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On (n,k)-quasiparanormal operators

by

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For positive integers n and k, an operator T is called (n,k)-quasiparanormal if

$$\|T^{1+n}(T^k x)\|^{1/(1+n)}\|T^k x\|^{n/(1+n)} \geq \|T(T^k x)\| \quad \text{for } x \in \mathcal{H}.$$

The class of (n,k)-quasiparanormal operators contains the classes of n-paranormal and k-quasiparanormal operators. We consider some properties of (n,k)-quasiparanormal operators: (1) inclusion relations and examples; (2) a matrix representation and SVEP (single valued extension property); (3) ascent and Bishop's property (β); (4) quasinilpotent part and Riesz idempotents for k-quasiparanormal operators.

1. Introduction. In this paper, an operator means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. As extensions of well-known hyponormal and paranormal operators, some operator classes were introduced in recent years. Let n, k be positive integers and T an operator.

(1) T belongs to k-*quasiclass* A if $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$ (see [19, 10]).
(2) T is called n-*paranormal* if $\|T^{1+n}x\|^{1/(1+n)}\|x\|^{n/(1+n)} \geq \|Tx\|$ for $x \in \mathcal{H}$ (see [23]).
(3) T is called k-*quasiparanormal* if $\|T^2(T^k x)\|^{1/2}\|T^k x\|^{1/2} \geq \|T(T^k x)\|$ for $x \in \mathcal{H}$ (see [16]).

Class A operators (defined by $|T^2| \geq |T|^2$) are paranormal by definition ([9], [20]). k-Quasiclass A contains class A and is contained in the class of k-quasiparanormal operators [10, Theorem 2.2]. n-Paranormal operators are normaloid [11, Theorem 1]. These operator classes have many interesting properties, such as inclusion relations, SVEP (single valued extension property), Bishop's property (β), finite ascent, properties of isolated spectral points, and so on. We refer to [2], [3], [19], [22], [8], [16].

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As an extension, for positive integers n and k, we call an operator T (n,k)-*quasiparanormal* if

$$\|T^{1+n}(T^k x)\|^{1/(1+n)}\|T^k x\|^{n/(1+n)} \geq \|T(T^k x)\| \quad \text{for } x \in \mathcal{H}. \tag{1.1}$$

The class of (n,k)-quasiparanormal operators contains the classes of n-paranormal and k-quasiparanormal (that is, $(1,k)$-quasiparanormal) operators (see Theorem 2.1 below).

In this work we consider some properties of (n,k)-quasiparanormal operators. In Section 2, some inclusion relations and examples related to (n,k)-quasiparanormal operators are discussed. In Section 3, a matrix representation is obtained and it is proved that (n,k)-quasiparanormal operators have SVEP (single valued extension property). In Section 4, we prove that they have finite ascent and Bishop's property (β). Section 5 is devoted to the quasinilpotent part and Riesz idempotents for k-quasiparanormal operators.

2. Inclusion relations and examples

THEOREM 2.1. *The following assertions hold:*

(1) *If T is (n,k)-quasiparanormal, then it is $(n,k+1)$-quasiparanormal.*
(2) *If T is (n,k)-quasiparanormal, then its restriction to each invariant subspace is also (n,k)-quasiparanormal.*
(3) *If T is k-quasiparanormal, then it is (n,k)-quasiparanormal.*
(4) *Assume $T^k\mathcal{H}$ is not dense. Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \qquad \text{on } [T^k\mathcal{H}] \oplus \ker T^{*k}$$

where $[T^k\mathcal{H}]$ is the closure of $T^k\mathcal{H}$. If T is (n,k)-quasiparanormal, then T_1 is n-paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Similar results hold for (p,k)-quasihyponormal operators (defined by $T^{*k}(T^*T)^p T^k \geq T^{*k}(TT^*)^p T^k$ where $p > 0$ and k is a positive integer) and k-quasiclass A operators ([19]).

Proof. (1) follows by taking $x = Tz$ in the definition, and (2) is clear.

