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Bad properties of the Bernstein numbers

by

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Abstract. We show that the classes $\mathfrak{L}_p^{\text{bern}} := \{T : (b_n(T)) \in l_p\}$ associated with the Bernstein numbers b_n fail to be operator ideals. Moreover, $\mathfrak{L}_p^{\text{bern}} \circ \mathfrak{L}_q^{\text{bern}} \not\subseteq \mathfrak{L}_r^{\text{bern}}$ for $1/r = 1/p + 1/q$.

Let T be a (bounded linear) operator from a Banach space X into a Banach space Y. Then the nth *Bernstein number* $b_n(T)$ is defined to be the supremum of all constants $c \geq 0$ for which there exists an operator A from an n-dimensional Banach space E_n into X such that $\|A\| \leq 1$ and

$$\|TAu\| \geq c\|u\| \quad \text{whenever } u \in E_n.$$

If $\dim(X) \geq n$, then it is enough that A ranges over the canonical embeddings of the n-dimensional subspaces E_n of X; that is, $\|Tx\| \geq c\|x\|$ for $x \in E_n$. It easily turns out that the b_n's are injective s-numbers in the original sense of [5, pp. 202–203, 207–208]; see the modifications in [7, p. 79] and [8, p. 327].

The Bernstein numbers were invented by Mityagin and Pełczyński [3, p. 370]. In the context of widths, the concept above and its naming go back to the work of Tikhomirov [10]; see also [4, p. 306].

According to Milman [2, p. 141],

$$\mathfrak{L}^{\text{bern}} := \{T\colon \lim_{n\to\infty} b_n(T) = 0\}$$

is a closed operator ideal. Later on, $\mathfrak{L}^{\text{bern}}$ was identified as the superideal associated with the ideal of strictly singular operators; see [9]. Therefore well-known results about approximation numbers, Gelfand numbers etc. raised some hope that

$$\mathfrak{L}_p^{\text{bern}} := \Big\{T\colon \sum_{n=1}^{\infty} b_n(T)^p < \infty\Big\} \quad \text{with } 0 < p < \infty$$

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could be a one-parameter scale of quasi-Banach ideals. Moreover, the following Hölder-type result $(1/r = 1/p + 1/q,\ 0 < p, q < \infty)$ was conjectured:

$$T \in \mathfrak{L}_p^{\mathrm{bern}}(X, Y) \quad \text{and} \quad S \in \mathfrak{L}_q^{\mathrm{bern}}(Y, Z) \quad \text{imply} \quad ST \in \mathfrak{L}_r^{\mathrm{bern}}(X, Z).$$

In this paper, the two questions are answered negatively. These observations help to understand why the Bernstein numbers play only a minor role within the theory of s-numbers.

Given non-negative scalar sequences (α_n) and (β_n), the symbol $\alpha_n \preceq \beta_n$ means that $\alpha_n \le c\beta_n$ for $n = 1, 2, \ldots$ and with some constant $c > 0$. We write $\alpha_n \asymp \beta_n$ if $\alpha_n \preceq \beta_n$ and $\beta_n \preceq \alpha_n$.

Proposition 1. *Let $1 \le w < \infty$ and $D_\sigma : (\xi_k) \mapsto (k^{-\sigma}\xi_k)$, where $\sigma \ge 0$. Then*

$$b_n(D_\sigma : l_w \to l_\infty) = \Big(\sum_{k=1}^{n} k^{\sigma w}\Big)^{-1/w} \asymp n^{-(\sigma + 1/w)}.$$

Proof. The problem can be reduced to considering finite-dimensional diagonal operators

$$D_\sigma^N : (\xi_1, \ldots, \xi_k, \ldots, \xi_N) \mapsto (\xi_1, \ldots, k^{-\sigma}\xi_k, \ldots, N^{-\sigma}\xi_N).$$

Indeed, $\|D_\sigma A u\| \ge c\|u\|$ implies $\|D_\sigma A_0 u\| \ge (c - \|A - A_0\|)\|u\|$. Thus, using an arbitrarily small perturbation, we may arrange that the operator $A : E_n \to l_w$ in the definition of $b_n(D_\sigma : l_w \to l_\infty)$ maps E_n into a subspace $\{(\xi_k) \in l_\infty : \xi_k = 0 \text{ for } k > N\}$. Hence

$$b_n(D_\sigma : l_w \to l_\infty) = \sup_{1 \le N < \infty} b_n(D_\sigma^N : l_w^N \to l_\infty^N).$$

However, we know from [5, p. 217] that

$$b_n(D_\sigma^N : l_w^N \to l_\infty^N) = \Big(\sum_{k=1}^{n} k^{\sigma w}\Big)^{-1/w} \quad \text{for } n = 1, \ldots, N. \ \blacksquare$$

The symbol $\mathfrak{P}_2$ stands for the Banach ideal of 2-*summing operators*, and π_2 denotes the underlying norm. The nth *Weyl number* of an operator T from X into Y is defined by $x_n(T) := \sup\{a_n(TA) : \|A : l_2 \to X\| \le 1\}$, where $a_n(TA)$ denotes the nth *approximation number* of TA. Concerning further details, the reader is referred to [7].

