STUDIA MATHEMATICA 222 (2) (2014)

Marcinkiewicz multipliers of higher variation and summability of operator-valued Fourier series

by

EARL BERKSON (Urbana, IL)

Abstract. Let $f \in V_r(\mathbb{T}) \cup \mathfrak{M}_r(\mathbb{T})$, where, for $1 \le r < \infty$, $V_r(\mathbb{T})$ (resp., $\mathfrak{M}_r(\mathbb{T})$) denotes the class of functions (resp., bounded functions) $g : \mathbb{T} \to \mathbb{C}$ such that g has bounded r-variation (resp., uniformly bounded r-variations) on $\mathbb{T}$ (resp., on the dyadic arcs of $\mathbb{T}$). In the author's recent article [New York J. Math. 17 (2011)] it was shown that if $\mathfrak{X}$ is a super-reflexive space, and $E(\cdot) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$ is the spectral decomposition of a trigonometrically well-bounded operator $U \in \mathfrak{B}(\mathfrak{X})$, then over a suitable non-void open interval of r-values, the condition $f \in V_r(\mathbb{T})$ implies that the Fourier series $\sum_{k=-\infty}^{\infty} \widehat{f}(k) z^k U^k$ ($z \in \mathbb{T}$) of the operator ergodic "Stieltjes convolution" $\mathfrak{S}_U : \mathbb{T} \to \mathfrak{B}(\mathfrak{X})$ expressed by $\int_{[0,2\pi]}^{\oplus} f(ze^{it})\, dE(t)$ converges at each $z \in \mathbb{T}$ with respect to the strong operator topology. The present article extends the scope of this result by treating the Fourier series expansions of operator ergodic Stieltjes convolutions when, for a suitable interval of r-values, f is a continuous function that is merely assumed to lie in the broader (but less tractable) class $\mathfrak{M}_r(\mathbb{T})$.

Since it is known that there are a trigonometrically well-bounded operator U_0 acting on the Hilbert sequence space $\mathfrak{X} = \ell^2(\mathbb{N})$ and a function $f_0 \in \mathfrak{M}_1(\mathbb{T})$ which cannot be integrated against the spectral decomposition of U_0, the present treatment of Fourier series expansions for operator ergodic convolutions is confined to a special class of trigonometrically well-bounded operators (specifically, the class of disjoint, modulus mean-bounded operators acting on $L^p(\mu)$, where μ is an arbitrary sigma-finite measure, and $1 < p < \infty$). The above-sketched results for operator-valued Stieltjes convolutions can be viewed as a single-operator transference machinery that is free from the power-boundedness requirements of traditional transference, and endows modern spectral theory and operator ergodic theory with the tools of Fourier analysis in the tradition of Hardy–Littlewood, J. Marcinkiewicz, N. Wiener, the (W. H., G. C., and L. C.) Young dynasty, and others. In particular, the results show the behind-the-scenes benefits of the operator ergodic Hilbert transform and its dual conjugates, and encompass the Fourier multiplier actions of $\mathfrak{M}_r(\mathbb{T})$-functions in the setting of A_p-weighted sequence spaces.

2010 *Mathematics Subject Classification*: 26A45, 40E05, 46E30, 47B40, 47A35.
Key words and phrases: spectral decomposition, spectral integral, Marcinkiewicz multiplier of higher variation, trigonometrically well-bounded operator, disjoint operator, modulus mean-bounded operator, Stieltjes convolution.

DOI: 10.4064/sm222-2-2

1. Introduction and notation. At the outset we sketch briefly the nature of the investigations below, fixing some notation in the process and postponing some of the precise definitions and descriptions of results until later. Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$, and let $\mathbb{R}$ be the real line; the symbols $\mathbb{N}$, $\mathbb{Z}$ will designate, respectively, the set of all positive integers and the set of all integers. The Banach algebra of all complex-valued continuous functions defined on $\mathbb{T}$ will be symbolized by $\mathfrak{C}(\mathbb{T})$. An overbar will be used to signify the complex conjugate $\overline{z}$ of any complex number z. The Banach algebra of all continuous linear operators mapping a Banach space $\mathfrak{X}$ into itself will be denoted by $\mathfrak{B}(\mathfrak{X})$, and the identity of $\mathfrak{B}(\mathfrak{X})$ will be designated by I. The symbol "K" with a (possibly empty) set of subscripts will signify a constant which depends only on those subscripts, and which may change in value from one occurrence to another.

The convergence (resp., $(C,1)$-summability) of a bilateral series $\sum_{k=-\infty}^{\infty} a_k$ will mean the convergence (resp., $(C,1)$-summability) of its sequence of "balanced" partial sums $\{\sum_{k=-n}^{n} a_k\}_{n=0}^{\infty}$. Where it exists, the Fourier transform (respectively, inverse Fourier transform) of a function F will be written as $\widehat{F}$ (respectively, $F^{\vee}$). For each $n \in \mathbb{Z}$, we shall let $\mathfrak{e}_n \in \mathrm{BV}(\mathbb{T})$ denote the corresponding character of $\mathbb{T}$: $\mathfrak{e}_n(z) \equiv z^n$. By the term *trigonometric polynomial* we shall mean a pointwise sum of the form $\sum_{n=-\infty}^{\infty} c_n \mathfrak{e}_n$ such that $\{c_n\}_{n=-\infty}^{\infty} \subseteq \mathbb{C}$ is finitely supported. For each non-negative integer N, we shall denote by κ_N the *Fejér kernel* of order N for $\mathbb{T}$:

$$\kappa_N(z) := \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) z^n, \qquad z \in \mathbb{T}.$$

For $1 \le r < \infty$, we symbolize by $V_r(\mathbb{T})$ (respectively, $\mathfrak{M}_r(\mathbb{T})$) the Banach algebra consisting of all complex-valued functions defined on and having finite r-variation on $\mathbb{T}$ (respectively, all bounded, complex-valued functions on $\mathbb{T}$ having uniformly bounded r-variations on the dyadic arcs of $\mathbb{T}$). The spaces $V_r(\mathbb{T})$ form a nested family that increases with r, and the same is true for the spaces $\mathfrak{M}_r(\mathbb{T})$.

Our present investigations take place under the following blanket assumptions: the Banach space of reference will be $L^p(\mu)$, where μ is an arbitrary σ-finite measure and $1 < p < \infty$, and $T \in \mathfrak{B}(L^p(\mu))$ will be an arbitrary invertible, disjoint operator that is modulus mean-bounded (i.e., the sequence of nth two-sided ergodic averages of the linear modulus of T is uniformly bounded). (These blanket assumptions are known to guarantee, in particular, that T is trigonometrically well-bounded [6]—that is, T has a necessarily unique "unitary-like" spectral decomposition consisting of spectral projections, as detailed in Definitions 2.1 and 2.2 below.)

In this setting we study, relative to the strong operator topology of $\mathfrak{B}(L^p(\mu))$, the $(C,1)$-summability and convergence properties of the Fourier series $\sum_{k=-\infty}^{\infty} \widehat{f}(k) z^k T^k$ associated with the class of operator-valued "Stieltjes convolution" functions $\mathfrak{F}_T : \mathbb{T} \to \mathfrak{B}(L^p(\mu))$ having the form

$$\mathfrak{F}_T(z) := \int_{[0,2\pi]}^{\oplus} f(ze^{it})\, dE(t), \tag{1.1}$$

where $f \in \mathfrak{C}(\mathbb{T}) \cap \mathfrak{M}_r(\mathbb{T})$ with r situated in a certain duly prescribed interval of positive length, and $E(\cdot) : \mathbb{R} \to \mathfrak{B}(L^p(\mu))$ is the idempotent-valued spectral decomposition of T. The present investigations continue the studies regarding $V_r(\mathbb{T})$ in [3], where the Banach space of reference was an arbitrary super-reflexive Banach space $\mathfrak{X}$, and $U \in \mathfrak{B}(\mathfrak{X})$ was an arbitrary trigonometrically well-bounded operator (whose spectral decomposition is denoted here by $\mathcal{E}(\cdot) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$). In this setup, where the space of reference is $\mathfrak{X}$, Theorem 4.1 of [3] established the convergence in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ of the Fourier series $\sum_{k=-\infty}^{\infty} \widehat{g}(k) z^k U^k$ for the functions $\mathfrak{G}_U : \mathbb{T} \to \mathfrak{B}(\mathfrak{X})$ having the form

$$\mathfrak{G}_U(z) := \int_{[0,2\pi]}^{\oplus} g(ze^{it})\, d\mathcal{E}(t),$$

where $g \in V_r(\mathbb{T})$, with r belonging to a certain prescribed interval of positive length.

In particular our current expansion of viewpoint from the classes $V_r(\mathbb{T})$ in the direction of the much broader (and less tractable) classes $\mathfrak{M}_r(\mathbb{T})$ further extends the scope of transference methods in analysis, while remaining free of the power-boundedness requirements imposed by traditional Calderón–Coifman–G. Weiss transference methods (initiated in [13], [16], [17]). While our blanket setting $T \in \mathfrak{B}(L^p(\mu))$ is more restrictive than the scenario $U \in \mathfrak{B}(\mathfrak{X})$ just alluded to, such restrictiveness is necessitated by the general limitations on spectral integration of $\mathfrak{M}_r(\mathbb{T})$ classes. For example, there are a trigonometrically well-bounded operator U_0 acting on the Hilbert sequence space $\mathfrak{X} = \ell^2(\mathbb{N})$ and a function belonging to $\mathfrak{M}_1(\mathbb{T})$ which cannot be integrated against the spectral decomposition of U_0 (see [9, proof of Theorem 6.1]). On the other hand, our present blanket setting provides an appropriate range of r-values which guarantee the existence of the requisite Stieltjes convolutions described in (1.1) for all $f \in \mathfrak{M}_r(\mathbb{T})$, and all $z \in \mathbb{T}$ (see Theorem 4.4 below).

In contrast to the class $V_r(\mathbb{T})$, the class $\mathfrak{M}_r(\mathbb{T})$ is not rotation invariant, and in our context this poses an obstacle to the relevant Fourier analysis of operators (and to various regularization techniques, in particular). This type of obstacle will be circumvented below by invoking techniques flowing from

[3, Theorem 2.1], which is designed for addressing such matters. Primarily to avoid technical distractions in the Fourier analysis of spectral integration, we shall often explicitly impose an auxiliary continuity condition on the integrands.

Before we enter into the precise description below, we discuss briefly how these considerations set up a lively interplay between modern spectral theory, classical harmonic analysis, operator ergodic theory (in particular, the operator ergodic Hilbert transform), and the Fourier multiplier theory associated with A_p-weighted spaces. The starting point of the discussion is the observation that, upon denoting by $G(\cdot)$ the spectral decomposition of projections for an arbitrary trigonometrically well-bounded operator V on a Banach space, we can (via spectral integration) regard the sequence $\{V^k\}_{k=-\infty}^{\infty}$ as being, in a *formal sense*, the sequence of Fourier–Stieltjes coefficients of dG (see (2.3) below). Although in our blanket setting in particular, the formal expression dE, where $E(\cdot)$ is the spectral decomposition of T, cannot be endowed with properties stemming from a countably additive measure, it nevertheless turns out that, for $f \in \mathfrak{C}(\mathbb{T}) \cap \mathfrak{M}_r(\mathbb{T})$, the sequence of Fourier coefficients, relative to the strong operator topology, of the Stieltjes convolution $\mathfrak{F}_T(\cdot)$ in (1.1) has the same form as occurs classically in the scalar-valued case when a function is convolved with a measure, i.e., in our blanket setting, $\widehat{\mathfrak{F}_T} = \{\widehat{f}(k)T^k\}_{k=-\infty}^{\infty}$ (Theorem 4.5 below). Operator-valued Stieltjes convolutions whose Fourier transforms take the latter form constitute a powerful tool furnishing operator-valued transference of classical Fourier series and their roles as expansions.

The ramifications for operator ergodic theory are readily seen. Since trigonometrically well-bounded operators on super-reflexive spaces can, in a sense, be characterized as invertible operators whose operator ergodic discrete Hilbert transform exists ([2, Theorem 4.3] and [8, (6.8)], see Theorem 2.3 below), when the foregoing Fourier series transference engendered by Stieltjes convolutions occurs, it furnishes wide scope for behind-the-scenes roles of the operator ergodic Hilbert transform. From the standpoint of operator-valued harmonic analysis such transference of classical Fourier series links operator theory to Fourier multiplier theory. One avenue of the latter state of affairs will be dealt with in detail in §5 below, when we specialize the scenario to the left bilateral shift and A_p-weighted sequence spaces. There, for the most part, we can involve the Stieltjes convolution $\mathfrak{F}_T(\cdot)$ of (1.1) in multiplier theory, while dropping the continuity requirement on $f \in \mathfrak{M}_r(\mathbb{T})$ (see Theorem 5.4).

In §2 and §3 we shall blend the requisite background items from spectral theory, operator ergodic theory, and real analysis methods in harmonic analysis. This will set the stage for our main results in §4 furnishing limit values in the strong operator topology for the relevant Fourier series (Theorem 4.7, which provides $(C,1)$-summability, and Theorem 4.12, which provides a handy Tauberian theorem for operator-valued Fourier series convergence). The above-described considerations are illustrated in §5 by examples, including applications to bilateral shift operators in the framework of A_p-weighted sequence spaces.

2. Spectral theory and disjoint operators: background. We now begin the detailed considerations by recalling their abstract spectral-theoretic ingredients. The notion of a trigonometrically well-bounded operator $U \in \mathfrak{B}(\mathfrak{X})$ (introduced in [4] and [5]) and its characterization by a "unitary-like" spectral representation, which will play key roles below, rest on the following vehicle for spectral decomposability.

DEFINITION 2.1. A *spectral family of projections* in a Banach space $\mathfrak{X}$ is an idempotent-valued function $E(\cdot) : \mathbb{R} \to \mathfrak{B}(\mathfrak{X})$ with the following properties:

(a) $E(\lambda)E(\tau) = E(\tau)E(\lambda) = E(\lambda)$ if $\lambda \le \tau$;
(b) $\|E\|_u := \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$;
(c) with respect to the strong operator topology, $E(\cdot)$ is right-continuous and has a left-hand limit $E(\lambda^-)$ at each point $\lambda \in \mathbb{R}$;
(d) $E(\lambda) \to I$ as $\lambda \to \infty$ and $E(\lambda) \to 0$ as $\lambda \to -\infty$, each limit being with respect to the strong operator topology.