(3) By assumption, for $x \in T^k\mathcal{H}$ and $Tx \neq 0$ we have

$$\frac{\|T^2x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|}. \tag{2.1}$$

Noting that $Tx \in T^{k+1}\mathcal{H} \subseteq T^k\mathcal{H}$, for $T^2x \neq 0$, (2.1) implies

$$\frac{\|T^3x\|}{\|T^2x\|} \geq \frac{\|T^2x\|}{\|Tx\|}.$$

By repeating this process, for $x \in T^k\mathcal{H}$ and $T^n x \neq 0$, we obtain

$$\|T\| \geq \cdots \geq \frac{\|T^{1+n}x\|}{\|T^n x\|} \geq \frac{\|T^n x\|}{\|T^{n-1}x\|} \geq \cdots \geq \frac{\|T^2 x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|x\|},$$
$$\|T^{1+n}x\|\,\|x\|^n \geq \|Tx\|^{1+n}.$$

If $T^n x = 0$, then there exists $y \in \mathcal{H}$ such that $y \in \ker T^{n+k}$. By definition of k-quasiparanormality, $\ker T^{2+k} = \ker T^{1+k}$. Thus $y \in \ker T^{n+k} = \ker T^{1+k}$ and $Tx = 0$. Hence $\|T^{1+n}x\|\,\|x\|^n \geq \|Tx\|^{1+n}$ for all $x \in T^k\mathcal{H}$.

(4) Observe that $T_1^{1+n} z = T^{1+n} z$ for $z \in [T^k\mathcal{H}]$. So T_1 is n-paranormal by (1.1). Let $x \in \ker T^{*k}$. Then

$$T^k x = \begin{pmatrix} T_1^k & \sum_{i=0}^{k} T_1^i T_2 T_3^{k-1-i} \\ 0 & T_3^k \end{pmatrix} (0 \oplus x) \in [T^k\mathcal{H}].$$

Hence $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. ∎

Later we give an example to show that the class of $(k+1)$-quasiparanormal operators strictly contains the class of k-quasiparanormal operators.

LEMMA 2.2. *T is (n,k)-quasiparanormal if and only if*

$$T^{*k}T^{*(1+n)}T^{1+n}T^k - (1+n)\mu^n T^{*k}T^*TT^k + n\mu^{1+n}T^{*k}T^k \geq 0 \tag{2.2}$$

for any $\mu > 0$.

Proof. The proof is similar to that of [23, Lemma 2.2]. Let T be (n,k)-quasiparanormal. By the generalized arithmetic-geometric mean inequality, we have

$$\begin{aligned} &\frac{1}{1+n}(\mu^{-n}|T^{1+n}|^2T^k x, T^k x) + \frac{n}{1+n}(\mu T^k x, T^k x) \\ &\qquad \geq (\mu^{-n}|T^{1+n}|^2T^k x, T^k x)^{1/(1+n)}(\mu T^k x, T^k x)^{n/(1+n)} \\ &\qquad = (|T^{1+n}|^2T^k x, T^k x)^{1/(1+n)}(T^k x, T^k x)^{n/(1+n)} \\ &\qquad \geq (|T|^2T^k x, T^k x) = (T^*TT^k x, T^k x). \end{aligned}$$

Conversely, if $x \in \mathcal{H}$ with $(|T^{1+n}|^2T^k x, T^k x) = 0$, multiplying (2.2) by μ^{-n} and letting $\mu \to 0$ we have $(T^*TT^k x, T^k x) = 0$, thus

$$\|T^{1+n}(T^k x)\|\,\|T^k x\|^n \geq \|T(T^k x)\|^{1+n}.$$

If $x \in \mathcal{H}$ with $(|T^{1+n}|^2T^k x, T^k x) > 0$, putting

$$\mu = \left(\frac{(|T^{1+n}|^2T^k x, T^k x)}{(T^k x, T^k x)}\right)^{1/(1+n)}$$

in (2.2) we have

$$(|T^{1+n}|^2T^k x, T^k x)^{1/(1+n)}(T^k x, T^k x)^{n/(1+n)} \geq (T^*TT^k x, T^k x)$$

for any $x \in \mathcal{H}$, so T is (n,k)-quasiparanormal. ∎

EXAMPLE 2.3. Denote by $w := (w_n)_{n\in\mathbb{N}}$ a bounded sequence of positive numbers. The corresponding unilateral weighted right shift operator on $l^2(\mathbb{N})$ with the canonical orthogonal basis $\{e_n\}_{n=0}^{\infty}$ is defined by $Tx = \sum_{n=0}^{\infty} w_n x_n e_{n+1}$ where $x := (x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. Then the following statements hold:

(1) T belongs to k-quasiclass A if and only if $w_k \le w_{k+1} \le w_{k+2} \le \cdots$.
(2) If $w_k > w_{k+1}$ and $w_{k+1} \le w_{k+2} \le \cdots$, then T is a $(k+1)$-quasiclass A operator but not a k-quasiclass A operator.
(3) T is (p,k)-quasihyponormal if and only if $w_{k-1} \le w_k \le w_{k+1} \le \cdots$.
(4) If $w_{k-1} > w_k$ and $w_k \le w_{k+1} \le \cdots$, then T is $(k+1)$-quasihyponormal but not k-quasihyponormal.
(5) T is k-quasiparanormal if and only if $w_k \le w_{k+1} \le w_{k+2} \le \cdots$.
(6) If $w_k > w_{k+1}$ and $w_{k+1} \le w_{k+2} \le \cdots$, then T is $(k+1)$-quasiparanormal but not k-quasiparanormal.
(7) If $w_0 > w_1$ and $w_1 \le w_2 \le \cdots$, then T is quasiparanormal but not paranormal.
(8) If $w_0 > w_1$ and $w_1 = w_2 = \cdots$, then T is quasiparanormal but not normaloid.

