinals, and $C(\omega^{\omega^\alpha})$ is isomorphic to a (complemented) subspace of $X\check{\otimes}Y$. Is*

- *$C(\omega^{\omega^\alpha})$ isomorphic to a (complemented) subspace of X or Y, or*
- *$C(\omega^{\alpha_n})$ uniformly isomorphic to a (complemented) subspace of one of X and Y and c_0 isomorphic to a subspace of the other?*

References

[1] D. E. Alspach, *$C(K)$ norming subsets of $C[0,1]^*$*, Studia Math. 70 (1981), 27–61.
[2] D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math. 3 (1965), 205–210.
[3] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), 151–164.
[4] —, —, *Spaces of continuous functions IV*, ibid. 19 (1960), 53–61.

[5] M. Cambern, *A generalized Banach–Stone theorem*, Proc. Amer. Math. Soc. 17 (1966), 396–400.
[6] P. Cembranos, *$C(K,E)$ contains a complemented copy of* c_0, ibid. 91 (1984), 556–558.
[7] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Math. Stud. 176, North-Holland, Amsterdam, 1993.
[8] J. Diestel and J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.
[9] F. J. Freniche, *Barrelledness of the space of vector valued and simple functions*, Math. Ann. 267 (1984), 479–486.
[10] E. M. Galego, *Banach spaces of continuous vector-valued functions of ordinals*, Proc. Edinburgh Math. Soc. 44 (2001), 49–62.
[11] —, *On isomorphic classifications of spaces of compact operators*, Proc. Amer. Math. Soc. 137 (2009), 3335–3342.
[12] —, *Complete isomorphic classifications of some spaces of compact operators*, ibid. 138 (2010), 725–736.
[13] —, *On spaces of compact operators on $C(K,X)$ spaces*, ibid. 139 (2011), 1383–1386.
[14] E. M. Galego and R. P. Salguedo, *Geometric relations between spaces of nuclear operators and spaces of compact operators*, ibid., to appear.
[15] S. P. Gul'ko and A. V. Os'kin, *Isomorphic classification of spaces of continuous functions on totally ordered bicompacta*, Funktsional. Anal. i Prilozhen. 9 (1975), no. 1, 61–62 (in Russian).
[16] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. 1, North-Holland, Amsterdam, 2001, 1–84.
[17] B. Josefson, *Weak sequential convergence in the dual of a Banach space does not imply norm convergence*, Ark. Mat. 13 (1975), 79–89.
[18] S. V. Kislyakov, *Classification of spaces of continuous functions of ordinals*, Siberian Math. J. 16 (1975), 226–231.
[19] D. Leung and F. Räbiger, *Complemented copies of c_0 in l^∞ sums of Banach spaces*, Illinois J. Math. 34 (1990), 52–58.
[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, Berlin, 1977.
[21] S. Mazurkiewicz et W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fund. Math. 1 (1920), 17–27.
[22] A. A. Milyutin, *Isomorphisms of spaces of continuous functions on compacts of power continuum*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 2 (1966), 150–156 (in Russian).
[23] A. Nissenzweig, *W^* sequential convergence*, Israel J. Math. 22 (1975), 266–272.
[24] H. P. Rosenthal, *The Banach space $C(K)$*, in: Handbook of the Geometry of Banach Spaces, Vol. 1, North-Holland, Amsterdam, 2001, 1547–1602.
[25] C. Samuel, *On spaces of operators on $C(Q)$ spaces (Q countable metric space)*, Proc. Amer. Math. Soc. 137 (2009), 965–970.
[26] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer, New York, 1970.

Dale E. Alspach
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078, U.S.A.
E-mail: alspach@math.okstate.edu

Elói Medina Galego
Department of Mathematics
University of São Paulo
São Paulo, Brazil 05508-090
E-mail: eloi@ime.usp.br

Received October 16, 2011
Revised version November 26, 2011 (7328)