If, in addition, there exist $a, b \in \mathbb{R}$ with $a \le b$ such that $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda \ge b$, $E(\cdot)$ is said to be *concentrated* on $[a, b]$.

Their definition endows spectral families of projections with properties reminiscent of, but weaker than, those that would be inherited from a countably additive Borel spectral measure on $\mathbb{R}$. Given a spectral family $E(\cdot)$ in the Banach space $\mathfrak{X}$ concentrated on a compact interval $J = [a, b]$, an associated notion of spectral integration can be developed as follows (for these and further basic details regarding spectral integration, see [32]). For each bounded function $\psi : J \to \mathbb{C}$ and each partition $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of J, where we take $\lambda_0 = a$ and $\lambda_n = b$, set

$$\mathcal{S}(\mathcal{P}; \psi, E) = \sum_{k=1}^{n} \psi(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}. \tag{2.1}$$

If the net $\{\mathcal{S}(\mathcal{P}; \psi, E)\}$ converges in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$ as $\mathcal{P}$ runs through the set of partitions of J directed to increase by refinement, then the strong limit is called the *spectral integral* of ψ with respect to $E(\cdot)$, and is denoted by $\int_J \psi(\lambda)\, dE(\lambda)$. In this case, we define $\int_J^{\oplus} \psi(\lambda)\, dE(\lambda)$ by writing

$$\int_J^{\oplus} \psi(\lambda)\, dE(\lambda) := \psi(a)E(a) + \int_J \psi(\lambda)\, dE(\lambda).$$

It can be shown that the spectral integral $\int_J \psi(\lambda)\, dE(\lambda)$ exists for each complex-valued function ψ having bounded variation on J (in symbols, $\psi \in \mathrm{BV}(J)$), and that the mapping $\psi \mapsto \int_J^{\oplus} \psi(\lambda)\, dE(\lambda)$ is an algebra homomorphism of $\mathrm{BV}(J)$ into $\mathfrak{B}(\mathfrak{X})$ satisfying

$$\Big\| \int_J^{\oplus} \psi(t)\, dE(t) \Big\| \le \|\psi\|_{\mathrm{BV}(J)} \|E\|_u, \tag{2.2}$$

where $\|\cdot\|_{\mathrm{BV}(J)}$ denotes the usual Banach algebra norm expressed by

$$\|\psi\|_{\mathrm{BV}(J)} := \sup_{x\in J} |\psi(x)| + \mathrm{var}(\psi, J).$$

Similarly, $\mathrm{BV}(\mathbb{T})$ denotes the Banach algebra of all $f \in \mathrm{BV}([0, 2\pi])$ such that $f(0) = f(2\pi)$. (Where there is no danger of confusion, we shall, as convenient, tacitly indulge in the conventional practice of identifying a function F defined on $\mathbb{T}$ with its (2π)-periodic counterpart $F(e^{i(\cdot)})$ defined on $\mathbb{R}$.)

We can now state the following "spectral theorem" characterization of a trigonometrically well-bounded operator on a Banach space $\mathfrak{X}$.

Definition 2.2. An operator $U \in \mathfrak{B}(\mathfrak{X})$ is said to be *trigonometrically well-bounded* if there is a spectral family $E(\cdot)$ in $\mathfrak{X}$ concentrated on $[0, 2\pi]$ such that $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda)$. In this case, it is possible to arrange that $E((2\pi)^-) = I$, and with this additional property the spectral family $E(\cdot)$ is uniquely determined by U, and is called the *spectral decomposition* of U.

If U is a trigonometrically well-bounded operator on a Banach space $\mathfrak{X}$ with spectral decomposition $E(\cdot)$, then in view of the multiplicativity of spectral integration for functions of bounded variation, we immediately see that, for all $k \in \mathbb{Z}$,

$$\int_{[0,2\pi]}^{\oplus} e^{ik\lambda}\, dE(\lambda) = U^k. \tag{2.3}$$

By, for instance, Theorem 2.1 and Corollary 2.2 of [8], we see that if U is a trigonometrically well-bounded operator on a Banach space $\mathfrak{X}$ with spectral decomposition $E(\cdot)$, then zU is also trigonometrically well-bounded for each $z \in \mathbb{T}$ (we shall denote the spectral decomposition of zU by $E_z(\cdot)$). The collection of spectral decompositions $E_z(\cdot)$, $z \in \mathbb{T}$, has the property

$$\eta(U) := \sup\{\|E_z\|_u : z \in \mathbb{T}\} < \infty. \tag{2.4}$$

Trigonometrically well-bounded operators have a variety of convenient alternative characterizations on Banach spaces—see, e.g., [4, Corollary 2.17], and [8, Proposition 2.5 and Theorem 5.2]. The following recent result in the super-reflexive space setting ([2, Theorem 4.3]) ties trigonometric well-boundedness there directly to the operator ergodic discrete Hilbert averages, and, as mentioned in §1, this state of affairs indicates that in many settings of modern analysis the class of trigonometrically well-bounded operators acts as a transference vehicle for the classical discrete Hilbert transform, allowing the latter to operate abstractly behind the scenes.

THEOREM 2.3. *Let $\mathfrak{X}$ be a super-reflexive Banach space, and let $U \in \mathfrak{B}(\mathfrak{X})$ be invertible. Then U is trigonometrically well-bounded if and only if*

$$\sup\Big\{\Big\|\sum_{0<|k|\le n}\frac{z^k}{k}U^k\Big\| : n\in\mathbb{N},\, z\in\mathbb{T}\Big\}<\infty.$$

We now focus our attention on the operator-ergodic side of our main setting. Throughout all that follows, except when otherwise noted, (Ω,μ) will be a σ-finite measure space, and $1<p<\infty$. An operator $S\in\mathfrak{B}(L^p(\mu))$ will be called *positive* (respectively, *disjoint*) provided that $Sf\ge 0$ μ-a.e. whenever $f\in L^p(\mu)$ and $f\ge 0$ μ-a.e. (respectively, provided that whenever $f,g\in L^p(\mu)$ and the pointwise product fg vanishes μ-a.e. on Ω, it follows that the pointwise product $(Sf)(Sg)$ vanishes μ-a.e. on Ω). Positive operators will also be referred to as *positivity-preserving*. Disjoint operators are also called *Lamperti operators* (or *separation-preserving operators*) in the literature. See [24] for a full account of their basic features, which we now summarize (for a more detailed summary than that presented here, see [6, §2]).

The separation-preserving property has the following alternative characterization: an operator $S\in\mathfrak{B}(L^p(\mu))$ is separation-preserving if and only if there is a positive operator $|S|\in\mathfrak{B}(L^p(\mu))$ with the property that

$$\text{for every } f\in L^p(\mu),\qquad |Sf|=|S|(|f|)\qquad \mu\text{-a.e. on } \Omega. \tag{2.5}$$

In this case, the condition in (2.5) uniquely characterizes $|S|$ among the operators belonging to $\mathfrak{B}(L^p(\mu))$, and $|S|$ is called the *linear modulus* of S; moreover, $|Sf|=\big||S|(f)\big|$ for all $f\in L^p(\mu)$, $|S|$ is also separation-preserving, S is positive if and only if $S=|S|$, and for all $z\in\mathbb{T}$, $|zS|=|S|$. This circle of ideas includes the following pleasant structural facts regarding invertibility (see, e.g., [6, Scholium (2.3)]): if $S\in\mathfrak{B}(L^p(\mu))$ is separation-preserving and invertible, then S^{-1} is separation-preserving, $|S|$ is also invertible, and

$$|S|^{-1}=|S^{-1}|.$$

The study of separation-preserving operators can be traced back as far as Banach (see [1, Chapter XI, §5]). One way to produce separation-preserving operators is to use the fact that if $S\in\mathfrak{B}(L^p(\mu))$ is an invertible positivity-preserving operator, then S is separation-preserving if and only if S^{-1} is also positivity-preserving (see [24, Proposition 3.1]). Another well-known source of separation-preserving operators is the fact that if $p \neq 2$ then any linear isometry of $L^p(\mu)$ into $L^p(\mu)$ is automatically separation-preserving ([26, Corollary 2.1]).

Henceforth let $T \in \mathfrak{B}(L^p(\mu))$ be an invertible separation-preserving operator, and let $\mathcal{A}$ be the algebra under pointwise operations consisting of all complex-valued measurable functions on Ω, identified modulo equality μ-a.e. Under these circumstances, for each $j \in \mathbb{Z}$, there exist $h_j \in \mathcal{A}$ with $|h_j| > 0$ on Ω, and an algebra automorphism Φ_j mapping $\mathcal{A}$ onto $\mathcal{A}$ such that:

(i) for every $f \in L^p(\mu)$, $T^j f$ is expressed by the pointwise product on Ω of the functions h_j and $\Phi_j(f)$;
(ii) Φ_j preserves the μ-a.e. convergence to a limit function for sequences contained in $\mathcal{A}$.

It follows from these properties that Φ_j is, in particular, a positive linear transformation on $\mathcal{A}$, that the sequences $\{h_j\}_{j=-\infty}^{\infty}$ and $\{\Phi_j\}_{j=-\infty}^{\infty}$ are uniquely determined, and that for $j \in \mathbb{Z}$, $f \in \mathcal{A}$, and $0 < \alpha < \infty$, we have

$$|\Phi_j(f)|^\alpha = \Phi_j(|f|^\alpha). \tag{2.6}$$

With the aid of the Radon–Nikodym Theorem it can further be seen that there is a unique sequence $\{J_j\}_{j=-\infty}^{\infty} \subseteq \mathcal{A}$ such that for each $j \in \mathbb{Z}$, $J_j > 0$ on Ω, and

$$\int_\Omega f\,d\mu = \int_\Omega J_j \Phi_j(f)\,d\mu \quad \text{for all } f \in L^1(\mu). \tag{2.7}$$

(This notation will be fixed henceforth.)

Application of the group property $T^{j+k} = T^j T^k$ to the uniquely determined sequences described above furnishes the following relationships, valid for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}$:

$$\Phi_{j+k}(f) = \Phi_j(\Phi_k(f)) \qquad \text{for every } f \in \mathcal{A}, \tag{2.8}$$

$$h_{j+k}(x) = h_j(x)(\Phi_j h_k)(x) \qquad \text{for } \mu\text{-almost all } x \in \Omega, \tag{2.9}$$

$$J_{j+k}(x) = J_j(x)[\Phi_j(J_k)](x) \qquad \text{for } \mu\text{-almost all } x \in \Omega. \tag{2.10}$$

For each non-negative integer n, we shall henceforth denote by $\mathfrak{E}_n(T)$ the nth two-sided ergodic average of T,

$$\mathfrak{E}_n(T) = \frac{1}{2n+1} \sum_{j=-n}^{n} T^j. \tag{2.11}$$

Note that when $T \in \mathfrak{B}(L^p(\mu))$ is an invertible separation-preserving operator, the above considerations also apply to the positive invertible operator $|T|$, which is automatically separation-preserving, and, in terms of the unique sequences associated above with T, is itself associated with the unique sequences $\{|h_j|\}_{j=-\infty}^{\infty}$, $\{\Phi_j\}_{j=-\infty}^{\infty}$, $\{J_j\}_{j=-\infty}^{\infty}$. Similarly, if $T \in$

$\mathfrak{B}(L^p(\mu))$ is an invertible separation-preserving operator, then so is zT for each $z \in \mathbb{T}$, and, in the preceding notation, zT is associated with the unique sequences $\{z^j h_j\}_{j=-\infty}^{\infty}$, $\{\Phi_j\}_{j=-\infty}^{\infty}$, $\{J_j\}_{j=-\infty}^{\infty}$. In this framework we say that T is *mean-bounded* (respectively, *modulus mean-bounded*) provided $\sup\{\|\mathfrak{E}_n(T)\| : n = 0, 1, \ldots\} < \infty$ (respectively, $\sup\{\|\mathfrak{E}_n(|T|)\| : n = 0, 1, \ldots\} < \infty$). In particular, an invertible, positive element of $\mathfrak{B}(L^p(\mu))$ that has a positive inverse is modulus mean-bounded if and only if it is mean-bounded. Standard background facts about disjoint operators such as the foregoing will be used without explicit mention in our considerations below.

We now turn our attention to describing how the above items link the operator ergodic theory of invertible disjoint operators with the rich theory of discrete A_p weights. The A_p condition for weight sequences takes the following form ([22]).

Definition 2.4. Suppose that $1 < p < \infty$. A *weight sequence* (that is, a bilateral sequence of positive real numbers) $w \equiv \{w_k\}_{k=-\infty}^{\infty}$ belongs to the class $A_p(\mathbb{Z})$ provided that there is a real constant C (called an $A_p(\mathbb{Z})$ *weight constant* for w) such that

$$\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k\right)\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k^{-1/(p-1)}\right)^{p-1} \le C,$$

whenever $L \in \mathbb{Z}$, $M \in \mathbb{Z}$, and $L \le M$.

We can now state the seminal dominated ergodic theorem of Martín-Reyes and de la Torre (see [29, §3], or [30, Theorem (2.4)]), which takes the following form when adapted to separation-preserving operators, and which will play a central role in the studies undertaken below.