Examples 2.3(1)–(2) are known ([10, Example 1.3], [13, Example 1.2]).

Proof. Obviously, it is sufficient to prove (3), (5) and (8).

(3) By calculation, $TT^* = 0 \oplus w_0^2 \oplus w_1^2 \oplus \cdots$, and for each positive integer m,

$$T^{*m}T^m = (w_0^2 \cdots w_{m-1}^2) \oplus (w_1^2 \cdots w_m^2) \oplus (w_2^2 \cdots w_{m+1}^2) \oplus \cdots \quad \text{on } l^2(\mathbb{N}). \tag{2.3}$$

Hence

$$T^{*k}(T^*T)^p T^k = (w_0^2 \cdots w_{k-1}^2 w_k^{2p}) \oplus (w_1^2 \cdots w_k^2 w_{k+1}^{2p}) \oplus (w_2^2 \cdots w_{k+1}^2 w_{k+2}^{2p}) \oplus \cdots \quad \text{on } l^2(\mathbb{N}), \tag{2.4}$$

$$T^{*k}(TT^*)^p T^k = (w_0^2 \cdots w_{k-1}^2 w_{k-1}^{2p}) \oplus (w_1^2 \cdots w_k^2 w_k^{2p}) \oplus (w_2^2 \cdots w_{k+1}^2 w_{k+1}^{2p}) \oplus \cdots \quad \text{on } l^2(\mathbb{N}), \tag{2.5}$$

So (3) holds.

(5) By Lemma 2.2 and (2.3), T is k-quasiparanormal if and only if, for any real number μ,

$$\begin{aligned} &w_k^2 w_{k+1}^2 - 2\mu w_k^2 + \mu^2 \ge 0,\\ &w_{k+1}^2 w_{k+2}^2 - 2\mu w_{k+1}^2 + \mu^2 \ge 0,\\ &w_{k+2}^2 w_{k+3}^2 - 2\mu w_{k+2}^2 + \mu^2 \ge 0, \quad \text{etc.} \end{aligned} \tag{2.6}$$

That is, T is k-quasiparanormal if and only if $w_k \le w_{k+1} \le w_{k+1} \le \cdots$.

(8) It is clear that $\|T\| = w_0$ and

$$r(T) = \lim_{m\to\infty} \|T^m\|^{1/m} = w_1,$$

therefore T is not normaloid. ∎

3. A matrix representation and SVEP

THEOREM 3.1. *Let T be (n,k)-quasiparanormal, $0 \neq \lambda \in \sigma_{\mathrm{p}}(T)$ and*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \qquad \text{on } \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}.$$

Then

$$T_{12}\left(\frac{T_{22}}{\lambda} + \cdots + \left(\frac{T_{22}}{\lambda}\right)^n - nI\right)T_{22}^k = 0, \tag{3.1}$$

and

$$\|T_{22}^{1+n}T_{22}^k x\|^{2/(1+n)} \cdot \|T_{22}^k x\|^{2n/(1+n)} \geq \|T_{12}T_{22}^k x\|^2 + \|T_{22}T_{22}^k x\|^2 \tag{3.2}$$

for any $x \in (\ker(T-\lambda))^{\perp}$. In particular, T_{22} is also (n,k)-quasiparanormal.

This is a generalization of [21, Theorem 2.1] and [23, Lemma 2.3].

Proof. Without loss of generality, we may assume that $\lambda = 1$. For each positive integer m,

$$T^{1+m} = \begin{pmatrix} 1 & T_{12}(I + T_{22} + \cdots + T_{22}^m) \\ 0 & T_{22}^{1+m} \end{pmatrix},$$

$$T^{*(1+m)}T^{1+m} = \begin{pmatrix} 1 & T_{12}(1+m) \\ (T_{12}(1+m))^* & |T_{12}(1+m)|^2 + |T_{22}^{1+m}|^2 \end{pmatrix}$$

