STUDIA MATHEMATICA 180 (2) (2007)

On S. Mazur's problems 8 and 88 from the Scottish Book

by

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Abstract. The paper discusses Problems 8 and 88 posed by Stanisław Mazur in the Scottish Book. It turns out that negative solutions to both problems are immediate consequences of the results of Peller [J. Operator Theory 7 (1982)]. We discuss here some quantitative aspects of Problems 8 and 88 and give answers to open problems discussed in a recent paper of Pełczyński and Sukochev in connection with Problem 88.

1. Introduction. We are going to discuss Problems 8 and 88 posed by Stanisław Mazur in the Scottish Book [SB]. Problem 88 asks whether a Hankel matrix in the injective tensor product $\ell^1 \check{\otimes} \ell^1$ of two spaces ℓ^1 must have finite sum of the moduli of its matrix entries. Problem 8 asks whether for an arbitrary sequence $\{z_n\}_{n\geq 0}$ in the space c of converging sequences there exist sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ in c such that

$$z_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k y_{n-k}, \qquad n \geq 0.$$

We give precise statements of the problems and all necessary definitions later.

It is known that both problems have negative solutions. Independently, solutions were obtained by Kwapień and Pełczyński [KP] and Eggermont and Leung [EL]. In a recent paper by Pełczyński and Sukochev [PS] in Section 6 certain quantitative results related to negative solutions of Problems 8 and 88 are obtained and certain open problems are raised.

It turns out, however, that the results of Section 5 of my earlier paper [P1] immediately imply negative solutions to Problems 8 and 88. Moreover, Section 5 of [P1] contains much stronger results. In particular, a complete description of the Hankel matrices ([1]) in the injective tensor product of two

2000 *Mathematics Subject Classification*: 47B35, 46M05, 47A07, 46E35.

Key words and phrases: Hankel matrix, injective tensor product, projective tensor product, Besov spaces, Fourier coefficients.

The author is partially supported by NSF grant DMS 0200712.

([1]) Note that Hankel matrices and Hankel operators play an important role in many areas of mathematics and applications; see [P2].

spaces ℓ^1 is obtained in [P1] in terms of the Besov space $B^1_{\infty,1}$. Unfortunately, I was not aware of Problems 8 and 88 when I was writing the paper [P1].

In Sections 3 and 4 of this paper we explain why the results of [P1] immediately imply negative solutions to Problems 8 and 88 and we give a solution to the problems raised in [PS].

In §2 we collect necessary information on tensor products and Besov spaces.

2. Preliminaries

2.1. *Projective and injective tensor products.* We define the *projective tensor product* $\ell^\infty \widehat{\otimes} \ell^\infty$ as the space of matrices $\{q_{jk}\}_{j,k\ge 0}$ of the form

$$q_{jk} = \sum_{n\ge 0} a_j^{(n)} b_k^{(n)}, \tag{2.1}$$

where $a^{(n)} = \{a_j^{(n)}\}_{j\ge 0}$ and $b^{(n)} = \{b_j^{(n)}\}_{j\ge 0}$ are sequences in ℓ^∞ such that

$$\sum_{n\ge 0} \|a^{(n)}\|_{\ell^\infty} \|b^{(n)}\|_{\ell^\infty} < \infty. \tag{2.2}$$

The norm of the matrix $\{q_{jk}\}_{j,k\ge 0}$ in $\ell^\infty \widehat{\otimes} \ell^\infty$ is defined as the infimum of the left-hand side of (2.2) over all sequences $a^{(n)} = \{a_j^{(n)}\}_{j\ge 0}$ and $b^{(n)} = \{b_j^{(n)}\}_{j\ge 0}$ satisfying (2.1).

Similarly, one can define the projective tensor products $c \widehat{\otimes} c$ and $c_0 \widehat{\otimes} c_0$, where c is the subspace of ℓ^∞ that consists of the converging sequences and c_0 is the subspace of c that consists of the sequences with zero limit.

We consider the space V^2 that is a kind of a “weak completion” of $\ell^\infty \widehat{\otimes} \ell^\infty$. V^2 consists of the matrices $Q = \{q_{jk}\}_{j,k\ge 0}$ for which

$$\sup_{n>0} \|P_n Q\|_{\ell^\infty \widehat{\otimes} \ell^\infty} < \infty,$$

where the projections P_n are defined by

$$(P_n Q)_{jk} = \begin{cases} q_{jk} & \text{if } j,k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $c \widehat{\otimes} c \subset \ell^\infty \widehat{\otimes} \ell^\infty \subset V^2$.