Theorem 2.5. *Suppose that (Ω, μ) is a σ-finite measure space, $1 < p < \infty$, and $T \in \mathfrak{B}(L^p(\mu))$ is invertible and separation-preserving. Then the following conditions are equivalent:*

(i) *The maximal operator $\mathcal{M}$ defined on $L^p(\mu)$ by*

$$\mathcal{M}f = \sup_{N\ge 0} \frac{1}{2N+1}\sum_{j=-N}^{N} |T^j f|$$

is of strong type (p,p) on $L^p(\mu)$, that is, there exists a real constant ω such that

$$\|\mathcal{M}f\|_{L^p(\mu)} \le \omega \|f\|_{L^p(\mu)} \quad \textit{for all } f \in L^p(\mu).$$

(ii) *T is modulus mean-bounded, that is,*

$$\gamma(T) := \sup\left\{\left\|\frac{1}{2N+1}\sum_{n=-N}^{N} |T|^n\right\| : N \ge 0\right\} < \infty. \tag{2.12}$$

(iii) *In terms of the preceding notation and terminology, there is a real number $\mathfrak{C} > 0$ such that, for μ-almost all $x \in \Omega$, the weight sequence $\{|h_k(x)|^{-p} J_k(x)\}_{k=-\infty}^{\infty}$ belongs to $A_p(\mathbb{Z})$ with $\mathfrak{C}$ as an $A_p(\mathbb{Z})$ weight constant.*

It was deduced in [6, Theorem (4.2)] that if T satisfies the hypotheses and the equivalent conditions in the conclusion of Theorem 2.5, then T is trigonometrically well-bounded. Except where otherwise noted, we shall henceforth confine our attention to this setup—that is, $T \in \mathfrak{B}(L^p(\mu))$ will be invertible, separation-preserving, and modulus mean-bounded—and we shall denote the spectral decomposition of the trigonometrically well-bounded operator T by $E(\cdot)$.

A close reading of [29, pp. 147–148] shows that when the hypotheses and equivalent conditions (i)–(iii) of Theorem 2.5 are satisfied, the constant $\mathfrak{C}$ in Theorem 2.5(iii) can be chosen so as to depend only on p and on $\gamma(T)$ in (2.12). We shall always make this choice (in symbols, $\mathfrak{C} = \mathfrak{C}(p, \gamma(T))$). Since, for each $z \in \mathbb{T}$,

$$\gamma(zT) = \gamma(T) = \gamma(|T|) = \gamma(T^{-1}), \tag{2.13}$$

we shall always have

$$\mathfrak{C}(p, \gamma(zT)) = \mathfrak{C}(p, \gamma(T)) = \mathfrak{C}(p, \gamma(|T|)) = \mathfrak{C}(p, \gamma(T^{-1})).$$

These two chains of equalities herald a certain amount of "keeping track" of constants that we shall indulge in from time to time in order to ensure that we can substitute zT or T^{-1} for T in various conclusions ahead without loss of generality.

We remark that, although on UMD spaces the power-boundedness of an operator U (that is, the condition $\sup_{n\in\mathbb{Z}} \|U^n\| < \infty$) is sufficient to ensure trigonometric well-boundedness of U ([12, Theorem 4.5]), power-boundedness is not a necessary condition for trigonometric well-boundedness in the general Banach space setting, or, for that matter, in the UMD space setting. Whenever a trigonometrically well-bounded operator is not power-bounded, its powers cannot implement the traditional transference methods of [17], which would require power-boundedness, and that is where weighted norm conditions like Theorem 2.5(iii) may come to the forefront, as in the trigonometric well-boundedness of T, together with associated estimates like (3.21) and (4.1) below. Some examples where T satisfies our blanket hypotheses without being power-bounded are taken up in Example 5.1 (including Proposition 5.2).

3. Modulus mean-bounded operators and higher variations. In this note we shall be concerned with the interplay between the invertible, disjoint modulus mean-bounded operator $T \in \mathfrak{B}(L^p(\mu))$ and the Marcinkiewicz r-classes $\mathfrak{M}_r(\mathbb{T})$, $1 \le r < \infty$, which are defined below in (3.15). We first describe the algebras (under pointwise operations) $V_r(\mathbb{T})$, consisting of all $f\colon \mathbb{T} \to \mathbb{C}$ having finite r-variation on $\mathbb{T}$:

$$\operatorname{var}_r(f, \mathbb{T}) := \sup\Big\{\sum_{k=1}^{N} |f(e^{ix_k}) - f(e^{ix_{k-1}})|^r\Big\}^{1/r} < \infty, \tag{3.1}$$

where the supremum is extended over all partitions $0 = x_0 < x_1 < \cdots < x_N = 2\pi$ of $[0, 2\pi]$. When endowed with the following norm, $V_r(\mathbb{T})$ becomes a unital Banach algebra:

$$\|f\|_{V_r(\mathbb{T})} := \sup\{|f(z)| : z \in \mathbb{T}\} + \operatorname{var}_r(f, \mathbb{T}).$$

The Banach algebra $V_1(\mathbb{T})$ coincides with $\mathrm{BV}(\mathbb{T})$, the usual Banach algebra consisting of the functions of bounded variation on $\mathbb{T}$. The class $V_r(\mathbb{T})$, $1 < r < \infty$, was introduced by Norbert Wiener in his article [33], which although primarily aimed at the initial Fourier analysis of $V_2(\mathbb{T})$, shows that if $1 \le r < \infty$, and $f \in V_r(\mathbb{T})$, then at each $t \in \mathbb{R}$, the one-sided limits

$$\lim_{x\to t^+} f(e^{ix}) \quad \text{and} \quad \lim_{x\to t^-} f(e^{ix}) \tag{3.2}$$

exist. (It follows from this by, e.g., [18, p. 15 or p. 18] that the function $x \in \mathbb{R} \mapsto f(e^{ix})$ has only countably many discontinuities.)

For any function $F : \mathbb{T} \to \mathbb{C}$, we shall define the complex-valued (2π)-periodic function $F^{\#}$ at each $t \in \mathbb{R}$ such that the one-sided limits described in (3.2) exist for F by writing

$$F^{\#}(t) = \frac{\lim_{x\to t^+} F(e^{ix}) + \lim_{x\to t^-} F(e^{ix})}{2}. \tag{3.3}$$

It is clear that whenever $1 \le r < \infty$, and $f \in V_r(\mathbb{T})$, we may regard the (2π)-periodic function $f^{\#} : \mathbb{R} \to \mathbb{C}$ as an element of $V_r(\mathbb{T})$, and also that after Fejér's Theorem is applied to (3.2), the observation on pp. 259, 260 of [34] that $V_r(\mathbb{T})$ is a subset of the integrated Lipschitz class $\mathrm{Lip}(r^{-1}, r)$ can be combined with Theorem 1 of [20] to infer that

(3.4) if $1 \le r < \infty$, and $f \in V_r(\mathbb{T})$, then for each $z = e^{it} \in \mathbb{T}$, the Fourier series of f converges at z to $f^{\#}(t)$.

(Prior to [34], Marcinkiewicz presented a self-contained treatment of $V_r(\mathbb{T})$, including [28, (3.4)].) It is elementary to see that for $1 \le r < \infty$, and $g : \mathbb{T} \to \mathbb{C}$, $\operatorname{var}_r(g, \mathbb{T})$ and $\|g\|_{V_r(\mathbb{T})}$ are decreasing extended real-valued functions of r by Jensen's inequality ([21, item 19, p. 28]), and so the class $V_r(\mathbb{T})$ increases with r.

For further discussion of key fundamentals of the r-variation, $1 \le r < \infty$, see, e.g., [3], [14], [15], [20], [31], [34]. In particular, the fact that $V_r(\mathbb{T}) \subseteq \mathrm{Lip}(r^{-1}, r)$, when taken in conjunction with Lemma 11 of [20], furnishes the following rate of decay for the Fourier coefficients of arbitrary

$f \in V_r(\mathbb{T})$:

$$\sup_{k\in\mathbb{Z}} |k|^{1/r}|\widehat{f}(k)| < \infty. \tag{3.5}$$

Thanks to the Closed Graph Theorem this last formula can be rephrased as follows. If $1 \le r < \infty$, and $f \in V_r(\mathbb{T})$, then

$$\sup_{k\in\mathbb{Z}} |k|^{1/r}|\widehat{f}(k)| \le K_r \|f\|_{V_r(\mathbb{T})}. \tag{3.6}$$

We remark that (3.5) can be shown to imply that for $1 < s < \infty$, $\bigcup_{1\le r<s} V_r(\mathbb{T})$ is not dense in the Banach space $V_s(\mathbb{T})$ (see [2, Remark 2.8(ii)]). In keeping with the notation introduced at the start of this article, for each $n \in \mathbb{Z}$, we denote by $\mathfrak{e}_n \in \mathrm{BV}(\mathbb{T})$ the corresponding character of $\mathbb{T}$: $\mathfrak{e}_n(z) \equiv z^n$. The reasoning in [34, pp. 275, 276] shows that for $1 \le r < \infty$,

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} \frac{\mathrm{var}_r(\mathfrak{e}_n, \mathbb{T})}{|n|^{1/r}} < \infty. \tag{3.7}$$

The analogous formulation to (3.1) is employed to define the r-variation of a complex-valued function on an arbitrary compact interval $J = [a,b]$ of $\mathbb{R}$, and thus to define analogously the corresponding unital Banach algebra $V_r(J)$. As in the case of $V_r(\mathbb{T})$, Jensen's inequality shows that for $1 \le r < \infty$ and $g : J \to \mathbb{C}$, $\mathrm{var}_r(g, J)$ and $\|g\|_{V_r(J)}$ are decreasing extended real-valued functions of r, and so the class $V_r(J)$ increases with r. It follows readily that for $1 \le r < \infty$, and each monotone function $G : J \to \mathbb{R}$, we have $\mathrm{var}_r(G, J) = \mathrm{var}_1(G, J)$. Also, as in the case of $V_r(\mathbb{T})$, if $F : J \to \mathbb{C}$ belongs to $V_r(J)$, then F has a left-hand (respectively, right-hand) limit at each point of the interval $(a, b]$ (respectively, $[a, b)$), and so the set of discontinuities of F on J is countable. It is an elementary consequence of Minkowski's inequality that if c is an interior point of the compact interval $J = [a,b]$, then for $1 \le r < \infty$ and any function $g : J \to \mathbb{C}$,

$$\mathrm{var}_r(g, [a,b]) \le \mathrm{var}_r(g, [a,c]) + \mathrm{var}_r(g, [c,b]). \tag{3.8}$$

If we replace absolute values by norms in the foregoing definitions of r-variation, we arrive at the corresponding definitions for vector-valued functions. Furthermore, for a vector-valued function f defined on $\mathbb{R}$ (including the scalar-valued case), the standard counterpart for $\mathbb{R}$ of r-variation is given by

$$\mathrm{var}_r(f, \mathbb{R}) = \sup_{-\infty<a<b<\infty} \mathrm{var}_r(f, [a,b]).$$

If $E(\cdot)$ is a spectral family of projections in an arbitrary Banach space $\mathfrak{X}$, and $1 \le q < \infty$, we shall also use the symbol $\mathrm{var}_q(E)$ to denote

$$\sup\{\mathrm{var}_q(E(\cdot)x, \mathbb{R}) : \|x\| \le 1\}.$$

Theorem 3.7 of [2] shows that some delicate features of spectral families can be blended in with the R. C. James inequality for super-reflexive Banach spaces developed in [23, Theorem 3] and with a fundamental feature of Young–Stieltjes integration (in [34, §10]) to establish the following mainstay of spectral integration for functions of higher variation.

THEOREM 3.1. *Let X be a super-reflexive Banach space, and let $E(\cdot)$ be the spectral decomposition of a trigonometrically well-bounded operator $U \in \mathfrak{B}(X)$. Then for some $q \in (1,\infty)$, we have*

$$\operatorname{var}_q(E) < \infty. \tag{3.9}$$

Let $r \in (1, q')$, where $q' = q(q-1)^{-1}$ is the conjugate index of q. Then the spectral integral $\int_{[0,2\pi]} \phi(e^{it})\, dE(t)$ exists for each $\phi \in V_r(\mathbb{T})$, and the mapping $\phi \mapsto \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t)$ is an identity-preserving algebra homomorphism of $V_r(\mathbb{T})$ into $\mathfrak{B}(\mathfrak{X})$ such that

$$\Big\| \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t) \Big\| \le K_{r,q} \operatorname{var}_q(E) \|\phi\|_{V_r(\mathbb{T})} \quad \textit{for all } \phi \in V_r(\mathbb{T}). \tag{3.10}$$

The following item (3.11) arises from the circle of ideas involved in Theorem 3.1 above (see [2, Proposition 3.2]). More specifically, it follows from (2.4) and [3, Theorem 3.2] that if $\mathfrak{X}$ is super-reflexive, $U \in \mathfrak{B}(\mathfrak{X})$ is trigonometrically well-bounded, and $E(\cdot)$ is the spectral decomposition of U, then there is $q_1 = q_1(U) \in (1,\infty)$ such that its conjugate index q_1' satisfies

$$\tau(U) := \sup\{\operatorname{var}_{q_1'}(E_z) : z \in \mathbb{T}\} < \infty, \tag{3.11}$$

where, as in (2.4), E_z denotes the spectral decomposition of zU. Notice that under these circumstances, we also have, for all $z, w \in \mathbb{T}$,

$$(E_w)_z = E_{zw}, \tag{3.12}$$

and hence for a given trigonometrically well-bounded operator U on a super-reflexive space, we can and always will choose the same value of q_1 for all zU, $z \in \mathbb{T}$, thereby giving us

$$q_1(U) = q_1(zU) \quad \text{and} \quad \tau(U) = \tau(zU) \quad \text{for each } z \in \mathbb{T}. \tag{3.13}$$

The dyadic points relevant to the study of (2π)-periodic functions are the terms of the sequence $\{t_k\}_{k=-\infty}^{\infty} \subseteq (0, 2\pi)$ given by

$$t_k = \begin{cases} 2^{k-1}\pi & \text{if } k \le 0, \\ 2\pi - 2^{-k}\pi & \text{if } k > 0. \end{cases} \tag{3.14}$$

The dyadic arcs Δ_k, $k \in \mathbb{Z}$, are specified by $\Delta_k = \{e^{ix} : x \in [t_k, t_{k+1}]\}$, and for $\phi : \mathbb{T} \to \mathbb{C}$, we shall write $\operatorname{var}_r(\phi, \Delta_k)$ to stand for $\operatorname{var}_r(\phi(e^{i(\cdot)}), [t_k, t_{k+1}])$. With this notation, $\mathfrak{M}_r(\mathbb{T})$, $1 \le r < \infty$, is defined as the class of all functions

$\phi : \mathbb{T} \to \mathbb{C}$ such that

$$\|\phi\|_{\mathfrak{M}_r(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\phi(z)| + \sup_{k\in\mathbb{Z}} \operatorname{var}_r(\phi, \Delta_k) < \infty. \tag{3.15}$$

The class $\mathfrak{M}_r(\mathbb{T})$ is called the *Marcinkiewicz r-class* for $\mathbb{T}$; with the norm $\|\cdot\|_{\mathfrak{M}_r(\mathbb{T})}$ just described, $\mathfrak{M}_r(\mathbb{T})$ is a unital Banach algebra under pointwise operations.