We now establish an analogue of the well-known estimate (see [7, p. 98])

$$\sqrt{n}\, x_n(T) \le \pi_2(T) \quad \text{for all 2-summing operators } T.$$

Lemma 1. *$\sqrt{n}\, b_n(T) \le \pi_2(T)$ for all 2-summing operators T.*

Proof. Choose some operator A as described in the definition of $b_n(T)$. If $c > 0$, then TA induces an isomorphism S between E_n and $F_n := TA(E_n)$, and we have $\|S^{-1}\| \le c^{-1}$. The situation is illustrated by the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle A}\big\uparrow & & \big\uparrow{\scriptstyle J} \\ E_n & \xrightarrow{S} & F_n \end{array}$$

in which J denotes the canonical embedding from F_n into Y.

Recall that the 2-summing norm is injective: $\pi_2(S) = \pi_2(JS)$. Another fundamental result says that $\pi_2(I_{E_n}) = \sqrt{n}$ for the identity map I_{E_n} of every n-dimensional Banach space E_n (see [7, pp. 45, 158]). Hence

$$\sqrt{n} = \pi_2(I_{E_n}) \le \|S^{-1}\|\pi_2(S) = \|S^{-1}\|\pi_2(JS) = \|S^{-1}\|\pi_2(TA) \le c^{-1}\pi_2(T).$$

This implies that $\sqrt{n}\,c \le \pi_2(T)$. ∎

We know from [7, p. 156] that

$$(\Lambda) \qquad |\lambda_{2n-1}(R)| \le e\Big(\prod_{k=1}^{n} x_k(R)\Big)^{1/n},$$

where $(\lambda_n(R))$ denotes the eigenvalue sequence of the Riesz operator R acting on a Banach space X.

Lemma 2. *$b_{2n-1}(T) \le e(\prod_{k=1}^{n} x_k(T))^{1/n}$ for all operators T.*

Proof. With the difference that n is replaced by $2n-1$, we consider the same diagram as above. Recall that the Weyl numbers are injective: $x_k(S) = x_k(JS)$. Applying the eigenvalue estimate (Λ) to the identity map of E_{2n-1}, we obtain

$$1 = \lambda_{2n-1}(I_{E_{2n-1}}) \le e\Big(\prod_{k=1}^{n} x_k(I_{E_{2n-1}})\Big)^{1/n} \le e\|S^{-1}\|\Big(\prod_{k=1}^{n} x_k(S)\Big)^{1/n}$$
$$= e\|S^{-1}\|\Big(\prod_{k=1}^{n} x_k(JS)\Big)^{1/n} = e\|S^{-1}\|\Big(\prod_{k=1}^{n} x_k(TA)\Big)^{1/n} \le ec^{-1}\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}.$$

Remark. Looking at the identity map of l_1^n yields $b_n(\mathrm{Id} : l_1^n \to l_1^n) = 1$ (trivial) and $x_n(\mathrm{Id} : l_1^n \to l_1^n) = 1/\sqrt{n}$ (see [1, p. 19]). Thus an inequality of the type $b_n(T) \le cx_n(T)$ cannot hold.

Since $b_n(\mathrm{Id} : l_2 \to c_0) = 1/\sqrt{n}$ and $x_n(\mathrm{Id} : l_2 \to c_0) = 1$, the situation in the converse direction is even worse: $x_{2n-1}(T) \le c(\prod_{k=1}^{n} b_k(T))^{1/n}$ fails to hold for any constant $c > 0$.

The next result goes back to Lubitz [1, p. 30]. Streamlined proofs can be found in [7, pp. 112–113].

Lemma 3. *Let $2 \le w < \infty$ and $D_\tau : (\xi_k) \mapsto (k^{-\tau}\xi_k)$, where $\tau > 1/w$. Then*

$$x_n(D_\tau : l_\infty \to l_w) \asymp n^{-\tau}.$$

Now we are prepared to establish a counterpart of Proposition 1.

