STUDIA MATHEMATICA 222 (2) (2014)

Extension of smooth subspaces in Lindenstrauss spaces

by

V. P. Fonf (Beer-Sheva) and P. Wojtaszczyk (Warszawa)

We dedicate this paper to the memory of Aleksander Pełczyński

Abstract. It follows from our earlier results [Israel J. Math., to appear] that in the Gurariy space G every finite-dimensional smooth subspace is contained in a bigger smooth subspace. We show that this property does not characterise the Gurariy space among Lindenstrauss spaces and we provide various examples to show that $C(K)$ spaces do not have this property.

The starting point of this paper is the following observation which easily follows from [3, Theorem 1.2] (see the proof below).

Observation. *Let L be a finite-dimensional smooth subspace of the Gurariy space G. Then there is a smooth subspace $M \subset G$ with $M \supsetneq L$.*

Recall that a point x of the unit sphere S_X of a Banach space X is called a *smooth point* of S_X if there is a unique linear functional $f \in S_{X^*}$ such that $f(x) = 1$. A subspace X of a Banach space Y is called *smooth* if any point $x \in S_X$ is a smooth point of S_X. A separable Banach space G is called a *Gurariy space* if given $\varepsilon > 0$ and an isometric embedding $T : L \to G$ of a finite-dimensional normed space L into G, for any finite-dimensional space $M \supset L$ there is an extension $\tilde{T} : M \to G$ with $\|\tilde{T}\|\,\|\tilde{T}^{-1}\| \le 1 + \varepsilon$. Such a space was constructed by Gurariy [4] and its isometric uniqueness was shown by Lusky [10] (see also [6]).

A Banach space X is called a *Lindenstrauss space* if its dual is isometric to an $L_1(\mu)$ for some measure μ. This class includes $C(K)$ spaces and was intensively studied in [9] and [8]. It is known (see [4]) that the Gurariy space is a Lindenstrauss space.

We say that a pair $L \subset M$ of normed spaces has the *unique Hahn–Banach extension property* (UHB for short) if every functional $f \in L^*$ has a unique

2010 *Mathematics Subject Classification*: Primary 46B20.
Key words and phrases: smooth subspaces, Gurariy space, Lindenstrauss spaces, spaces of continuous functions.

DOI: 10.4064/sm222-2-3

extension $\hat{f} \in M^*$ with $\|\hat{f}\| = \|f\|$. For instance if M is smooth and $L \subset M$, $\dim L < \infty$, then this pair has UHB.

In the proof of the Observation we use the following theorem, which is the main result of [3].

Theorem 1. *Let X be a separable Banach space. The following are equivalent:*

(a) $X = G$.
(b) *Let $L \subset M$ with $\dim L < \infty$ and $\operatorname{codim}_M L = 1$ be a pair with property UHB and let $T : L \to X$ be an isometric embedding of L into X. Then there is an isometric extension $\tilde{T} : M \to X$ of T.*

Proof of the Observation. Put $M_1 = L \oplus \mathbb{R}$ and define a norm on M_1 as

$$\|(x,t)\| = (\|x\|^2 + t^2)^{1/2}, \quad x \in L,\ t \in \mathbb{R}.$$

Since L is smooth it easily follows that so is M_1, and hence the pair $L \subset M_1$ has UHB. By Theorem 1(b) (for $T = \mathrm{Id}$) there is an isometric extension $\tilde{T} : M_1 \to G$ of T. Putting $M = \tilde{T}(M_1)$ finishes the proof.

Now we briefly describe the paper. First we note (Theorem 3) that the property of the space G stated in the Observation does not characterise the Gurariy space among Lindenstrauss spaces. Next we investigate spaces $C(K)$, an important class of Lindenstrauss spaces, and we show that they contain finite-dimensional smooth spaces which cannot be enlarged to smooth spaces.

Recall that a Banach space X is called *polyhedral* if the unit ball of any finite-dimensional subspace $E \subset X$ is a polytope (i.e. finite intersection of closed half-spaces).