The classes $\mathfrak{M}_r(\mathbb{T})$, which consist of Fourier multipliers in various weighted and unweighted settings, have been featured in a number of studies—e.g., [10], [11], and [15]. It is readily seen from (3.8) that for $\psi \in \mathfrak{M}_r(\mathbb{T})$ and J any compact subinterval of the open interval $(0, 2\pi)$, $\operatorname{var}_r(\psi(e^{i(\cdot)}), J) < \infty$, whence the domain of definition of the function $\psi^{\#}$ contains $\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}$. In Proposition 4.9 below, the Principle of Localization for Fourier series will be used along with [35] to deduce a counterpart for the classes $\mathfrak{M}_r(\mathbb{T})$, $1 \le r < \infty$, to the pointwise convergence (3.4) of the Fourier series for $V_r(\mathbb{T})$ functions.

Notice that for any function $f : \mathbb{T} \to \mathbb{C}$,

$$\|f\|_{\mathfrak{M}_r(\mathbb{T})} \le \|f\|_{V_r(\mathbb{T})},$$

and so $V_r(\mathbb{T}) \subseteq \mathfrak{M}_r(\mathbb{T})$. Moreover, the class $\mathfrak{M}_r(\mathbb{T})$ increases as r increases, since by Jensen's inequality, $\|\cdot\|_{\mathfrak{M}_r(\mathbb{T})}$ is a decreasing function of r. It is readily seen that if ϕ belongs to $\mathfrak{M}_r(\mathbb{T})$, then so does the function ϕ^* specified on $\mathbb{T}$ by writing

$$\phi^*(z) := \overline{\phi(z^{-1})}. \tag{3.16}$$

Additionally, $\|\phi\|_{\mathfrak{M}_r(\mathbb{T})} = \|\phi^*\|_{\mathfrak{M}_r(\mathbb{T})}$, and the Fourier transform $(\phi^*)^\wedge$ satisfies

$$(\phi^*)^\wedge(k) = \overline{\phi^\wedge(k)} \quad \text{for all } k \in \mathbb{Z}. \tag{3.17}$$

The next fact (which does not seem to be readily available in the literature) describes how the decay rate of the Fourier coefficients of the general function $\psi \in \mathfrak{M}_r(\mathbb{T})$ tries to imitate the behavior stipulated in (3.6) for the decay rate of the Fourier coefficients of the general $V_r(\mathbb{T})$-function.

THEOREM 3.2. *If $1 \le r < \infty$, and $\psi \in \mathfrak{M}_r(\mathbb{T})$, then for each $n \in \mathbb{Z}\setminus\{0\}$, the nth Fourier coefficient of ψ satisfies*

$$|\widehat{\psi}(n)| \le K_r \|\psi\|_{\mathfrak{M}_r(\mathbb{T})} |n|^{-1/r} \log(\pi|n|). \tag{3.18}$$

Proof. Given $n \in \mathbb{Z} \setminus \{0\}$, choose $k_n \in \mathbb{N}$ such that

$$2^{k_n}\pi^{-1} \le |n| < 2^{k_n+1}\pi^{-1}.$$

Thus $|n|^{-1}$ lies in the dyadic interval $(\pi 2^{-k_n-1}, \pi 2^{-k_n}] = (t_{-k_n}, t_{-k_n+1}]$, and so we have, in particular,

$$k_n \le \frac{\log(\pi|n|)}{\log 2}. \tag{3.19}$$

Upon denoting the characteristic function of an interval $J \subseteq (0, 2\pi)$ by χ_J, we see with the aid of (3.6) and repeated use of (3.8) that, for $n \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}
2\pi|\widehat{\psi}(n)| &\le \Big| \int_0^{t_{-k_n}} \psi(e^{iu}) e^{-inu}\, du \Big| \\
&\quad + \Big| \int_{t_{-k_n}}^{t_{k_n+1}} \psi(e^{iu}) e^{-inu}\, du \Big| + \Big| \int_{t_{k_n+1}}^{2\pi} \psi(e^{iu}) e^{-inu}\, du \Big| \\
&\le 2\|\psi\|_u \pi 2^{-k_n-1} + K_r \|\chi_{[t_{-k_n}, t_{k_n+1}]}\psi\|_{V_r(\mathbb{T})} |n|^{-1/r} \\
&\le 2\|\psi\|_u |n|^{-1} + K_r |n|^{-1/r} \Big(5\|\psi\|_u + \operatorname*{var}_r(\psi, {}_{[t_{-k_n}, t_{k_n+1}]}) \Big) \\
&\le 2\|\psi\|_u |n|^{-1} + K_r |n|^{-1/r} \Big\{ 5\|\psi\|_u + (2k_n + 1) \sup_{k \in \mathbb{Z}} \operatorname{var}_r(\psi, \Delta_k) \Big\} \\
&\le 2\|\psi\|_u |n|^{-1} + K_r |n|^{-1/r} (2k_n + 1) \|\psi\|_{\mathfrak{M}_r(\mathbb{T})}.
\end{aligned}$$

An application of (3.19) to this last inequality shows that, for arbitrary $n \in \mathbb{Z} \setminus \{0\}$,

$$|\widehat{\psi}(n)| \le K\|\psi\|_u |n|^{-1} + K_r \|\psi\|_{\mathfrak{M}_r(\mathbb{T})} |n|^{-1/r} \log(\pi|n|). \ \blacksquare$$

REMARK 3.3. In the case $r = 1$, the discussion in [19, §8] observes (in what is now archaic nomenclature and without proof) the following specialized form of the decay rate that has been stipulated for $1 \le r < \infty$ in (3.18) above:

$$\text{(3.20)} \qquad \text{for } \psi \in \mathfrak{M}_1(\mathbb{T}), \quad \widehat{\psi}(n) = O(|n|^{-1} \log |n|) \quad \text{as } |n| \to \infty.$$

This fact prompts Hardy and Littlewood to raise the question of whether $\log |n|$ can be dropped from (3.20), as would be the case if ψ were of bounded variation on all of $\mathbb{T}$. They answer this question in the negative by explicitly constructing a counterexample in the form of a function $\psi_0 \in \mathfrak{M}_1(\mathbb{T})$ such that $\psi_0(e^{i(\cdot)})$ is an even function on $\mathbb{R}$, and such that ψ_0 fails to satisfy

$$\widehat{\psi_0}(n) = O(|n|^{-1}) \quad \text{as } |n| \to \infty.$$

We begin the study of the interplay between our blanket hypotheses for the operator T and the classes $\mathfrak{M}_r(\mathbb{T})$ by recalling Theorem 10 of [10].

THEOREM 3.4. *Suppose that (Ω, μ) is a σ-finite measure space, $1 < p < \infty$, and $T \in \mathfrak{B}(L^p(\mu))$ is an invertible, disjoint, modulus mean-bounded operator, and let $\gamma(T)$ be as in* (2.12):

$$\gamma(T) = \sup\Big\{ \Big\| \frac{1}{2N+1} \sum_{n=-N}^{N} |T|^n \Big\| : N \ge 0 \Big\} < \infty.$$

Then T is a trigonometrically well-bounded operator (whose spectral decomposition we denote by $E(\cdot)$), and there is a corresponding index $q_2 = q_2(p, \gamma(T)) \in (1, \infty)$ with the property that for every $r \in [1, q_2)$, the spectral integral $\int_{[0,2\pi]} \phi(e^{it})\, dE(t)$ exists for each $\phi \in \mathfrak{M}_r(\mathbb{T})$, and the mapping $\phi \mapsto \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t)$ is an identity-preserving algebra homomorphism of $\mathfrak{M}_r(\mathbb{T})$ into $\mathfrak{B}(L^p(\mu))$ such that

$$\Big\| \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t) \Big\| \le K_{p,r,\gamma(T)} \|\phi\|_{\mathfrak{M}_r(\mathbb{T})} \quad \textit{for all } \phi \in \mathfrak{M}_r(\mathbb{T}). \tag{3.21}$$

Theorem 3.4 permits us to formulate and prove the following convergence theorem of independent interest for families of spectral integrals having functions of a Marcinkiewicz r-class as integrands. This result parallels for Marcinkiewicz r-classes and for $E(\cdot)$ in the present setup the result in [2, Theorem 3.11] for spectral integrals with integrands belonging to appropriate classes V_r in the setting of arbitrary spectral families of projections on super-reflexive Banach spaces.

THEOREM 3.5. *Assume the hypotheses of Theorem 3.4. Let $q_3(T) \in (1, \infty)$ be the minimum of $q_2(p, \gamma(T)) \in (1, \infty)$ in Theorem 3.4 and $q_1(T) \in (1, \infty)$ whose conjugate index figures in (3.11). Whenever $r \in [1, q_3)$, and $\{\phi_\gamma\}_{\gamma \in \Gamma} \subseteq \mathfrak{M}_r(\mathbb{T})$ is a net such that:*

(i) *for each $\gamma \in \Gamma$, the function $t \in (0, 2\pi) \mapsto \phi_\gamma(e^{it})$ is left-continuous on $(0, 2\pi)$,*
(ii) *the net $\{\phi_\gamma\}_{\gamma \in \Gamma}$ converges pointwise on $\mathbb{T}$ to a function $\phi : \mathbb{T} \to \mathbb{C}$,*
(iii) $\sup\{\|\phi_\gamma\|_{\mathfrak{M}_r(\mathbb{T})} : \gamma \in \Gamma\} < \infty$,

then $\phi \in \mathfrak{M}_r(\mathbb{T})$, and the net $\{\int_{[0,2\pi]}^{\oplus} \phi_\gamma(e^{it})\, dE(t)\}_{\gamma \in \Gamma}$ converges in the strong operator topology of $\mathfrak{B}(L^p(\mu))$ to $\int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t)$.

Proof. It is clear from (ii) and (iii) that $\phi \in \mathfrak{M}_r(\mathbb{T})$. By (iii) and (3.21),

$$\sup\Big\{\Big\| \int_{[0,2\pi]}^{\oplus} \phi_\gamma(e^{it})\, dE(t) \Big\| : \gamma \in \Gamma\Big\} < \infty. \tag{3.22}$$

It is also readily seen from the properties of a spectral decomposition that, in terms of the dyadic points of $(0, 2\pi)$ in (3.14), the linear manifold

$$\bigcup_{n \in \mathbb{N}} \{E(t_n) - E(t_{-n})\} L^p(\mu)$$

is norm-dense in $\{I - E(0)\} L^p(\mu)$. In view of this and (3.22), the proof reduces to showing that for each $f \in L^p(\mu)$ and each $n \in \mathbb{N}$, the net

$$\Big\{ \int\limits_{[0,2\pi]} \phi_\gamma(e^{it})\, dE(t)\{E(t_n) - E(t_{-n})\}\Big\} f$$

converges in the norm topology of $L^p(\mu)$ to $\{\int_{[0,2\pi]} \phi(e^{it})\, dE(t)\{E(t_n) - E(t_{-n})\}\} f$, as γ runs through Γ. In other words, it suffices for the present theorem to show that for each $f \in L^p(\mu)$ and each $n \in \mathbb{N}$,

$$\lim_\gamma \Big\| \int\limits_{[t_{-n},t_n]} \phi_\gamma(e^{it})\, dE(t) f - \int\limits_{[t_{-n},t_n]} \phi(e^{it})\, dE(t) f \Big\| = 0. \tag{3.23}$$

Since $n \in \mathbb{N}$ is fixed here, we obviously can iterate (3.8) to get

$$\sup\{\mathrm{var}_r(\phi_\gamma, [t_{-n}, t_n]) : \gamma \in \Gamma\} \le 2n \sup\{\|\phi_\gamma\|_{\mathfrak{M}_r(\mathbb{T})} : \gamma \in \Gamma\} < \infty,$$

and from this fact, together with (i), (ii), and Theorem 3.11 of [2], we immediately deduce (3.23) when $1 < r < q_3(T)$ (for the case $r = 1$ of (3.23), apply the well-known standard fact about spectral integration for nets of BV functions recounted in [2, Theorem 2.2]). ∎

4. Operator-valued Fourier analysis with modulus mean-bounded operators. To start this section, we recall a few needed items from the multiplier theory for discrete A_p-weighted ℓ^p-spaces. For $1 < p < \infty$, and $w := \{w_k\}_{k=-\infty}^{\infty} \in A_p(\mathbb{Z})$, denote by $\ell^p(w)$ the corresponding Banach space consisting of all complex-valued sequences $x \equiv \{x_k\}_{k=-\infty}^{\infty}$ such that

$$\|x\|_{\ell^p(w)} := \Big\{ \sum_{k=-\infty}^{\infty} |x_k|^p w_k \Big\}^{1/p} < \infty.$$

DEFINITION 4.1. Suppose that $1 < p < \infty$, and $w \in A_p(\mathbb{Z})$. We say that $\psi \in L^\infty(\mathbb{T})$ is a *multiplier for* $\ell^p(w)$ (in symbols, $\psi \in M_{p,w}(\mathbb{T})$) provided that convolution by its inverse Fourier transform defines a bounded operator on $\ell^p(w)$. Specifically, we require:

(i) for each $x := \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$ and each $j \in \mathbb{Z}$, the series

$$(\psi^\vee * x)(j) := \sum_{k=-\infty}^{\infty} \psi^\vee(j-k) x_k$$

converges absolutely, and

(ii) the mapping $S_\psi^{(p,w)} : x \mapsto \psi^\vee * x$ is a bounded linear mapping of $\ell^p(w)$ into $\ell^p(w)$.