PROPOSITION 2. *Let $2 \le w < \infty$ and $D_\tau : (\xi_k) \mapsto (k^{-\tau}\xi_k)$, where $\tau > 1/w$. Then*

$$b_n(D_\tau : l_\infty \to l_w) \asymp n^{-\tau}.$$

Proof. By Lemmas 2 and 3, we have

$$b_{2n-1}(D_\tau : l_\infty \to l_w) \le e\Big(\prod_{k=1}^{n} x_k(D_\tau : l_\infty \to l_w)\Big)^{1/n} \asymp \Big(\prod_{k=1}^{n} k^{-\tau}\Big)^{1/n}.$$

Hence it follows from $n^n/n! < e^n$ that

$$b_n(D_\tau : l_\infty \to l_w) \preceq (n!)^{-\tau/n} \preceq n^{-\tau}.$$

The converse estimate is trivial. ■

In the following, it is more convenient to work with the class $\mathfrak{L}_{p,\infty}^{\rm bern}$ that consists of all operators T for which

$$\lambda_{p,\infty}^{\rm bern}(T) := \sup_{1\le n<\infty} n^{1/p} b_n(T)$$

is finite. Note that $\mathfrak{L}_p^{\rm bern} \subset \mathfrak{L}_{p,\infty}^{\rm bern} \subset \mathfrak{L}_{p_0}^{\rm bern}$ for $0 < p < p_0 < \infty$.

Now we are ready for the *final construction*, which is borrowed from [6, pp. 362–363]:

Given $p > 0$, we choose an exponent w such that $\max\{2, p\} < w < \infty$. Let $\sigma := 1/p - 1/w > 0$ and $\tau := 1/p > 1/w$. Form the direct sum $l_w \oplus l_\infty$ equipped with the norm

$$\|(x, y)\| := (\|x \,|\, l_w\|^w + \|y \,|\, l_\infty\|^w)^{1/w}.$$

Define the canonical maps

$$J_w : x \mapsto (x, \mathsf{o}), \quad Q_w : (x, y) \mapsto x \quad \text{and} \quad J_\infty : y \mapsto (\mathsf{o}, y), \quad Q_\infty : (x, y) \mapsto y.$$

Moreover, put

$$S : (x, y) \mapsto (\mathsf{o}, D_\sigma x) \quad \text{and} \quad T : (x, y) \mapsto (D_\tau y, \mathsf{o}).$$

Then

$$S = J_\infty D_\sigma Q_w, \quad D_\sigma = Q_\infty S J_w \quad \text{and} \quad T = J_w D_\tau Q_\infty, \quad D_\tau = Q_w T J_\infty.$$

Propositions 1 and 2 tell us that both operators S and T belong to $\mathfrak{L}_{p,\infty}^{\rm bern}$ with $1/p = \sigma + 1/w = \tau$. We have

$$S + T : (e_k, k^{1/2w} e_k) \mapsto k^{-(\tau+\sigma)/2}(e_k, k^{1/2w} e_k),$$

where e_k denotes the kth unit sequence.

Let E_n be the linear space of all n-tuples equipped with the norm

$$\|(\xi_k)\| := \Big(\sum_{k=1}^{n} |\xi_k|^w + \sup_{1\le k\le n} k^{1/2}|\xi_k|^w\Big)^{1/w}.$$

Then

$$J_n : (\xi_k) \mapsto \sum_{k=1}^{n} \xi_k (e_k, k^{1/2w} e_k)$$

defines an isometric embedding from E_n into $l_w \oplus l_\infty$. Moreover, it follows from

$$(S+T)J_n : (\xi_k) \mapsto \sum_{k=1}^{n} k^{-(\tau+\sigma)/2} \xi_k (e_k, k^{1/2w} e_k)$$

that

$$\begin{aligned} \|(S+T)J_n(\xi_k)\| &= \Big(\sum_{k=1}^{n} |k^{-(\tau+\sigma)/2}\xi_k|^w + \sup_{1\le k\le n} k^{1/2} |k^{-(\tau+\sigma)/2}\xi_k|^w \Big)^{1/w} \\ &\ge n^{-(\tau+\sigma)/2} \|(\xi_k)\|. \end{aligned}$$

Hence

$$b_n(S+T) \ge n^{-(\tau+\sigma)/2} = n^{-1/p+1/2w}.$$

This means that subject to a suitable choice of w, the sum $S+T$ does not belong to $\mathfrak{L}^{\mathrm{bern}}_{p_0-\varepsilon,\infty}$ whenever $0 < \varepsilon < p_0$, where $1/p_0 := 1/p - 1/4$ if $0 < p < 2$ and $p_0 := 2p$ if $2 \le p < \infty$. In both cases p_0 is larger than p. ∎

The upshot of the foregoing results is

Theorem 1. *The classes $\mathfrak{L}^{\mathrm{bern}}_p$ and $\mathfrak{L}^{\mathrm{bern}}_{p,\infty}$ fail to be operator ideals.*

We proceed with another negative fact.