Proposition 2. *Let X be a polyhedral space, V be arbitrary Banach space, $E \subset X \oplus_\infty V$ be a finite-dimensional smooth space, and P be the coordinate projection from $X \oplus_\infty V$ onto V. Then $P|E$ is an isometry into V.*

Proof. Let $\bar{V} = P(E)$ and $\bar{X} = (I-P)(E)$. Then $E \subset \bar{X} \oplus_\infty \bar{V}$ and let ι denote this identity embedding. Then $\iota^* : \bar{X}^* \oplus_1 \bar{V}^* \to E^*$ is an onto map. Since E is smooth, E^* is strictly convex, so every point in S_{E^*} is an extreme point. We have $\operatorname{ext} B_{\bar{X}^* \oplus_1 \bar{V}^*} = \operatorname{ext} B_{\bar{X}^*} \cup \operatorname{ext} B_{\bar{V}^*}$; but $\bar{X}$ is a finite-dimensional polyhedral space, so $\operatorname{ext} B_{\bar{X}^*}$ is a finite set. This implies that $\iota^*(\operatorname{ext} B_{\bar{V}^*})$ is dense in S_{E^*}, in particular it is norming. Thus for $e \in E$ we have

$$\begin{aligned}\|e\| &= \sup_{g^* \in \operatorname{ext} B_{\bar{V}^*}} \iota^*(0,g^*)(e) = \sup_{g^* \in \operatorname{ext} B_{\bar{V}^*}} (0,g^*)(\iota(e)) \\ &= \sup_{g^* \in \operatorname{ext} B_{\bar{V}^*}} g^*(P(e)) = \|P(e)\|.\end{aligned}$$

Now we can prove our first main result.

Theorem 3. *Let X be a separable polyhedral Lindenstrauss space. Then the* (*Lindenstrauss*) *space $Y = X \oplus_\infty G$ has the smooth extension property, i.e. for any finite-dimensional smooth subspace $E \subset Y$ there is a finite-dimensional smooth subspace $M \subset Y$ with $M \supsetneq E$.*

Proof. It follows from Proposition 2 that $E_1 = P(E)$ is a smooth subspace of G, where P is the coordinate projection onto G. By the Observation there exists a smooth subspace $M_1 \subset G$ with $E_1 \subsetneq M_1$. Define $T : E_1 \to X$ as $T = (I-P)P^{-1}$, where $P^{-1} : E_1 \to E$. Clearly, $\|T\| \le 1$. Since X is a polyhedral Lindenstrauss space, by the Lazar–Lindenstrauss theorem (see [7] and [9]) the (finite-dimensional, hence compact) operator T has a norm-preserving extension $\tilde{T} : M_1 \to X$, $\|\tilde{T}\| = \|T\| \le 1$. Define

$$M = \{x + y : y \in M_1,\ x = \tilde{T}y\} \subset X \oplus_\infty G.$$

Clearly, M is isometric to M_1 and hence smooth. We now check that $E \subset M$. Take $z \in E$ and put $y = Pz \in E_1 \subset M_1$ and $x = (I-P)z$. To prove that $z \in M$ we need to verify that $x = \tilde{T}y$. However,

$$\tilde{T}y = Ty = (I-P)P^{-1}y = (I-P)P^{-1}Pz = x,$$

which finishes the proof.

Remark. The space Y from Theorem 3 is not isometric to G. To see this, just note that $w^*\text{-cl}\operatorname{ext} B_{G^*} = B_{G^*}$ (see [8]), but it is easy to see that $w^*\text{-cl}\operatorname{ext} B_{Y^*} \ne B_{Y^*}$. However, the space Y is isomorphic to G. Indeed, $Y = X \oplus_\infty G$ where X is a Lindenstrauss space. Clearly, the infinite sum $(\sum X)_{c_0}$ is a Lindenstrauss space too. By [11], G contains it as a complemented subspace, so

$$Y = X + G \sim X + (\textstyle\sum X)_{c_0} + V) \sim (\textstyle\sum X)_{c_0} + V) \sim G.$$

Problem. *Assume that a separable Lindenstrauss space X has the smooth extension property. Is it true that X isomorphic to G?*

Now we consider the problem of extension of smooth subspaces of $C(K)$ spaces. We will need the following general fact.

Proposition 4. *Let M be a smooth finite-dimensional subspace of a Banach space X and let L be a proper subspace of M. Then*

$$\operatorname{ext} B_{X^*}|_M \supset S_{M^*} \tag{1}$$

and

$$S_{M^*}|_L = B_{L^*}. \tag{2}$$

Proof. It is well known that a finite-dimensional space M is smooth if and only if M^* is strictly convex, i.e. $\operatorname{ext} B_{M^*} = S_{M^*}$, and (1) follows from the Krein–Milman theorem.

The second assertion is obvious.

We start with the case $C(S^n)$ where S^n stands for the n-dimensional unit sphere, i.e. the boundary of the unit ball of the real $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ (e.g. S^1 is the unit circle in the plane).