where $T_{12}(1+m) = T_{12}(I + T_{22} + \cdots + T_{22}^m)$. Then

$$\begin{aligned} 0 &\leq T^{*k}T^{*(1+n)}T^{1+n}T^k - (1+n)\mu^n T^{*k}T^*TT^k + n\mu^{1+n}T^{*k}T^k \\ &= \begin{pmatrix} X(\mu) & Z(\mu) \\ (Z(\mu))^* & Y(\mu) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} X(\mu) &= 1 - (1+n)\mu^n + n\mu^{1+n}, \\ Z(\mu) &= T_{12}(n+k+1) - (1+n)\mu^n T_{12}(k+1) + n\mu^{1+n}T_{12}(k), \\ Y(\mu) &= |T_{12}(n+k+1)|^2 + |T_{22}^{n+k+1}|^2 - (1+n)\mu^n(|T_{12}(k+1)|^2 + |T_{22}^{k+1}|^2) \\ &\quad + n\mu^{1+n}(|T_{12}(k)|^2 + |T_{22}^k|^2). \end{aligned}$$

Put $\mu = 1$ in (2.2). Then

$$0 \le T^{*(n+k+1)}T^{n+k+1} - (1+n)T^{*(k+1)}T^{k+1} + nT^{*k}T^k$$
$$= \begin{pmatrix} 0 & Z(1) \\ (Z(1))^* & Y(1) \end{pmatrix}.$$

Hence $Z(1) = T_{12}T_{22}^k(T_{22} + \cdots + T_{22}^n - nI) = 0$, that is, (3.1) holds.

Next we prove (3.2). For each $\mu \neq 1$ there exists a contraction $D(\mu)$ such that $Z(\mu) = (X(\mu))^{1/2}D(\mu)(Y(\mu))^{1/2}$ (see [24, Lemma 1.4] and [7]). Thus

$$X(\mu)Y(\mu) \ge X(\mu)(Y(\mu))^{1/2}(D(\mu))^*D(\mu)(Y(\mu))^{1/2} = |Z(\mu)|^2.$$

This together with (3.1) implies that

$$\begin{aligned} Y(\mu) &\ge \frac{1}{X(\mu)}|Z(\mu)|^2 = \frac{1}{X(\mu)}|X(\mu)T_{12}(k) + (1+n)(1-\mu^n)T_{12}T_{22}^k|^2 \\ &= X(\mu)|T_{12}(k)|^2 + (1+n)(1-\mu^n)(T_{12}(k)^*T_{12}T_{22}^k + (T_{12}T_{22}^k)^*T_{12}(k)) \\ &\quad + \frac{1}{X(\mu)}(1+n)^2(1-\mu^n)^2|T_{12}T_{22}^k|^2. \end{aligned}$$

That is, for $\mu \neq 1$,

$$\text{(3.3)} \quad |T_{22}^{n+k+1}|^2 - (1+n)\mu^n(|T_{12}T_{22}^k|^2 + |T_{22}^{k+1}|^2) + n\mu^{1+n}|T_{22}^k|^2 \ge \frac{(1-\mu^n)^2 - X(\mu)}{X(\mu)}(1+n)^2|T_{12}T_{22}^k|^2.$$

As in [23, Lemma 2.3], let $f(\mu) = (1-\mu^n)^2 - X(\mu)$ on $(0,\infty)$. If $n = 1$ then $f(\mu) \ge 0$ is clear. If $n \ge 2$, then $f(1) = 0$, $f'(1) = 0$ and $f''(1) > 0$. Therefore $f(\mu) \ge 0$ on $(0,\infty)$ and

$$\text{(3.4)} \quad |T_{22}^{n+k+1}|^2 - (1+n)\mu^n(|T_{12}T_{22}^k|^2 + |T_{22}^{k+1}|^2) + n\mu^{1+n}|T_{22}^k|^2 \ge 0$$

for $\mu \neq 1$ by (3.3). It is clear that (3.3) holds for each real number μ by the continuity of μ. Similar to the proof that (2.2) implies the (n,k)-quasiparanormality of T in the proof of Lemma 2.2, (3.2) follows from (3.4). ∎

Corollary 3.2. *If T is (n,k)-quasiparanormal and $\lambda \neq 0$, then* $\ker(T_{22} - \lambda) = \{0\}$ *where T_{22} is as in Theorem* 3.1.