Theorem 2. *If $0 < p, q < \infty$ and $1/r = 1/p + 1/q$, then*

$$\mathfrak{L}^{\mathrm{bern}}_p \circ \mathfrak{L}^{\mathrm{bern}}_q \not\subseteq \mathfrak{L}^{\mathrm{bern}}_r \quad \text{and} \quad \mathfrak{L}^{\mathrm{bern}}_{p,\infty} \circ \mathfrak{L}^{\mathrm{bern}}_{q,\infty} \not\subseteq \mathfrak{L}^{\mathrm{bern}}_{r,\infty}.$$

Proof. Choose w such that $\max\{2, p, q\} < w < \infty$. Let $\sigma := 1/p - 1/w$ and $\tau := 1/q$. Then Propositions 1 and 2 tell us that $D_\sigma \in \mathfrak{L}^{\mathrm{bern}}_{p,\infty}(l_w, l_\infty)$ and $D_\tau \in \mathfrak{L}^{\mathrm{bern}}_{q,\infty}(l_\infty, l_w)$. On the other hand, it follows from

$$b_n(D_\sigma D_\tau : l_\infty \to l_\infty) = b_n(D_\tau D_\sigma : l_w \to l_w) = n^{-(\sigma+\tau)} = n^{-1/r+1/w}$$

that the Bernstein numbers of the products $D_\sigma D_\tau$ and $D_\tau D_\sigma$ are rather large. ∎

Finally, we recall that s-numbers s_n are said to be *additive* if

$$s_{m+n-1}(S+T) \le s_m(S) + s_n(T) \quad \text{for } X \xrightarrow{S,T} Y \text{ and } m, n = 1, 2, \ldots$$

and *multiplicative* if

$$s_{m+n-1}(ST) \le s_m(S) s_n(T) \quad \text{for } X \xrightarrow{T} Y \xrightarrow{S} Z \text{ and } m, n = 1, 2, \ldots.$$

The first property ensures that the classes $\mathfrak{L}^{(s)}_p := \{T : (s_n(T)) \in l_p\}$ are operator ideals, while the second one implies the Hölder-type multiplication formula $\mathfrak{L}_p^{(s)} \circ \mathfrak{L}_q^{(s)} \subseteq \mathfrak{L}_r^{(s)}$ for $1/r = 1/p + 1/q$. Thus we can give a negative answer to a problem posed more than 30 years ago; see [5, p. 222].

THEOREM 3. *The Bernstein numbers are neither additive nor multiplicative.*

Direct proof. In view of [5, p. 217],

$$b_n(\mathrm{Id} : l_2^N \to l_\infty^N) = 1/\sqrt{n} \quad \text{for } n = 1, \ldots, N.$$

On the other hand, using Lemma 1, we obtain

$$\sqrt{n}\, b_n(D_{1/2}^N : l_\infty^N \to l_2^N) \le \pi_2(D_{1/2}^N : l_\infty^N \to l_2^N) = \sqrt{\frac{1}{1} + \cdots + \frac{1}{N}} \le \sqrt{1 + \log N}.$$

Now we may proceed as in the *final construction*: Replace D_σ and D_τ by $\mathrm{Id} : l_2^N \to l_\infty^N$ and $D_{1/2}^N : l_\infty^N \to l_2^N$, respectively. Put $N = 2n$.

The concept of Bernstein numbers can be modified as follows. We define $l_n(T)$ as the supremum of all constants $c \ge 0$ for which there exists an operator A from l_2^n into X such that $\|A\| \le 1$ and

$$\|TAu\| \ge c\|u\| \quad \text{whenever } u \in l_2^n.$$

This quantity could be called the nth *Dvoretzky number* of the operator T. The l_n's are s-numbers in the sense of [8, p. 327]. That is, the formula $s_n(\mathrm{Id} : E_n \to E_n) = 1$ is not required to hold for all n-dimensional Banach spaces E_n but only for l_2^n. In general, we have

$$l_n(\mathrm{Id} : E_n \to E_n) = d(E_n, l_2^n)^{-1},$$

where $d(E_n, l_2^n)$ denotes the Banach–Mazur distance between E_n and l_2^n. One easily obtains the inequalities $l_n(T) \le b_n(T)$ and $l_n(T) \le x_n(T)$, which may be strict.

The same reasoning as above shows that the Dvoretzky numbers fail to be multiplicative. Unfortunately, the counterexample does not work in the case of additivity. Nevertheless, I have strong doubts that the classes

$$\mathfrak{L}_p^{\mathrm{dvor}} := \Big\{T : \sum_{n=1}^{\infty} l_n(T)^p < \infty\Big\} \quad \text{with } 0 < p < \infty$$

are operator ideals.

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