We then call $S_\psi^{(p,w)}$ the *multiplier transform corresponding to* ψ.

The elements of $M_{p,w}(\mathbb{T})$ are identified modulo equality a.e. on $\mathbb{T}$. Straightforward reasoning shows that $M_{p,w}(\mathbb{T})$ is an algebra under pointwise operations, and that the mapping $\psi \mapsto S_\psi^{(p,w)}$ is an algebra isomorphism of $M_{p,w}(\mathbb{T})$ into $\mathfrak{B}(\ell^p(w))$. Hence $M_{p,w}(\mathbb{T})$ is a unital normed algebra under the norm

$$\|\psi\|_{M_{p,w}(\mathbb{T})} \equiv \|S_\psi^{(p,w)}\|_{\mathfrak{B}(\ell^p(w))}.$$

In fact, when furnished with this norm, $M_{p,w}(\mathbb{T})$ is a unital Banach algebra [7, Theorem 2.10]. (For the multiplier properties of the classes $\mathfrak{M}_r(\mathbb{T})$ in the setting of $A_p(\mathbb{Z})$-weighted sequence spaces see [10, Theorems 9 and 10, and Remark 2].)

The next item describes multiplier regularization for the function classes $M_{p,w}(\mathbb{T})$ (see [6, Theorem (5.2)]).

THEOREM 4.2. *Suppose that* $1 < p < \infty$, $w \in A_p(\mathbb{Z})$, $\phi \in M_{p,w}(\mathbb{T})$, *and* $k \in L^1(\mathbb{T})$. *Then the convolution* $k * \phi$ *belongs to* $M_{p,w}(\mathbb{T})$, *and*

$$\|k * \phi\|_{M_{p,w}(\mathbb{T})} \le \|k\|_{L^1(\mathbb{T})} \|\phi\|_{M_{p,w}(\mathbb{T})}.$$

The method of proof for Theorem 11 of [10] will now be adapted to obtain the following control over operator-valued $(C,1)$-means in our blanket setting. We include this transference theorem for its independent interest, although its estimate in (4.1) will not be needed to establish either of our two main results (Theorems 4.7 and 4.12), and could be bypassed in §5, when we take up applications of the present section to operator-valued Fourier analysis for shifts on the weighted-norm spaces $\ell^p(w)$.

THEOREM 4.3. *Suppose that* (Ω,μ) *is a* σ*-finite measure space,* $1 < p < \infty$, *and* $T \in \mathfrak{B}(L^p(\mu))$ *is invertible, disjoint, and modulus mean-bounded. Then the index* $q_2 = q_2(p,\gamma(T)) \in (1,\infty)$ *described in Theorem* 3.4 *has the property that whenever* $r \in [1,q_2)$ *and* $\psi \in \mathfrak{M}_r(\mathbb{T})$, *the uniform* $(C,1)$*-boundedness of the series* $\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) z^k T^k$ *holds. Specifically, for all* $n \ge 0$ *and all* $z \in \mathbb{T}$,

$$\left\| \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \widehat{\psi}(k) z^k T^k \right\|_{\mathfrak{B}(L^p(\mu))} \le K_{p,r,\gamma(T)} \|\psi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{4.1}$$

Proof. For each non-negative integer n, we continue to denote by κ_n the Fejér kernel of order n for $\mathbb{T}$: $\kappa_n(z) \equiv \sum_{k=-n}^{n} (1 - |k|/(n+1)) z^k$, $z \in \mathbb{T}$. It is elementary that $\kappa_n \ge 0$ and $\|\kappa_n\|_{L^1(\mathbb{T})} = 1$. We first consider the case $z = 1$. Free use will now be made of the notation for and properties of invertible separation-preserving operators described in the latter part of §1. For $\psi \in \mathfrak{M}_r(\mathbb{T})$, under the present hypotheses, we fix $n \ge 0$, and then apply the reasoning for the proof of Theorem 11 of [10] to the trigonometric polynomial $\kappa_n * \psi$, starting with use of (2.7) above. By virtue of the multiplier properties of $\mathfrak{M}_r(\mathbb{T})$ for $A_p(\mathbb{Z})$-weighted sequence spaces (described in [10, Theorems 9 and 10 and Remark 2]), this reasoning shows that for

$f \in L^p(\mu)$, $L \in \mathbb{N}$, and $\chi_{L,n}$ denoting the characteristic function, defined on $\mathbb{Z}$, of $\{k \in \mathbb{Z} : |k| \le L+n\}$, we have

$$\Big\| \sum_{\nu=-n}^{n} \widehat{\kappa_n}(\nu)\widehat{\psi}(\nu)T^\nu f\Big\|_{L^p(\mu)}^p$$
$$= \frac{1}{2L+1}\int_\Omega \sum_{j=-L}^{L} J_j\,|h_j|^{-p}\Big|\sum_{\nu=-\infty}^{\infty}(\kappa_n * \psi)^\vee(\nu)\chi_{L,n}(j-\nu)h_{j-\nu}\Phi_{j-\nu}f\Big|^p\,d\mu$$
$$\le K_{p,r,\gamma(T)}\,\|\psi\|_{\mathfrak{M}_r(\mathbb{T})}^p\frac{2L+2n+1}{2L+1}\|f\|_{L^p(\mu)}^p.$$

Upon letting $L \to \infty$, we obtain (4.1) in the special case $z = 1$. For the general $w \in \mathbb{T}$, we replace T with wT and apply the conclusion for the special case $z = 1$, while taking due account of (2.13) for w. ∎

For a function $\mathfrak{f} : \mathbb{T} \to \mathbb{C}$ and $z \in \mathbb{T}$ we signify by $\mathfrak{f}_z$ the corresponding "rotate" of $\mathfrak{f}$, specified for all $w \in \mathbb{T}$ by $\mathfrak{f}_z(w) = \mathfrak{f}(zw)$. Although none of the function classes $\mathfrak{M}_r(\mathbb{T})$, $1 \le r < \infty$, is a rotation-invariant class in this sense, the fact that under our blanket hypotheses we have $\gamma(T) = \gamma(zT)$ for all $z \in \mathbb{T}$ permits one to apply the general Banach space result [3, Theorem 2.1] to deduce readily (see [3, (d) on pp. 29, 30]) the following generalization of Theorem 3.4 above.

Theorem 4.4. *Under the hypotheses and notation of Theorem* 3.4, *the corresponding index $q_2 = q_2(p,\gamma(T)) \in (1,\infty)$ in its conclusion has the property that for every $r \in [1,q_2)$, each $\phi \in \mathfrak{M}_r(\mathbb{T})$, and every $z \in \mathbb{T}$, the spectral integral $\Phi_T(z) \equiv \int_{[0,2\pi]}^{\oplus} \phi_z(e^{it})\,dE(t)$ exists, equals $\int_{[0,2\pi]}^{\oplus}\phi(e^{it})\,dE_z(t)$, and satisfies the estimate*

$$\Big\|\int_{[0,2\pi]}^{\oplus}\phi_z(e^{it})\,dE(t)\Big\| \le K_{p,r,\gamma(T)}\|\phi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{4.2}$$

The next result ties our considerations in with vector-valued Fourier series.

Theorem 4.5. *Suppose that (Ω,μ) is a σ-finite measure space, $1 < p < \infty$, and $T \in \mathfrak{B}(L^p(\mu))$ is invertible, disjoint, and modulus mean-bounded. Let $E(\cdot)$ be the spectral decomposition of T, and let $q_3(T) \in (1,\infty)$ be the minimum of $q_2(p,\gamma(T)) \in (1,\infty)$ in Theorem* 3.4 *and $q_1(T) \in (1,\infty)$ whose conjugate index figures in* (3.11). *If $r \in [1,q_3)$ and $\psi \in \mathfrak{M}_r(\mathbb{T})$ is a continuous function on $\mathbb{T}$, we invoke Theorem* 4.4 *above to define the operator-valued function $\Psi_T : \mathbb{T} \to \mathfrak{B}(L^p(\mu))$ specified by writing*

$$\Psi_T(z) = \int_{[0,2\pi]}^{\oplus}\psi(ze^{it})\,dE(t) \quad \text{for all } z \in \mathbb{T}.$$

Then

$$\sup_{z\in\mathbb{T}} \|\Psi_T(z)\|_{\mathfrak{B}(L^p(\mu))} < \infty, \tag{4.3}$$

and for each $f \in L^p(\mu)$ and each $k \in \mathbb{Z}$, the kth Fourier coefficient of the bounded vector-valued function $\Psi_T(\cdot)f : \mathbb{T} \to L^p(\mu)$ is expressed by

$$(\widehat{\Psi_T(\cdot)f})(k) = \widehat{\psi}(k)T^k f. \tag{4.4}$$

Proof. Let $\psi \in \mathfrak{M}_r(\mathbb{T})$ be a continuous function. For $n \in \mathbb{N}$, we denote by χ_n the characteristic function of the union of the singleton subset $\{1\}$ of $\mathbb{T}$ with the subarc of $\mathbb{T} \setminus \{1\}$ specified in terms of the dyadic points notation of (3.14) by $\{e^{it} : t \in (t_{-n}, t_n]\}$, and we define ψ_n to be $\psi\chi_n$. Notice that since $\mathfrak{M}_r(\mathbb{T})$ is a Banach algebra, for all $n \in \mathbb{N}$ we have

$$\|\psi_n\|_{\mathfrak{M}_r(\mathbb{T})} \le \|\chi_n\|_{\mathfrak{M}_r(\mathbb{T})}\|\psi\|_{\mathfrak{M}_r(\mathbb{T})} \le K\|\psi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

By virtue of Theorem 3.5, (3.11), and (3.13) applied to $S = zT$ for each $z \in \mathbb{T}$, the sequence $\{\int_{[0,2\pi]}^{\oplus} \psi_n(e^{it})\, dE_z(t)\}_{n=1}^{\infty}$ converges in the strong operator topology to $\int_{[0,2\pi]}^{\oplus} \psi(e^{it})\, dE_z(t)$, and we see from (4.2) that for each $n \in \mathbb{N}$,

$$\Big\|\int_{[0,2\pi]}^{\oplus} \psi_n(e^{it})\, dE_z(t)\Big\| \le K_{p,r,\gamma(T)}\|\psi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

It now follows by Theorem 4.4 that for each $z \in \mathbb{T}$, $\{\int_{[0,2\pi]}^{\oplus} \psi_n(ze^{it})\, dE(t)\}$ converges in the strong operator topology to $\int_{[0,2\pi]}^{\oplus} \psi(ze^{it})\, dE(t)$, with

$$\sup\Big\{\Big\|\int_{[0,2\pi]}^{\oplus} \psi_n(ze^{it})\, dE(t)\Big\| : z \in \mathbb{T},\, n \in \mathbb{N}\Big\} < \infty.$$

Hence by Bounded Convergence, we deduce for each $f \in L^p(\mu)$ and each $k \in \mathbb{Z}$ that the kth Fourier coefficient of the function $\Psi_T(\cdot)f : \mathbb{T} \ni z \mapsto \int_{[0,2\pi]}^{\oplus} \psi(ze^{it})\, dE(t)f \in L^p(\mu)$ is, as $n \to \infty$, the limit in the norm topology of the kth Fourier coefficient of the function $(\Psi_n)_T(\cdot)f : \mathbb{T} \ni z \mapsto \int_{[0,2\pi]}^{\oplus} \psi_n(ze^{it})\, dE(t)f$. Since each ψ_n is in $V_r(\mathbb{T})$, we can now apply Theorem 4.1(a) of [3] to it in order to infer that the kth Fourier coefficient of $(\Psi_n)_T(\cdot)f$ is

$$\widehat{\psi_n}(k)T^k f.$$

The desired conclusion is now evident, since another application of Bounded Convergence to this context shows that, for each $k \in \mathbb{Z}$,

$$\lim_n \widehat{\psi_n}(k) = \widehat{\psi}(k). \ \blacksquare$$

The stage is now set for the following preliminary version of our first main result in Theorem 4.7.

THEOREM 4.6. *Suppose that* (Ω,μ) *is a* σ*-finite measure space,* $1 < p < \infty$*, and* $T \in \mathfrak{B}(L^p(\mu))$ *is invertible, disjoint, and modulus mean-bounded. Let* $E(\cdot)$ *be the spectral decomposition of* T*, and let* $q_3(T) \in (1,\infty)$ *be the minimum of* $q_2(p,\gamma(T)) \in (1,\infty)$ *in Theorem* 3.4 *and* $q_1(T) \in (1,\infty)$ *whose conjugate index figures in* (3.11)*. If also* $r \in [1, q_3(T))$*, and* $\psi \in \mathfrak{M}_r(\mathbb{T})$ *is a continuous function on* $\mathbb{T}$*, and* $f \in L^p(\mu)$*, then, in the notation of Theorem* 4.5*, the function* $\Psi_T(\cdot)f$ *is continuous on* $\mathbb{T}$ *with respect to the norm topology of* $L^p(\mu)$*.*

Proof. It suffices to show that the function $\mathbb{T} \ni z \mapsto \Psi_T(z)f$ is continuous at $z = 1$ with respect to the norm topology of $L^p(\mu)$, since this outcome at $z = 1$ would then automatically hold for wT and $E_w(\cdot)$ in place of T and $E(\cdot)$, where w is an arbitrary point of $\mathbb{T}$. Hence we would have

$$\lim_{z\to 1}\Big\| \int_{[0,2\pi]}^{\oplus} \psi(ze^{it})\, dE_w(t)f - \int_{[0,2\pi]}^{\oplus} \psi(e^{it})\, dE_w(t)f\Big\|_{L^p(\mu)} = 0.$$

By virtue of Theorem 4.4, this can be rewritten as

$$\lim_{z\to 1}\Big\| \int_{[0,2\pi]}^{\oplus} \psi(e^{it})\, d(E_w)_z(t)f - \int_{[0,2\pi]}^{\oplus} \psi(we^{it})\, dE(t)f\Big\|_{L^p(\mu)} = 0.$$

After taking account of (3.12) in the first integral on the left, we further apply Theorem 4.4 there to get

$$\lim_{z\to 1}\Big\| \int_{[0,2\pi]}^{\oplus} \psi(wze^{it})\, dE(t)f - \int_{[0,2\pi]}^{\oplus} \psi(we^{it})\, dE(t)f\Big\|_{L^p(\mu)} = 0.$$

So we now take up this reduction to $z = 1$, by showing that the $\mathfrak{B}(L^p(\mu))$-valued function $\mathbb{T} \ni z \mapsto \Psi_T(z)$ is continuous at $z = 1$ with respect to the strong operator topology of $\mathfrak{B}(L^p(\mu))$. We continue with the notation $\{t_k\}_{k=-\infty}^{\infty}$ for the dyadic points as specified in (3.14), and observe that by virtue of the uniform boundedness in operator norm expressed in (4.3), and the norm density in $\{I - E(0)\}L^p(\mu)$ of the linear manifold $\bigcup_{k=1}^{\infty}\{E(t_k) - E(t_{-k})\}L^p(\mu)$, the proof of the present theorem further reduces to establishing that for each fixed $k \in \mathbb{N}$, and each fixed $f \in L^p(\mu)$, we have

$$\lim_{z\to 1} \|\Psi_T(z)\{E(t_k) - E(t_{-k})\}f - \Psi_T(1)\{E(t_k) - E(t_{-k})\}f\| = 0. \tag{4.5}$$

Thus, by expressing the generic point $z \in \mathbb{T}$ in the form $z = e^{i\theta}$, where $\theta \in \mathbb{R}$, we now need only show that for each fixed $k \in \mathbb{N}$ we have, with respect to the strong operator topology of $\mathfrak{B}(L^p(\mu))$,

$$\lim_{\theta\to 0} \int_{[0,2\pi]}^{\oplus} \chi_k(t)\psi(e^{i\theta}e^{it})\, dE(t) = \int_{[0,2\pi]}^{\oplus} \chi_k(t)\psi(e^{it})\, dE(t), \tag{4.6}$$

where $\chi_k(\cdot)$ denotes the characteristic function of the interval $(t_{-k}, t_k]$.