Proof. Let $x \in \ker(T_{22} - \lambda)$. Then $\|(T-\lambda)x\|^2 = \|T_{12}x\|^2 \le 0$ by (3.2). Hence $x \in \ker(T-\lambda) \cap (\ker(T-\lambda))^\perp = \{0\}$ and $\ker(T_{22} - \lambda) = \{0\}$. ∎

Corollary 3.3. *If T is (n,k)-quasiparanormal and $\lambda\mu \neq 0$, then* $\ker(T-\lambda) \perp \ker(T-\mu)$ *for $\lambda \neq \mu$.*

Proof. Let

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{on } \ker(T-\lambda) \oplus (\ker(T-\lambda))^\perp$$

and $x = x_1 \oplus x_2 \in \ker(T-\mu)$. Then

$$0 = (T-\mu)x = [(\lambda-\mu)x_1 + T_{12}x_2] \oplus (T_{22}-\mu)x_2.$$

By $(T_{22}-\mu)x_2 = 0$ and (3.2), we have $\|T_{12}x_2\|^2 = 0$. So $x_1 = 0$ for $\lambda \neq \mu$, which implies $x \in (\ker(T-\lambda))^{\perp}$ and so $\ker(T-\lambda) \perp \ker(T-\mu)$. ∎

COROLLARY 3.4. *If T is (n,k)-quasiparanormal, then T has SVEP.*

Corollary 3.4 follows easily from Corollary 3.3 and the result below.

LEMMA 3.5. *If* $\ker(T-\lambda) \perp \ker(T-\mu)$ *for any two different nonzero eigenvalues λ and μ of T, then T has SVEP.*

Lemma 3.5 is a generalization of [22, Proposition 3.1].

Proof. Let f be an analytic function such that $(T-\lambda)f(\lambda) = 0$ on an open set $\mathcal{D}$. By assumption, $f(\lambda) \in \ker(T-\lambda)$ for each $\lambda \in \mathcal{D}$. Thus $f(\lambda) \perp f(\mu)$ for any two different nonzero complex numbers λ and μ in $\mathcal{D}$. Hence, for any sequence $\{\mu_n\}$ of nonzero complex numbers such that $\mu_n \to \lambda$,

$$\|f(\lambda)\|^2 = \lim_{\mu_n \to \lambda} (f(\lambda), f(\mu_n)) = 0. \ \blacksquare$$

4. Ascent and Bishop's property (β)**.** An operator T is said to have *totally finite ascent* if $T-\lambda$ has finite ascent for every $\lambda \in \mathbb{C}$.

THEOREM 4.1. *Let T be an (n,k)-quasiparanormal operator.*

(1) $\ker T^{1+k} = \ker T^{2+k}$ *and* $\ker(T-\lambda) = \ker(T-\lambda)^2$ *where $\lambda \neq 0$. In particular, T has totally finite ascent and SVEP.*
(2) *T has Bishop's property (β).*

Theorem 4.1 is a generalization of [8, Theorem 2.5] and [16, Theorem 2.6]. To prove it, we need the following lemmas.

LEMMA 4.2 ([21, 23]). *Let T be n-paranormal,* $0 \neq \lambda \in \sigma_{\mathrm{p}}(T)$ *and*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{on } \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}.$$

Then $\ker(T_{22}-\lambda) = \{0\}$,

$$T_{12}\left(\frac{T_{22}}{\lambda} + \cdots + \left(\frac{T_{22}}{\lambda}\right)^n\right) = nT_{12},$$

$$\|T_{22}^{1+n}x\|^{2/(1+n)}\|x\|^{2n/(1+n)} \geq \|T_{12}x\|^2 + \|T_{22}x\|^2$$

for any $x \in (\ker(T-\lambda))^{\perp}$*. In particular, T_{22} is also n-paranormal.*

LEMMA 4.3 ([8]). *If T is n-paranormal, then* $\ker(T-\lambda) = \ker(T-\lambda)^2$ *for each* $\lambda \in \mathbb{C}$.

This is [8, Lemma 2.3]; we give a proof for convenience.

Proof. Assume $0 \neq \lambda \in \sigma_{\mathrm{p}}(T)$ because the cases of $\lambda = 0$ and of $\lambda \notin \sigma_{\mathrm{p}}(T)$ are clear. Let $0 \neq x \in \ker(T-\lambda)^2$ and $x = x_1 \oplus x_2 \in \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}$. Then

$$0 = (T-\lambda)^2 x = \begin{pmatrix} 0 & T_{12}(T_{22}-\lambda) \\ 0 & (T_{22}-\lambda)^2 \end{pmatrix} x = T_{12}(T_{22}-\lambda)x_2 \oplus (T_{22}-\lambda)^2 x_2.$$

Since $\ker(T_{22}-\lambda) = \{0\}$ by Lemma 4.2, it follows that $x_2 = 0$ and $x = x_1 \in \ker(T-\lambda)$. ■

Lemma 4.4. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } M \oplus M^{\perp}.$$

If T_1 and T_2 have Bishop's property (β), then so does T.