THEOREM 5. *The space $C(S^n)$ contains an $(n+1)$-dimensional smooth subspace H consisting of C^1 functions. However in any $(n+2)$-dimensional smooth subspace of $C(S^n)$ the subspace of C^1 functions has dimension at most n. In particular H is not contained in a bigger smooth subspace.*

Proof. The space H consists of all restrictions to S^n of linear functionals on $\mathbb{R}^{n+1} \supset S^n$. It is isometric to ℓ_2^{n+1} (so smooth) and clearly consists of C^∞ functions. To prove the second claim suppose that there exists a smooth $(n+2)$-dimensional subspace $M \subset C(S^n)$ and an $(n+1)$-dimensional subspace $L \subset M$ which consists of C^1 functions. Now let $r : \operatorname{ext} B_{C(S^n)^*} \to B_{L^*}$ be the restriction map, $r(\mu) = \mu|L$. From Proposition 4 we see that it is an onto map. It is known that $\operatorname{ext} B_{C(S^n)^*}$ consists of $\pm$ point evaluations, thus we can identify it with $\pm S^n$. Let us fix a basis $\phi_1, \ldots, \phi_{n+1}$ in L with biorthogonal functionals $\phi_1^*, \ldots, \phi_{n+1}^*$. For $\ell \in L$ we have

$$r(\pm s)(\ell) = \pm \sum_{j=1}^{n+1} \phi_j^*(\ell)\phi_j(s), \qquad s \in S^n.$$

Thus the map $\Phi(\pm s) = \pm \sum_{j=1}^{n+1} \phi_j(s)\phi_j^*$ maps the union of two disjoint copies of S^n onto the unit ball of the $(n+1)$-dimensional space L^*. But this is a C^1 map (because the functions ϕ_j are C^1), which contradicts Sard's theorem. The proof of the theorem is complete.

The following theorem is in a sense a generalization of Theorem 5.

THEOREM 6. *Every separable $C(K)$ space with nonseparable dual contains every finite-dimensional smooth space E in such a way that no bigger subspace is smooth.*

Proof. By our assumptions on $C(K)$ we see that K is a metrizable compact space (since $C(K)$ is separable). Moreover, K is uncountable (if K were countable then $C(K)^* = l_1$, contradicting that $C(K)^*$ is nonseparable). Let $\phi : K \to S_{E^*}$ be a continuous map from K onto the unit sphere of E^*. Such a map exists. To see this, note e.g. that K contains a Cantor set, so we can map this subset onto a cube of proper dimension. Next we extend this map to K. Then we wrap this cube onto S_{E^*} ([1]).

([1]) This argument is standard and the result is well known. It is a special case of a more general and well known fact that if K_1 is any Peanian (i.e. metrizable, connected and locally connected) compact and K is an uncountable metrizable compact, then there is a continuous map from K onto K_1.

Next we define an isometric embedding $I_\phi : E \hookrightarrow C(K)$ by the formula $I_\phi(e)(k) = \phi(k)(e)$ for $e \in E$ and $k \in K$. Clearly, $L = I_\phi(E)$ is a smooth finite-dimensional subspace of $C(K)$. Moreover,

$$\|\delta_k|_L\| = 1, \quad k \in K. \tag{3}$$

Assume that there is a smooth subspace $M \subset C(K)$ with $L \subsetneq M$. Then by Proposition 4 we have $\operatorname{ext} B_{C(K)^*}|_L = \{\pm\delta_k : k \in K\}|_L = B_{L^*}$, contradicting (3). The proof is complete.

Now we present an analogous observation about infinite-dimensional smooth subspaces. Before we proceed we must recall some classical topological results essentially due to Keller [5].

Theorem 7 (Keller).

(a) *The closed unit ball B_{X^*} of the dual of a separable Banach space X, when equipped with the weak* topology, is homeomorphic to the Hilbert cube $Q = [0,1]^\infty$.*
(b) *The Hilbert cube is homogeneous, i.e. for any $p, q \in Q$ there exists a homeomorphism ϕ of Q such that $\phi(p) = q$.*

The proofs of this can be found in [5] and in more modern exposition in [1, Chap. 3, Ths. 3.1 and 4.1].

To prove Theorem 9 we also need the following easy lemma.

Lemma 8. *If L is a smooth Banach space then $\operatorname{ext} B_{L^*}$ is norm dense in S_{L^*}.*

Proof. If $f \in S_{L^*}$ attains its norm, say at $x \in S_L$, then it is the only supporting functional for x and so by the Krein–Milman theorem it must be an extreme point of B_{L^*}. The Bishop–Phelps theorem (see e.g. [2, Corollary 3.3]) finishes the proof of the lemma.

Remark. Instead of the Bishop–Phelps theorem we can apply the Hahn–Banach theorem and deduce that the set $\operatorname{ext} B_{L^*}$ is w^*-dense in S_{L^*} (even in B_{L^*}), which is enough for our purposes.