We proceed to show that, with $k \in \mathbb{N}$ fixed as above, the following holds:

$$\sup\{\|\chi_k(\cdot)\psi(e^{i\theta}e^{i(\cdot)})\|_{V_r([0,2\pi])} : |\theta| < 2^{-k-2}\pi\} < \infty. \tag{4.7}$$

The desired result (4.6) will then follow immediately from this upon application of Theorem 3.5. Observe that

$$\begin{aligned}\operatorname{var}_r(\chi_k(\cdot)\psi(e^{i\theta}e^{i(\cdot)}), [0,2\pi]) &\le \operatorname{var}_r(\chi_k(\cdot)\psi(e^{i\theta}e^{i(\cdot)}), [t_{-k}, t_k]) + \|\psi\|_\infty \\ &\le \operatorname{var}_r(\psi(e^{i\theta}e^{i(\cdot)}), [t_{-k}, t_k]) + 2\|\psi\|_\infty.\end{aligned}$$

Since θ is constrained to satisfy $|\theta| < 2^{-k-2}\pi$, we see that

$$\operatorname{var}_r(\psi(e^{i\theta}e^{i(\cdot)}), [t_{-k}, t_k]) \le \operatorname{var}_r(\psi(e^{i(\cdot)}), [t_{-k-1}, t_{k+1}]),$$

and this completes the proof. ∎

The following theorem, our first main result, is an immediate consequence of Theorem 4.6 by virtue of the vector-valued version of Fejér's Theorem.

Theorem 4.7. *Under the hypotheses and notation of Theorem* 4.6, *the sequence of* $(C,1)$*-means*

$$\left\{\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right) z^k \widehat{\psi}(k) T^k f\right\}_{n=0}^{\infty}$$

of the Fourier series of $\Psi_T(\cdot)f$ $(f \in L^p(\mu))$ *converges to* $\Psi_T(\cdot)f$ *uniformly in* $z \in \mathbb{T}$, *with respect to the norm topology of* $L^p(\mu)$.

An obvious question arising from Theorem 4.7 is whether its implicit pointwise $(C,1)$-summability can be improved to pointwise Fourier series convergence on $\mathbb{T}$, relative to the strong operator topology. (As indicated in §1, such strong operator-valued Fourier series convergence has recently been shown in [3, Theorem 4.1] for trigonometrically well-bounded operators on super-reflexive Banach spaces, provided the relevant function classes are confined to $V_r(\mathbb{T})$, with the parameter r lying in an appropriate range.) Although this question for Stieltjes convolutions defined by spectral integration in association with continuous $\mathfrak{M}_r(\mathbb{T})$-functions currently remains unanswered, the next theorem furnishes a partial response in the affirmative direction.

Theorem 4.8. *Suppose that* (Ω,μ) *is a* σ*-finite measure space,* $1<p<\infty$, *and* $T \in \mathfrak{B}(L^p(\mu))$ *is invertible, disjoint, and modulus mean-bounded. Let* $E(\cdot)$ *be the spectral decomposition of* T, *and let* $q_3(T) \in (1,\infty)$ *be the minimum of* $q_2(p,\gamma(T)) \in (1,\infty)$ *in Theorem* 3.4 *and* $q_1(T) \in (1,\infty)$ *whose conjugate index figures in* (3.11). *If also* $r \in [1, q_3(T))$, *and* $\psi \in \mathfrak{M}_r(\mathbb{T})$ *is a continuous function on* $\mathbb{T}$, *and* $f \in L^p(\mu)$, *then, with all convergence in the* *strong operator topology,*

$$\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) T^k \{E(2\pi - \delta) - E(\delta)\} \tag{4.8}$$

converges to $\Psi_T(1)\{E(2\pi - \delta) - E(\delta)\}$ *for each* $\delta \in (0, \pi)$*, and hence*

$$\lim_{\delta \to 0} \sum_{k=-\infty}^{\infty} \widehat{\psi}(k) T^k \{E(2\pi - \delta) - E(\delta)\} = \Psi_T(1) - \psi(1)E(0).$$

Proof. Let ξ_δ denote the characteristic function of the arc $\{e^{it} : \delta < t \le 2\pi - \delta\}$, constrain θ to satisfy $|\theta| < \delta/2$, and define $\phi \in V_r(\mathbb{T})$ by writing $\phi = \xi_{\delta/2}\psi$. Using standard manipulations with spectral integrals for achieving a localization, we find that

$$\int_{[0,2\pi]}^{\oplus} \psi(e^{i\theta} e^{it})\, dE(t) \{E(2\pi - \delta) - E(\delta)\} = \int_{[0,2\pi]}^{\oplus} \phi(e^{i\theta} e^{it})\, dE(t) \{E(2\pi - \delta) - E(\delta)\}.$$

Now apply Theorem 4.1 of [3] to infer the strong convergence at $e^{i\theta}$ of the Fourier series for Φ_T. Then use localization to obtain the convergence of the Fourier series at $z = 1$ specified in (4.8), invoking its known $(C, 1)$-summability spelled out by Theorem 4.7 to identify its sum. ∎

To continue with the motif of pointwise Fourier series convergence, let us recall the following classical result regarding $\mathfrak{M}_r(\mathbb{T})$.

PROPOSITION 4.9. *If* $1 \le r < \infty$*, and* $\psi \in \mathfrak{M}_r(\mathbb{T})$*, then for each* $z = e^{it} \in \mathbb{T} \setminus \{1\}$*, the Fourier series of* ψ *converges at* z *to*

$$\psi^{\#}(t) \equiv 2^{-1}\Big\{ \lim_{x \to t^+} \psi(e^{ix}) + \lim_{x \to t^-} \psi(e^{ix}) \Big\}.$$

Moreover, if $\psi^{\#}(0) \equiv 2^{-1}\{\lim_{x \to 0^+} \psi(e^{ix}) + \lim_{x \to 0^-} \psi(e^{ix})\}$ *exists, then the Fourier series of* ψ *converges at* $z = 1$ *to* $\psi^{\#}(0)$.

Proof. In the special case when $1 \le r < \infty$ and $\psi \in V_r(\mathbb{T})$, as noted in (3.4), the Fourier series of ψ converges pointwise on $\mathbb{T}$ to $\psi^{\#}(t)$. If now $\psi \in \mathfrak{M}_r(\mathbb{T})$, then for each $N \in \mathbb{N}$, let ζ_N denote the characteristic function defined on $[0, 2\pi]$ of the open interval having the dyadic points t_{-N} and t_N as its end-points, and let $\psi_N \in V_r(\mathbb{T})$ be specified by writing $\psi_N = \zeta_N\, \psi$. Applying the Principle of Localization for Fourier Series (see, e.g., [25, p. 54]) to ψ and ψ_N, we can now see that the Fourier series of ψ converges pointwise to $\psi^{\#}(t)$ on each open interval (t_{-N}, t_N), $N \in \mathbb{N}$. The final assertion of Proposition 4.9 follows from [35, Theorem 1] (which is readily seen to imply that it suffices for the convergence of the Fourier series of ψ at $z = 1$ that this Fourier series merely be (C, ς)-summable at $z = 1$ for some $\varsigma \in (-1.\infty)$). ∎

We now turn to the realm of Tauberian theorems to seek a method for converting the $(C,1)$-summability of Theorem 4.7 to strong Fourier series convergence pointwise by way of convenient side condition(s) on the Fourier coefficients of the continuous function $\psi \in \mathfrak{M}_r(\mathbb{T})$ alone. The spirit of the following special case of the "fixed sector" Tauberian theorem of Lukács [27] (more specifically, the manipulations involved in the reasoning in [27]) will now be shown to afford such an approach, culminating in our second main theorem (Theorem 4.12).

THEOREM 4.10. *Suppose that α, β are real numbers such that $\alpha \le \beta < \alpha + \pi$, and let $\mathbb{S}_{\alpha,\beta}$ be the sector specified by writing $\mathbb{S}_{\alpha,\beta} = \{re^{i\theta} \in \mathbb{C} : r \ge 0, \alpha \le \theta \le \beta\}$. If $\sum_{k=-\infty}^{\infty} a_k$ is a $(C,1)$-summable series of complex numbers such that*

$$\{a_k : k \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{S}_{\alpha,\beta},$$

then $\sum_{k=-\infty}^{\infty} a_k$ is convergent in $\mathbb{C}$.

Notice that $\mathbb{S}_{\alpha,\beta}$ is a closed convex subset of $\mathbb{C}$ which is invariant under multiplication by all non-negative real numbers. Consequently, whenever $w_1 \in \mathbb{S}_{\alpha,\beta}$ and $w_2 \in \mathbb{S}_{\alpha,\beta}$, then the sector $\mathbb{S}_{\alpha,\beta}$ must also contain $w_1 + w_2 = 2(\frac{w_1+w_2}{2})$. For our operator-theoretic environment we shall employ the following tool, to be used farther down in conjunction with the methods in [27].

LEMMA 4.11. *Suppose that (Ω, μ) is a σ-finite measure space, $1 < p < \infty$, and $T \in \mathfrak{B}(L^p(\mu))$ is invertible, disjoint, and modulus mean-bounded. Let $E(\cdot)$ be the spectral decomposition of T, and let $q_4(T) \in (1,\infty)$ be the minimum of $q_2(p,\gamma(T)) \in (1,\infty)$ in Theorem 3.4, $q_1(T) \in (1,\infty)$, and $q_1(|T|) \in (1,\infty)$. If $r \in [1, q_4(T))$, while $\psi \in \mathfrak{M}_r(\mathbb{T})$ is a continuous function on $\mathbb{T}$ such that $\widehat{\psi}(k) \ge 0$ for $k \in \mathbb{Z} \setminus \{0\}$, and $f \in L^p(\mu)$, then, with respect to the norm topology of $L^p(\mu)$, the Fourier series of $\Psi_T(\cdot)f$ converges at $z = 1$ to $\Psi_T(1)f$. Moreover, the following estimate holds:*

$$\sup_{n \in \mathbb{N}} \Big\| \sum_{k=-n}^{n} \widehat{\psi}(k) T^k \Big\|_{\mathfrak{B}(L^p(\mu))} \le K_{p,r,\gamma(T)} \|\psi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{4.9}$$

Proof. In view of Theorem 4.7, the sum of the Fourier series of $\Psi_T(\cdot)f$ at $z = 1$ will be identified as required once we prove that

$$\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) T^k f \tag{4.10}$$

converges in the norm topology of $L^p(\mu)$. Denote the conjugate index of p by p'. We first demonstrate (4.9), noting to begin with that elementary considerations with scalar series having all but finitely many terms non-negative, taken in conjunction with an application of Theorem 4.7 to $|T|$ in place of T, provide that whenever $F \in L^p(\mu)$, $G \in L^{p'}(\mu)$ are non-negative-valued functions, the series

$$\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) \int_{\Omega} (|T|^k F) G \, d\mu$$

converges to $\int_{\Omega} (\Psi_{|T|}(1) F) G \, d\mu$. It now follows by invoking (4.2) that

$$\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) \int_{\Omega} (|T|^k F) G \, d\mu \le K_{p,r,\gamma(T)} \|\psi\|_{\mathfrak{M}_r(\mathbb{T})} \|F\|_{L^p(\mu)} \|G\|_{L^{p'}(\mu)}.$$

Hence for $n \in \mathbb{N}$ we have

$$\sum_{0<|k|\le n} \widehat{\psi}(k) \int_{\Omega} (|T|^k F) G \, d\mu \le K_{p,r,\gamma(T)} \|\psi\|_{\mathfrak{M}_r(\mathbb{T})} \|F\|_{L^p(\mu)} \|G\|_{L^{p'}(\mu)}.$$

It follows readily that

$$\sup_{n\in\mathbb{N}} \Big\| \sum_{0<|k|\le n} \widehat{\psi}(k) |T|^k \Big\|_{\mathfrak{B}(L^p(\mu))} \le K_{p,r,\gamma(T)} \|\psi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{4.11}$$

Combining this with the pointwise a.e. estimate

$$\Big| \sum_{0<|k|\le n} \widehat{\psi}(k) T^k f \Big| \le \sum_{0<|k|\le n} \widehat{\psi}(k) |T|^k (|f|), \tag{4.12}$$

we get (4.9).