[19, Lemma 11] and [16, Theorem 2.6] give this result for k-quasiclass A and k-quasiparanormal operators. The proof of Lemma 4.4 is similar to [19, Lemma 4] or [16, Theorem 2.6], so we omit it here.

Proof of Theorem 4.1. (1) By definition, $\ker T^{1+n+k} = \ker T^{1+k}$, so that $\ker T^{2+k} = \ker T^{1+k}$. Assume $0 \neq \lambda \in \sigma_{\mathrm{p}}(T)$ because the case $\lambda \notin \sigma_{\mathrm{p}}(T)$ is obvious. Let $0 \neq x \in \ker(T-\lambda)^2$, $x = x_1 \oplus x_2 \in [T^k\mathcal{H}] \oplus \ker T^{*k}$ and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } [T^k\mathcal{H}] \oplus \ker T^{*k}.$$

Then

$$\begin{aligned} 0 = (T-\lambda)^2 x &= \begin{pmatrix} T_1-\lambda & T_2 \\ 0 & T_3-\lambda \end{pmatrix}^2 x \\ &= \begin{pmatrix} (T_1-\lambda)^2 x_1 + ((T_1-\lambda)T_2 + T_2(T_3-\lambda))x_2 \\ (T_3-\lambda)^2 x_2 \end{pmatrix}. \end{aligned}$$

Consequently, $x_2 = 0$ because $T_3 - \lambda$ is invertible by Theorem 2.1(4). Thus $(T_1-\lambda)^2 x_1 = 0$ and $(T_1-\lambda)x_1 = 0$ by Lemma 4.3. Therefore $(T-\lambda)x = (T-\lambda)(x_1 \oplus 0) = (T_1-\lambda)x_1 = 0$.

(2) Since quasinilpotent and n-paranormal operators have Bishop's property (β) [8, Theorem 2.5], the assertion follows by Theorem 2.1(4). ■

5. Quasinilpotent part and Riesz idempotents of k-quasiparanormal operators. The *quasinilpotent part* of T is defined by $\mathcal{H}_0(T) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}$. In general, $\mathcal{H}_0(T)$ is not closed [1, p. 43]. Let $\rho(T)$ and $p_0(T)$ denote the resolvent set and the set of poles of the resolvent of T respectively. For an isolated spectral point $\lambda \in \operatorname{iso}\sigma(T)$, let

$E_\lambda(T)$ be the Riesz idempotent for λ, denoted by E for short. The operator T is called *isoloid* if $\operatorname{iso}\sigma(T) \subset \sigma_{\rm p}(T)$, and *polaroid* if $\operatorname{iso}\sigma(T) \subset p_0(T)$.

It is known that $E\mathcal{H} = \mathcal{H}_0(T-\lambda)$, so $\mathcal{H}_0(T-\lambda)$ is closed [1, p. 157].

THEOREM 5.1. *Let T be a k-quasiparanormal operator and $\lambda \in \mathbb{C}$.*

(1) $\mathcal{H}_0(T) = \ker T^{k+1}$, *and if $\lambda \neq 0$, then* $\mathcal{H}_0(T-\lambda) = \ker(T-\lambda)$.

(2) *Let*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \qquad \text{on } \ker(T-\lambda) \oplus [(T-\lambda)^*\mathcal{H}].$$

If $0 \neq \lambda \in \operatorname{iso}\sigma(T)$ *and* $\ker(T_{22})^* = 0$, *then* $E = E^*$.

LEMMA 5.2. *Let m be a positive integer and $\lambda \in \operatorname{iso}\sigma(T)$.*

(1) *The following assertions are equivalent:*

(a) $E\mathcal{H} = \ker(T-\lambda)^m$.
(b) $\ker E = (T-\lambda)^m\mathcal{H}$.

(2) *If $\lambda \in p_0(T)$ and the order of λ is m, the following assertions are equivalent:*

(a) *E is self-adjoint.*
(b) $\ker(T-\lambda)^m = \ker(T-\lambda)^{*m}$.
(c) $\ker(T-\lambda)^m \subseteq \ker(T-\lambda)^{*m}$.

Proof. Let $\mathcal{H} = E\mathcal{H} + \ker E$, a topological direct sum. Then $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$ and $\lambda \notin \sigma(T|_{\ker E})$ (see [5, Chapter VII] and [14]).