Theorem 9. *Let X be a separable, smooth infinite-dimensional Banach space. There exists a subspace $Y \subset C(\Delta)$ isometric to X which is not contained in a bigger smooth subspace.*

Proof. Let $\Delta := \{0,1\}^\infty$ be the Cantor set and let $\phi((\epsilon_i)_{i=1}^\infty) = \sum_{i=1}^\infty \epsilon_i 2^{-i}$ be the classical Cantor map from Δ onto $[0,1]$. Since Δ is homeomorphic to Δ^∞, taking ϕ coordinatwise we get the natural map Φ from Δ onto the Hilbert cube $Q := [0,1]^\infty$. It is easy and well known that there exists a subset $F \subset [0,1]$ of cardinality continuum such that $\#\phi^{-1}(t) = 1$ for $t \in F$. This implies that the set $\mathcal{F} = \prod_{i=1}^\infty F \subset Q$ has cardinality continuum and for $p \in \mathcal{F}$ we have $\#\Phi^{-1}(p) = 1$.

Next with the help of Theorem 7(a) we construct a continuous map Ψ from Δ onto B_{X^*} (equipped with the weak* topology). Moreover without loss of generality by Theorem 7(b) we can assume that $\#\Psi^{-1}(0) = 1$. Using this map we define an isometric embedding

$$\iota(x)(t) = \Psi(t)(x) \tag{4}$$

of X into $C(\Delta)$. Put $Y = \iota(X)$.

Now suppose that there exists a smooth subspace L such that $C(\Delta) \supset L \supsetneq Y$.

The set $\operatorname{ext} B_{C(\Delta)^*}|_L$ is a w^*-compact subset of B_{L^*} which by the Krein–Milman theorem contains $\operatorname{ext} B_{L^*}$, and so by Lemma 8 it contains the unit sphere S_{L^*}. Since L is infinite-dimensional, this implies that $\operatorname{ext} B_{C(\Delta)^*}|_L = B_{L^*}$. When we restrict $\operatorname{ext} B_{C(\Delta)^*}$ further to Y, we get a map $\xi(\pm\delta_t) = \pm\Psi(t)$. Clearly, $\xi^{-1}(0) = \{\pm\Psi^{-1}(0)\}$ is a set of cardinality at most 2. On the other hand, the restriction of B_{L^*} to Y maps a whole interval of functionals to 0. This contradiction shows that L cannot be smooth. The proof is complete.

Remark. It was suggested by the referee that maybe in Theorem 9 one can replace Δ by any uncountable compact set.

Acknowledgements. We would like to thank the anonymous referee for very useful and informative comments. The first named author was partially supported by the Israel Science Foundation, Grant 209/09 and by European Grant Spade 2. The second named author was partially supported by the Center for Advanced Studies in Mathematics of the Ben-Gurion University of the Negev, the HPC Infrastructure for Grand Challenges of Science and Engineering Project, co-financed by the European Regional Development Fund under the Innovative Economy Operational Programme, and Polish NCN grant DEC2011/03/B/ST1/04902.

References

[1] Cz. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Monografie Mat. 58, PWN, Warszawa, 1975.

[2] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64, Longman, 1993.

[3] V. P. Fonf and P. Wojtaszczyk, *Characteristic properties of the Gurariy space*, Israel J. Math., to appear.

[4] V. I. Gurariĭ, *Space of universal disposition, isotropic spaces and the Mazur problem on rotations of Banach spaces*, Sibirsk. Mat. Zh. 7 (1966), 1002–1013 (in Russian).

[5] O.-H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann. 105 (1931), 748–758.

[6] W. Kubiś and S. Solecki, *A proof of uniqueness of the Gurariĭ space*, Israel J. Math. 195 (2013), 449–456.

[7] A. J. Lazar, *Polyhedral Banach spaces and extensions of compact operators*, Israel J. Math. 7 (1969), 357–364.

[8] A. J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices*, Acta Math. 126 (1971), 165–193.

[9] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964), 112 pp.

[10] W. Lusky, *The Gurarij spaces are unique*, Arch. Math. (Basel) 27 (1976), 627–635.

[11] P. Wojtaszczyk, *Some remarks on the Gurarij space*, Studia Math. 41 (1972), 207–210.

V. P. Fonf
Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
Beer-Sheva 84105, Israel
E-mail: fonf@math.bgu.ac.il

P. Wojtaszczyk
Interdisciplinary Centre for Mathematical
and Computational Modelling
University of Warsaw
Prosta 69
02-838 Warszawa, Poland
and
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: wojtaszczyk@icm.edu.pl

Received July 23, 2013
Revised version May 20, 2014 (7816)