To complete the proof of this lemma, we now show that for each $f \in L^p(\mu)$, the series in (4.10) converges in the norm topology of $L^p(\mu)$. From (4.11) in conjunction with Fatou's Lemma, we infer that the series defined pointwise μ-a.e. on Ω by

$$\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) |T|^k (|f|) \tag{4.13}$$

itself defines a function belonging to $L^p(\mu)$. Otherwise expressed, we have now established the pointwise absolute convergence μ-a.e. of the series $\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) T^k f$, and so from (4.12) together with the Dominated Convergence Theorem we deduce the convergence of $\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) T^k f$ in the norm topology of $L^p(\mu)$. ■

The stage is now set for our second main result, which is stated as the following full-fledged "operator-theoretic" Tauberian theorem.

THEOREM 4.12. *Suppose that (Ω, μ) is a σ-finite measure space, $1 < p < \infty$, and $T \in \mathfrak{B}(L^p(\mu))$ is invertible, disjoint, and modulus mean-bounded. Let $q_4(T) \in (1, \infty)$ be as in Lemma 4.11. Suppose that $r \in [1, q_4(T))$, and $\psi \in \mathfrak{M}_r(\mathbb{T})$ is a continuous function on $\mathbb{T}$. Suppose that α, β are real numbers* *satisfying:*

(i) $\alpha \le \beta < \alpha + \pi$, *and*

(ii) $\widehat{\psi}(\mathbb{Z} \setminus \{0\}) \subseteq \mathbb{S}_{\alpha,\beta}$ (*in the notation of Theorem* 4.10).

Then for each $f \in L^p(\mu)$, and each $z \in \mathbb{T}$, the Fourier series of $\Psi_T(\cdot)f$ converges at z to $\Psi_T(z)f$ with respect to the norm topology of $L^p(\mu)$. Moreover, the following estimate holds:

$$\sup\Big\{\Big\|\sum_{k=-n}^{n}\widehat{\psi}(k)z^kT^k\Big\|_{\mathfrak{B}(L^p(\mu))} : n \in \mathbb{N}\Big\} \le K_{p,r,\gamma(T)}\left(\sec\left(\frac{\beta-\alpha}{2}\right)\right)\|\psi\|_{\mathfrak{M}_r(\mathbb{T})} \tag{4.14}$$

for each $z \in \mathbb{T}$.

Proof. Once the convergence in the norm topology of the Fourier series is established at each z, the identification of its sum as $\Psi_T(z)f$ follows immediately from Theorem 4.7. So the demonstration will now address the required Fourier series convergence and the desired estimate in (4.14). Let us note at the outset that it is enough to establish each of these conclusions in the special case $z = 1$, since we could then replace T by zT, for arbitrary $z \in \mathbb{T}$. So we shall establish these two conclusions while confining our attention to the case $z = 1$. The approach we shall follow is an adaptation of the Tauberian-type scalar reasoning in [27] (we shall rely on the above Lemma 4.11 rather than the Tauberian theorem of Landau applied by [27] to scalar series).

Put $\zeta = (a+\beta)/2$. Clearly the desired series convergence is equivalent to that of the series $\sum_{k=-\infty}^{\infty} e^{-i\zeta}\widehat{\psi}(k)T^kf$, and the corresponding desired estimates are identical. Put $\theta_0 = (\beta-\alpha)/2$. Thus, $0 \le \theta_0 < \pi/2$, and $\{e^{-i\zeta}\widehat{\psi}(k) : k \in \mathbb{Z}\setminus\{0\}\} \subseteq \mathbb{S}_{-\theta_0,\theta_0}$. Hence for both the desired conclusions we can assume without loss of generality that

$$\{\widehat{\psi}(k) : k \in \mathbb{Z}\setminus\{0\}\} \subseteq \mathbb{S}_{-\theta_0,\theta_0}, \quad \text{where } 0 \le \theta_0 < \pi/2.$$

It follows that for $k \in \mathbb{Z}\setminus\{0\}$, the real part of $\widehat{\psi}(k)$ (denoted $\Re(\widehat{\psi}(k))$) satisfies $\Re(\widehat{\psi}(k)) \ge 0$. In terms of the function ψ^* defined by (3.16), we have $\Re(\widehat{\psi}) \equiv \big(\frac{\psi+\psi^*}{2}\big)^{\wedge}$, and so an application of Lemma 4.11 shows that

$$\sum_{k=-\infty}^{\infty}\Re(\widehat{\psi}(k))T^kf \tag{4.15}$$

converges in the norm topology of $L^p(\mu)$, and that

$$\sup\Big\{\Big\|\sum_{k=-n}^{n}\Re(\widehat{\psi}(k))T^k\Big\|_{\mathfrak{B}(L^p(\mu))} : n \in \mathbb{N}\Big\} \le K_{p,r,\gamma(T)}\|\psi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{4.16}$$

We now pass to the series $\sum_{k=-\infty}^{\infty} e^{i\theta_0}\widehat{\psi}(k)T^k f$, working in this situation with the imaginary parts

$$\Im(e^{i\theta_0}\widehat{\psi}(k)) := \left(\frac{e^{i\theta_0}\psi - (e^{i\theta_0}\psi)^*}{2i}\right)^{\wedge}(k).$$

Since $\Im(e^{i\theta_0}\widehat{\psi}(k)) \ge 0$ for $k \in \mathbb{Z}\setminus\{0\}$, an application of Lemma 4.11 provides here the convergence in the norm topology of the series

$$\sum_{k=-\infty}^{\infty} \Im(e^{i\theta_0}\widehat{\psi}(k))T^k f, \tag{4.17}$$

along with the estimate

$$\sup\Big\{\Big\|\sum_{k=-n}^{n} \Im(e^{i\theta_0}\widehat{\psi}(k))T^k\Big\|_{\mathfrak{B}(L^p(\mu))} : n \in \mathbb{N}\Big\} \le K_{p,r,\gamma(T)}\|\psi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{4.18}$$

Since for all $k \in \mathbb{Z}$, $\Im(e^{i\theta_0}\widehat{\psi}(k)) = \Re(\widehat{\psi}(k))\sin\theta_0 + \Im(\widehat{\psi}(k))\cos\theta_0$, it is clear from the convergence in the norm topology of the series in (4.15) and (4.17) that $\sum_{k=-\infty}^{\infty} \widehat{\psi}(k)T^k f$ converges in the norm topology. Moreover, by taking due account of the estimates (4.16), (4.18) we find that

$$\sup\Big\{\Big\|\sum_{k=-n}^{n} \Im(\widehat{\psi}(k))T^k\Big\|_{\mathfrak{B}(L^p(\mu))} : n \in \mathbb{N}\Big\} \le K_{p,r,\gamma(T)}(\sec\theta_0)\|\psi\|_{\mathfrak{M}_r(\mathbb{T})},$$

and the relevant form of (4.14) follows from this and (4.16). ∎

5. Examples and some applications to Fourier multiplier theory. In this section, we present some examples illustrating (and expanding on) our blanket framework (the hypotheses and equivalent conditions (i)–(iii) in Theorem 2.5).

Example 5.1. For the background details of this genre of examples based on shift operators, we refer the reader to [11] (mainly to Theorem 3.3, Corollary 3.5, and Proposition 3.8 therein). Suppose that $1 < p < \infty$ and $w = \{w_k\}_{k=-\infty}^{\infty} \in A_p(\mathbb{Z})$. For the σ-finite measure space of this example, we take $\mathbb{Z}$, with each $k \in \mathbb{Z}$ a point mass having measure w_k. The corresponding L^p-space is thereby specialized to be $\ell^p(w)$. In this example, we also specialize the operator T of our blanket hypotheses to be the left bilateral shift $\mathcal{L} \in \mathfrak{B}(\ell^p(w))$, which is the positive, invertible, disjoint, and mean-bounded operator defined on $\ell^p(w)$ by putting $\mathcal{L}x := \{x_{k+1}\}_{k=-\infty}^{\infty}$ for each $x = \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$. (We remark in passing that, in the context of $\mathcal{L}$, for each $\nu \in \mathbb{Z}$, the $A_p(\mathbb{Z})$ weight sequence specified in Theorem 2.5(iii) above specializes to become $\{w_{\nu+k}w_\nu^{-1}\}_{k=-\infty}^{\infty}$.) It is easily seen that, for each $n \in \mathbb{Z}$,

$$\|\mathcal{L}^n\|_{\mathfrak{B}(\ell^p(w))} = \sup\Big\{\Big(\frac{w_{k-n}}{w_k}\Big)^{1/p} : k \in \mathbb{Z}\Big\}. \tag{5.1}$$

From this, we can readily produce a collection of concrete weights $w \in A_p(\mathbb{Z})$ such that $\sup_{n\in\mathbb{Z}} \|\mathcal{L}^n\|_{\mathfrak{B}(\ell^p(w))} = \infty$. Specifically, let $\alpha \in \mathbb{R}$ with $0 < \alpha < p-1$. Then (as covered in [11, Proposition 3.8]) the $A_p(\mathbb{Z})$ condition is satisfied by the weight sequence $\mathfrak{w}^{(\alpha)}$ defined by $\mathfrak{w}^{(\alpha)}(0) = 1$, and $\mathfrak{w}^{(\alpha)}(k) = |k|^\alpha$ for $k \in \mathbb{Z} \setminus \{0\}$. Elementary calculations proceeding from (5.1) readily show that, for all $n \in \mathbb{Z}$,

$$\|\mathcal{L}^n\|_{\mathfrak{B}(\ell^p(\mathfrak{w}^{(\alpha)}))} = (|n|+1)^{\alpha/p}.$$

So, in particular, an operator T satisfying the hypotheses and the equivalent conditions (i)–(iii) stated in the conclusion of Theorem 2.5 need not be power-bounded. An additional genre of examples illustrating that the trigonometrically well-bounded operator T of our blanket hypotheses need not be power-bounded is furnished by the following proposition (see [6, Proposition at the bottom of p. 1177]).

PROPOSITION 5.2. *Suppose that $1 < p < \infty$, $(Y, \mathcal{S}, \lambda)$ is a non-atomic measure space such that $0 < \lambda(Y) < \infty$, and τ is a one-to-one mapping of Y onto Y which is measure-preserving and ergodic for $(Y, \mathcal{S}, \lambda)$. Then there is a finite measure ρ defined on $\mathcal{S}$ and equivalent to λ such that the composition mapping $\mathfrak{T} : g \mapsto g(\tau(\cdot))$ is a trigonometrically well-bounded operator on $L^p(\rho)$ such that $\mathfrak{T}$ and $\mathfrak{T}^{-1}$ are positive, $\mathfrak{T}$ is separation-preserving and mean-bounded, and satisfies*

$$\sup_{n\in\mathbb{Z}} \|\mathfrak{T}^n\|_{\mathfrak{B}(L^p(\rho))} = \infty.$$

The remainder of this section will be devoted to examining the operator-ergodic Fourier analysis associated with functions of the Marcinkiewicz r-classes, when we specialize our blanket hypotheses for $T \in \mathfrak{B}(L^p(\mu))$ to the framework of the left bilateral shift $\mathcal{L} \in \mathfrak{B}(\ell^p(w))$, $w = \{w_k\}_{k=-\infty}^{\infty} \in A_p(\mathbb{Z})$. To illustrate the ties among our blanket hypotheses, spectral theory, and multiplier theory for weighted spaces, we begin the discussion by recalling the following proposition. It should be mentioned here that, besides illustrating the preceding sections, this specialized context can sometimes provide valuable results that, as in the proof of Theorem 4.3 above, transfer to our general blanket context of $T \in \mathfrak{B}(L^p(\mu))$.

PROPOSITION 5.3. *Suppose that $1<p<\infty$, and $w = \{w_k\}_{k=-\infty}^{\infty} \in A_p(\mathbb{Z})$. Then the left shift $\mathcal{L}$ is a trigonometrically well-bounded operator on $\ell^p(w)$, and the spectral decomposition $\mathcal{E}(\cdot)$ of $\mathcal{L}$ can be described as follows. For $0 \le t < 2\pi$, denote by $\phi_{[0,t]}$ the characteristic function defined on $\mathbb{T}$ of the arc $\{e^{is} : 0 \le s \le t\}$. Then $\phi_{[0,t]}$ is a multiplier for $\ell^p(w)$ whose corresponding multiplier transform $S_{\phi_{\phi_{[0,t]}}}$ coincides with $\mathcal{E}(t)$. (In particular, $\mathcal{E}(0) = 0$.) If $f : \mathbb{T} \to \mathbb{C}$ is a bounded function such that f is continuous a.e. on $\mathbb{T}$ with* *respect to Haar measure, and the spectral integral*

$$\int_{[0,2\pi]} f(e^{it})\, d\mathcal{E}(t) \tag{5.2}$$

exists, then f is a multiplier for $\ell^p(w)$ whose corresponding multiplier transform $S_f^{(p,w)}$ coincides with the spectral integral in (5.2).