(1) (a)⇒(b): We have

$$(T-\lambda)^m\mathcal{H} = (T-\lambda)^m(E\mathcal{H} + \ker E) = (T-\lambda)^m \ker E = \ker E.$$

(b)⇒(a): We have

$$\ker(T-\lambda)^m = \ker(T|_{E\mathcal{H}}-\lambda)^m + \ker(T|_{\ker E}-\lambda)^m = \ker(T|_{E\mathcal{H}}-\lambda)^m.$$

On the other hand,

$$\begin{aligned} \ker E = (T-\lambda)^m\mathcal{H} &= (T-\lambda)^m(E\mathcal{H} + \ker E) \\ &= (T|_{E\mathcal{H}}-\lambda)^m E\mathcal{H} + (T|_{\ker E}-\lambda)^m \ker E \\ &= (T|_{E\mathcal{H}}-\lambda)^m E\mathcal{H} + \ker E. \end{aligned} \tag{5.1}$$

Hence $(T|_{E\mathcal{H}}-\lambda)^m E\mathcal{H} = \{0\}$ and $E\mathcal{H} = \ker(T-\lambda)^m$.

(2) (a)⇒(b): By (1),

$$\ker(T-\lambda)^m = E\mathcal{H} = (\ker E)^\perp = ((T-\lambda)^m\mathcal{H})^\perp = \ker(T-\lambda)^{*m}.$$

(b)⇒(c) is obvious.

(c)⇒(a): (c) ensures $\ker(T-\lambda)^m \perp (T-\lambda)^m\mathcal{H}$, that is, $E\mathcal{H} \perp \ker E$. ∎

If T is hyponormal and $\lambda \in \operatorname{iso}\sigma(T)$, then $E\mathcal{H} = \ker(T-\lambda)$ ([17, Theorem 2]) and $\ker(T-\lambda)$ is a reducing space of T by definition. Thus

$\ker(T-\lambda) = \ker(T-\lambda)^*$ and E is self-adjoint by Lemma 5.2. This can be regarded as an alternative proof of [18, Theorem C] without using condition G_1.

LEMMA 5.3. *Let T be k-quasiparanormal.*

(1) *If $\sigma(T) = \{\lambda\}$, then $T^{1+k} = 0$ when $\lambda = 0$, and $T = \lambda$ when $\lambda \neq 0$.*
(2) *If $\lambda \in \operatorname{iso}\sigma(T)$, then $\lambda \in p_0(T)$, and the order of λ is no more than $1+k$ when $\lambda = 0$, and 1 when $\lambda \neq 0$.*

Lemma 5.3 implies that k-quasiparanormal operators are polaroid and isoloid. Paranormal operators can be regarded as 0-quasiparanormal operators. Lemma 5.3 holds for paranormal operators ([6, Lemma 2.1], [12], [4, Theorem 2.1]). [16, Lemma 2.8] yields the case $\lambda \neq 0$ of Lemma 5.3.

Proof. (1) If $T^k\mathcal{H}$ is dense, then T is paranormal and the assertion holds by [6, Lemma 2.1]. Assume $T^k\mathcal{H}$ is not dense. Then $\sigma(T) = \{\lambda\}$ implies $\lambda = 0$ by Theorem 2.1. So $\sigma(T_1) = \{0\}$ and $T_1 = 0$ (T_1 is as in Theorem 2.1). Thus

$$T^{1+k} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{1+k} \end{pmatrix} = 0$$

by Theorem 2.1.

(2) By Theorem 2.1(2), $T|_{E\mathcal{H}}$ is k-quasiparanormal and $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$. So $(T|_{E\mathcal{H}})^{1+k} = 0$ when $\lambda = 0$, and $T|_{E\mathcal{H}} = \lambda$ when $\lambda \neq 0$. That is, $E\mathcal{H} = \ker T^{1+k}$ when $\lambda = 0$, and $E\mathcal{H} = \ker(T-\lambda)$ when $\lambda \neq 0$. The assertion follows from Lemma 5.2(1) immediately. ∎

Proof of Theorem 5.1. (1) By Theorem 4.1, T has Dunford's property C [15, Proposition 1.2.19], that is, the local spectral subspace $X_T(F)$ of T is closed for every closed set $F \subseteq \mathbb{C}$. Thus $\mathcal{H}_0(T-\lambda) = X_{T-\lambda}(\{0\})$ is closed [1, Theorem 2.20] and $\sigma(S) \subseteq \{\lambda\}$ where $S = T|_{\mathcal{H}_0(T-\lambda)}$ [15, Proposition 1.2.20]. Moreover, S is k-quasiparanormal by Theorem 2.1.

If $\sigma(S)$ is empty, then $\mathcal{H}_0(T-\lambda) = \{0\}$ and $\ker(T-\lambda) = \{0\}$. If $\sigma(S)$ is not empty, then $\sigma(S) = \{\lambda\}$. By Lemma 5.3, $S^{1+k} = 0$ when $\lambda = 0$, and $S = \lambda$ when $\lambda \neq 0$. So the assertion follows.