Proof. Use Scholium (5.13) of [6] to adapt the method of proof for Theorem 4.3 in [9] to the present circumstances. ∎

We now fix $q_3 = q_3(\mathcal{L}) \in (1,\infty)$ in accordance with Theorem 4.6 (note also that since $\mathcal{L}$ is positivity-preserving, $q_4(\mathcal{L})$ in Lemma 4.11 coincides with $q_3(\mathcal{L})$). By using Theorem 4.4 in conjunction with Proposition 5.3, we easily arrive at the following variant of Theorem 9 of [10]: if $r \in [1, q_3(\mathcal{L}))$, and $\psi \in \mathfrak{M}_r(\mathbb{T})$, and $z \in \mathbb{T}$, then ψ_z is a multiplier for $\ell^p(w)$ whose multiplier transform $S_{\psi_z}^{(p,w)} \in \mathfrak{B}(\ell^p(w))$ coincides with $\int_{[0,2\pi]} \psi(ze^{it})\, d\mathcal{E}(t) = \Psi_{\mathcal{L}}(z)$, and satisfies

$$\|S_{\psi_z}^{(p,w)}\| \le K_{p,r,\gamma(\mathcal{L})} \|\psi\|_{\mathfrak{M}_r(\mathbb{T})}. \tag{5.3}$$

We continue henceforth to let $r \in [1, q_3(\mathcal{L}))$, and to let ψ belong to $\mathfrak{M}_r(\mathbb{T})$. In specializing the considerations of §4 to the discrete weighted setting, we shall (except in Theorem 5.6 below) be able to avoid the imposition of any extra continuity condition on ψ (in contrast to the continuity hypothesis on ψ imposed in Theorems 4.5–4.7). This is a dividend of the direct spatial role played by the rich multiplier theory available in the weighted space setting. For each $z \in \mathbb{T}$, we define the linear isometry V_z of $\ell^p(w)$ onto $\ell^p(w)$ by writing, for each $x \equiv \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$,

$$V_z(x) = \{z^{-k} x_k\}_{k=-\infty}^{\infty}. \tag{5.4}$$

It is elementary to verify by direct calculations that

$$S_{\psi_{z^{-1}}}^{(p,w)} = V_{z^{-1}} S_{\psi}^{(p,w)} V_z,$$

and consequently we have

$$\|S_{\psi_{z^{-1}}}^{(p,w)}\|_{\mathfrak{B}(\ell^p(w))} = \|S_{\psi}^{(p,w)}\|_{\mathfrak{B}(\ell^p(w))}. \tag{5.5}$$

For each fixed $m \in \mathbb{Z}$, let us denote by $\mathfrak{y}^{(m)} = \{\mathfrak{y}_k^{(m)}\}_{k=-\infty}^{\infty} \in \ell^p(w)$ the vector whose coordinates are defined in terms of Kronecker's delta by $\mathfrak{y}_k^{(m)} = \delta_{m,k}$. Notice in particular that for each $z \in \mathbb{T}$, $\ell^p(w) \ni S_{\psi_{z^{-1}}}^{(p,w)}(\mathfrak{y}^{(m)}) = \{z^{k-m}\psi^{\vee}(k-m)\}_{k=-\infty}^{\infty}$. Next let us define the function $\mathfrak{Y}_m : \mathbb{T} \ni z \mapsto \ell^p(w)$ by writing $\mathfrak{Y}_m(z) = S_{\psi_{z^{-1}}}^{(p,w)}(\mathfrak{y}^{(m)})$. Then $\mathfrak{Y}_m$ is a continuous mapping of $\mathbb{T}$ into $\ell^p(w)$, since for $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\sum_{|k|>N} |\psi^\vee(k-m)|^p w_k < \varepsilon,$$

and consequently, for arbitrary $z_0 \in \mathbb{T}$, $z \in \mathbb{T}$, we have

$$\|\mathfrak{Y}_m(z) - \mathfrak{Y}_m(z_0)\|_{\ell^p(w)}^p \le \sum_{k=-N}^{N} |(z^{k-m} - z_0^{k-m})\psi^\vee(k-m)|^p w_k + 2^p\varepsilon.$$

This establishes the pointwise continuity of $\mathfrak{Y}_m$, and since the vectors $\mathfrak{y}^{(m)}$, $m \in \mathbb{Z}$, span a linear manifold that is dense in the norm topology of $\ell^p(w)$, we can combine this continuity with (5.5) to deduce that the mapping $z \mapsto S_{\psi_z}^{(p,w)}$ is continuous on $\mathbb{T}$ with respect to the strong operator topology of $\mathfrak{B}(\ell^p(w))$.

It is readily seen by direct calculation that for each $\nu \in \mathbb{Z}$, the mapping $\widehat{\mathfrak{Y}_m}(\nu)$, which is defined by $\ell^p(w)$-valued Bochner integration as $(2\pi)^{-1}\int_0^{2\pi} \mathfrak{Y}_m(e^{i\theta})e^{-i\nu\theta}\, d\theta$, can be expressed by

$$\widehat{\mathfrak{Y}_m}(\nu) = \widehat{\psi}(-\nu)\mathfrak{y}^{(m+\nu)} = \widehat{\psi}(-\nu)\mathcal{L}^{-\nu}(\mathfrak{y}^{(m)}).$$

Straightforward calculations proceeding from this show that for each $x \in \ell^p(w)$, the function $\mathbb{T} \ni z \mapsto S_{\psi_z}^{(p,w)}x$ has vector-valued Fourier coefficient sequence $\{\widehat{\psi}(k)\mathcal{L}^k(x)\}_{k=-\infty}^{\infty}$. Recalling (4.1) gives us

$$\text{(5.6)} \qquad \sup\left\{\left\|\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\widehat{\psi}(k)z^k\mathcal{L}^k\right\|_{\mathfrak{B}(\ell^p(w))} : n \ge 0,\ z \in \mathbb{T}\right\} \le K_{p,r,\gamma(\mathcal{L})}\|\psi\|_{\mathfrak{M}_r(\mathbb{T})},$$

and so we have arrived at the following theorem.

THEOREM 5.4. *Let $\mathcal{L}$ be the left bilateral shift on $\ell^p(w)$, where $1<p<\infty$, and $w := \{w_k\}_{k=-\infty}^{\infty}$ is a weight sequence belonging to $A_p(\mathbb{Z})$, let $\mathcal{E}(\cdot)$ be the spectral decomposition of $\mathcal{L}$, and let $q_3 = q_3(\mathcal{L}) \in (1,\infty)$ be the minimum of $q_2(p,\gamma(\mathcal{L})) \in (1,\infty)$ as described in Theorem* 3.4 *and $q_1(\mathcal{L}) \in (1,\infty)$ whose conjugate index satisfies* (3.11). *Then, in the preceding notation, the following hold whenever $r \in [1, q_3(\mathcal{L}))$ and $\psi \in \mathfrak{M}_r(\mathbb{T})$:*

(i) *The $\mathfrak{B}(\ell^p(w))$-valued function $\Psi_{\mathcal{L}}$ defined on $\mathbb{T}$ by writing*

$$\Psi_{\mathcal{L}}(z) := \int_{[0,2\pi]} \psi(ze^{it})\, d\mathcal{E}(t) = S_{\psi_z}^{(p,w)}$$

satisfies (5.3) *and* (5.5).

(ii) *The operator-valued Fourier transform* (*with respect to the strong operator topology of $\mathfrak{B}(\ell^p(w))$*) *of the function $\Psi_{\mathcal{L}}(\cdot)$ in* (i) *is given by $\{\widehat{\psi}(k)\mathcal{L}^k\}_{k=-\infty}^{\infty}$.*

(iii) *The mapping* $\Psi_{\mathcal{L}}$ *in* (i) *is continuous on* $\mathbb{T}$ *with respect to the strong operator topology of* $\mathfrak{B}(\ell^p(w))$, *and hence by the vector-valued version of Fejér's Theorem, for each* $x \in \ell^p(w)$ *the* $(C,1)$*-means of the Fourier series of* $\Psi_{\mathcal{L}}(\cdot)x$ *converge uniformly on* $\mathbb{T}$ *to* $\Psi_{\mathcal{L}}(\cdot)x$ *with respect to the norm topology of* $\ell^p(w)$.

(iv) *The global estimate for* $(C,1)$*-averages expressed by* (5.6) *holds.*

Remark 5.5. The estimate in (5.6) also follows immediately (without reliance on (4.1)) from Theorem 5.4(ii),(iii) combined with (5.3).

We close with the following Tauberian-type theorem, which directly specializes Theorem 4.12 to the present context.

Theorem 5.6. *Assume the hypotheses of Theorem* 5.4, *with* $r \in [1, q_3(\mathcal{L}))$, *and* $\psi \in \mathfrak{M}_r(\mathbb{T})$. *Suppose further that the function* ψ *is continuous on* $\mathbb{T}$, *and that, in the notation of Theorem* 4.12, $\{\widehat{\psi}(k) : k \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{S}_{\alpha,\beta}$ *for some real numbers* α, β *such that* $\alpha \le \beta < \alpha + \pi$. *Then for each* $x := \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$, *the Fourier series*

$$\sum_{k=-\infty}^{\infty} z^k \widehat{\psi}(k) \mathcal{L}^k x$$

of $\Psi_{\mathcal{L}}(\cdot)x$ *converges at every* $z \in \mathbb{T}$ *to* $\Psi_{\mathcal{L}}(z)x$ *in the norm topology of* $\ell^p(w)$. *Moreover,*

$$\sup\Big\{\Big\| \sum_{k=-n}^{n} z^k \widehat{\psi}(k) \mathcal{L}^k \Big\|_{\mathfrak{B}(\ell^p(w))} : n \in \mathbb{N},\ z \in \mathbb{T}\Big\} \le K_{p,r,\gamma(\mathcal{L})} \Big(\sec \frac{\beta-\alpha}{2}\Big) \|\psi\|_{\mathfrak{M}_r(\mathbb{T})}.$$

References

[1] S. Banach, *Théorie des Opérations Linéaires*, Monograf. Mat. 1, Warszawa, 1932.

[2] E. Berkson, *Spectral theory and operator ergodic theory on super-reflexive Banach spaces*, Studia Math. 200 (2010), 221–246.

[3] E. Berkson, *Rotation methods in operator ergodic theory*, New York J. Math. 17 (2011), 21–39.

[4] E. Berkson and T. A. Gillespie, *AC functions on the circle and spectral families*, J. Operator Theory 13 (1985), 33–47.

[5] E. Berkson and T. A. Gillespie, *Fourier series criteria for operator decomposability*, Integral Equations Operator Theory 9 (1986), 767–789.

[6] E. Berkson and T. A. Gillespie, *Mean-boundedness and Littlewood–Paley for separation-preserving operators*, Trans. Amer. Math. Soc. 349 (1997), 1169–1189.

[7] E. Berkson and T. A. Gillespie, *Multipliers for weighted L^p-spaces, transference, and the q-variation of functions*, Bull. Sci. Math. 122 (1998), 427–454.

[8] E. Berkson and T. A. Gillespie, *Spectral decompositions, ergodic averages, and the Hilbert transform*, Studia Math. 144 (2001), 39–61.

[9] E. Berkson and T. A. Gillespie, *Shifts as models for spectral decomposability on Hilbert space*, J. Operator Theory 50 (2003), 77–106.
[10] E. Berkson and T. A. Gillespie, *The q-variation of functions and spectral integration from dominated ergodic estimates*, J. Fourier Anal. Appl. 10 (2004), 149–177.
[11] E. Berkson and T. A. Gillespie, *Operator means and spectral integration of Fourier multipliers*, Houston J. Math. 30 (2004), 767–814.
[12] E. Berkson, T. A. Gillespie, and P. S. Muhly, *Abstract spectral decompositions guaranteed by the Hilbert transform*, Proc. London Math. Soc. (3) 53 (1986), 489–517.
[13] A. P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. 59 (1968), 349–353.
[14] V. V. Chistyakov and O. E. Galkin, *On maps of bounded p-variation with $p > 1$*, Positivity 2 (1998), 19–45.
[15] R. Coifman, J. L. Rubio de Francia, et S. Semmes, *Multiplicateurs de Fourier de $L^p(\mathbb{R})$ et estimations quadratiques*, C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), 351–354.
[16] R. R. Coifman and G. Weiss, *Operators associated with representations of amenable groups, singular integrals induced by ergodic flows, the rotation method and multipliers*, Studia Math. 47 (1973), 285–303.
[17] R. R. Coifman and G. Weiss, *Transference Methods in Analysis*, CBMS Reg. Conf. Ser. Math. 31, Amer. Math. Soc., Providence, RI, 1976.
[18] A. Froda, *Sur la distribution des propriétés de voisinage des fonctions de variables réelles*, thèse, Hermann, Paris, 1929.
[19] G. H. Hardy and J. E. Littlewood, *Notes on the theory of series (VII): On Young's convergence criterion for Fourier series*, Proc. London Math. Soc. (2) 28 (1928), 301–311.
[20] G. H. Hardy and J. E. Littlewood, *A convergence criterion for Fourier series*, Math. Z. 28 (1928), 612–634.
[21] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1952.
[22] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.
[23] R. C. James, *Super-reflexive spaces with bases*, Pacific J. Math. 41 (1972), 409–419.
[24] C.-H. Kan, *Ergodic properties of Lamperti operators*, Canad. J. Math. 30 (1978), 1206–1214.
[25] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover, New York, 1976.
[26] J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. 8 (1958), 459–466.
[27] F. Lukács, *Bemerkung zu einem Konvergenzsatze des Herrn Landau*, Arch. Math. Physik (3) 23 (1915), 367–368.
[28] J. Marcinkiewicz, *On a class of functions and their Fourier series*, C. R. Soc. Sci. Varsovie 26 (1934), 71–77; reprinted in: Józef Marcinkiewicz, Collected Papers, A. Zygmund (ed.), Inst. Math., Polish Acad. Sci. and PWN, Warszawa, 1964, 36–41.
[29] F. J. Martín-Reyes and A. de la Torre, *The dominated ergodic theorem for invertible, positive operators*, in: Semesterbericht Funktionalanalysis Tübingen, Sommersemester 1985, 143–150.
[30] F. J. Martín-Reyes and A. de la Torre, *The dominated ergodic estimate for mean bounded, invertible, positive operators*, Proc. Amer. Math. Soc. 104 (1988), 69–75.
[31] J. E. Porter, *Helly's selection principle for functions of bounded p-variation*, Rocky Mountain J. Math. 35 (2005), 675–679.

[32] P. G. Spain, *On well-bounded operators of type* (B), Proc. Edinburgh Math. Soc. (2) 18 (1972), 35–48.
[33] N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, J. Math. Phys. 3 (1924), 72–94.
[34] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. 67 (1936), 251–282.
[35] L. C. Young, *A convergence criterion for Fourier series*, Quart. J. Math. Oxford Ser. 9 (1938), 115–120.

Earl Berkson
Department of Mathematics
University of Illinois
1409 W. Green Street
Urbana, IL 61801, U.S.A.
E-mail: berkson@math.uiuc.edu

Received March 21, 2013
Revised version December 12, 2013 (7765)