(2) By Lemmas 5.2 and 5.3, λ is a simple pole of the resolvent of T and it is sufficient to prove $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$, that is, $T_{12} = 0$.

In fact, $\lambda \in \operatorname{iso}\sigma(T) \subset \rho(T_{22}) \cup \operatorname{iso}\sigma(T_{22})$. Since T_{22} is k-quasiparanormal and isoloid by Theorem 3.1 and Lemma 5.3, this together with $\ker(T_{22}-\lambda) = 0$ (Corollary 3.2) implies that $\lambda \in \rho(T_{22})$. Hence $T_{12}T_{22}^k = 0$ by Theorem 3.1, and $T_{12} = 0$ by the assumption $\ker(T_{22})^* = 0$. Therefore $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$. ∎

An operator T is called *algebraically (n,k)-quasiparanormal* if there exists a nonconstant complex polynomial h such that $h(T)$ is (n,k)-quasiparanormal. For $\lambda \in \sigma(T)$, let

$$h(T) - h(\lambda) = c(T-\lambda)^m (T-\lambda_1)^{m_1} \cdots (T-\lambda_l)^{m_l} \tag{5.2}$$

where $c \neq 0$, $\{m_i : i = 1, \dots, l\}$ is a subset of nonnegative integers, and $\lambda, \lambda_1, \dots, \lambda_l$ are different complex numbers.

The following assertions follow easily from the properties of (n,k)-quasiparanormal operators and polynomials (cf. [3]).

(1) If T is algebraically (n,k)-quasiparanormal then so is $T-\lambda$ for $\lambda \in \mathbb{C}$.
(2) If T is algebraically (n,k)-quasiparanormal then the restriction $T|_{\mathcal{M}}$ is also algebraically (n,k)-quasiparanormal.

Corollary 5.4. *Let T be algebraically k-quasiparanormal.*

(1) *If $\sigma(T) = \{\lambda\}$, then $(T-\lambda)^{m(1+k)} = 0$ when $h(\lambda) = 0$, and $(T-\lambda)^m = 0$ when $h(\lambda) \neq 0$.*
(2) *If $\lambda \in \operatorname{iso}\sigma(T)$, then $\lambda \in p_0(T)$, and the order of λ is no more than $m(1+k)$ when $h(\lambda) = 0$, and m when $h(\lambda) \neq 0$.*

Corollary 5.4 says that k-quasiparanormal operators are polaroid and isoloid.

Proof. (1) Since $\sigma(T) = \{\lambda\}$, we have $\sigma(h(T)) = \{h(\lambda)\}$ and $\{\lambda_i : i = 1, \dots, l\} \subseteq \rho(T)$. This together with (5.2) and Lemma 5.3 implies that $(T-\lambda)^{m(1+k)} = 0$ when $h(\lambda) = 0$, and $(T-\lambda)^m = 0$ when $h(\lambda) \neq 0$.

(2) By assumption, $h(T|_{E\mathcal{H}}) = h(T)|_{E\mathcal{H}}$ is k-quasiparanormal, that is, $T|_{E\mathcal{H}}$ is algebraically k-quasiparanormal. Moreover $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$, hence by (1) we have $(T|_{E\mathcal{H}} - \lambda)^{m(1+k)} = 0$ when $h(\lambda) = 0$, and $(T|_{E\mathcal{H}} - \lambda)^m = 0$ when $h(\lambda) \neq 0$. So $E\mathcal{H} = \ker(T-\lambda)^{m(1+k)}$ when $h(\lambda) = 0$, and $E\mathcal{H} = \ker(T-\lambda)^m$ when $h(\lambda) \neq 0$. Therefore the assertion holds by Lemma 5.2. ∎

Let $H(\sigma(T))$ be the set of all functions analytic on some open neighborhood $\mathcal{U}$ of $\sigma(T)$. It is well-known that if h is a nonconstant polynomial and $h(T)$ has SVEP, then T has SVEP [1, Theorem 2.40]. Thus algebraically k-quasiparanormal operators have SVEP by Corollary 3.4 or Theorem 4.1. The following result follows from Corollary 5.4 and [2, Theorems 3.12 and 3.14].

Corollary 5.5. *Let $f \in H(\sigma(T))$.*

(1) *If T is algebraically k-quasiparanormal, then Weyl type theorem (gW) holds for $f(T)$.*
(2) *If T^* is algebraically k-quasiparanormal, then Weyl type theorems (gW), (gaW), (gw) hold for $f(T)$ where f is nonconstant on each connected component of $\mathcal{U}$.*

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