The *injective tensor product* $\ell^1 \check{\otimes} \ell^1$ of two spaces ℓ^1 is, by definition, the space of matrices $Q = \{q_{jk}\}_{j,k\ge 0}$ such that

$$\|Q\|_{\ell^1 \check{\otimes} \ell^1} = \sup \left| \sum_{j,k=0}^{N} q_{jk} x_j y_k \right| < \infty,$$

where the supremum is taken over all sequences $\{x_j\}_{j\ge 0}$ and $\{y_k\}_{k\ge 0}$ in the unit ball of ℓ^∞ and over all positive integers N. The space $\ell^1 \check{\otimes} \ell^1$ can be naturally identified with the space of bounded linear operators from c_0 to ℓ^1 (note that every bounded operator from c_0 to ℓ^1 is compact).

2.2. *Besov spaces.* In this paper we consider only Besov spaces of functions analytic in the unit disk $\mathbb{D}$. Besov spaces $B^s_{p,q}$ admit many equivalent descriptions. We give a definition in terms of dyadic Fourier expansions. We define the polynomials W_n, $n \ge 0$, as follows. If $n \ge 1$, then $\widehat{W}_n(2^n) = 1$, $\widehat{W}_n(k) = 0$ for $k \notin (2^{n-1}, 2^{n+1})$, and $\widehat{W}_n$ is a linear function on $[2^{n-1}, 2^n]$ and on $[2^n, 2^{n+1}]$. We put $W_0(z) = 1 + z$. It is easy to see that

$$\|W_n\|_{L^1} \le 3/2, \quad n \ge 0,$$

and

$$f = \sum_{n\ge 0} f * W_n$$

for an arbitrary analytic function f in $\mathbb{D}$.

For $1 \le p, q \le \infty$, and $s \in \mathbb{R}$, we define the Besov space $B^s_{p,q}$ as the space of analytic functions in $\mathbb{D}$ satisfying

$$f \in B^s_{p,q} \Leftrightarrow \{2^{ns}\|f * W_n\|_{L^p}\}_{n\ge 0} \in \ell^q. \tag{2.3}$$

If $q = \infty$, the space $B^s_{p,q}$ is nonseparable. We denote by $b^s_{p,\infty}$ the closure of the set of polynomials in $B^s_{p,\infty}$. It is easy to verify that

$$f \in b^s_{p,\infty} \Leftrightarrow \{2^{ns}\|f * W_n\|_{L^p}\}_{n\ge 0} \in c_0.$$

Besov spaces admit many other descriptions (see [Pe] and [P2]).

3. Problem 8. To state Mazur's Problem 8 of the Scottish Book [SB], consider the bilinear form $\mathcal{B}$ on $c \times c$ defined by

$$\mathcal{B}(\{x_n\}_{n\ge 0}, \{y_n\}_{n\ge 0}) = \{z_n\}_{n\ge 0},$$

where

$$z_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k y_{n-k}, \quad n \ge 0.$$

It is easy to see that B maps $c \times c$ into c. Stanisław Mazur asked in Problem 8 *whether $\mathcal{B}$ maps $c \times c$ onto c.*

As mentioned in the Introduction, a negative solution to Problem 8 follows immediately from the results of §5 of [P1]. To state Theorem 5.1 of [P1], we define an operator $\mathcal{A}$ on the space of matrices. Let $Q = \{q_{jk}\}_{j,k\ge 0}$. Then $\mathcal{A}Q$ is the sequence defined by

$$\mathcal{A}Q = \{z_n\}_{n\ge 0}, \quad \text{where} \quad z_n = \frac{1}{n+1}\sum_{j+k=n} q_{jk}.$$

THEOREM (5.1 of [P1]). *$\mathcal{A}$ maps the space V^2 onto the space of Fourier coefficients of the Besov space $B^0_{1,\infty}$.*

Recall that the space V^2 is defined in Section 2. In particular, it follows from Theorem 5.1 of [P1] that

$$\mathcal{A}(c \mathbin{\widehat{\otimes}} c) \subset \mathcal{A}(\ell^\infty \mathbin{\widehat{\otimes}} \ell^\infty) \subset \mathcal{A}(V^2) \subset \{\{\widehat{f}(n)\}_{n\ge 0} : f \in B^0_{1,\infty}\},$$

and so

$$\mathcal{B}(c \times c) \subset \{\{\widehat{f}(n)\}_{n\ge 0} : f \in B^0_{1,\infty}\}.$$

It is easy to see that

$$c \not\subset \{\{\widehat{f}(n)\}_{n\ge 0} : f \in B^0_{1,\infty}\}.$$

Indeed, if $f \in B^0_{1,\infty}$, then it follows immediately from (2.3) and from [R, §8.6] that

$$\sup_{n\ge 0} \sum_{k=0}^{n} |\widehat{f}(2^n + 2^k)|^2 < \infty.$$

This gives a negative solution to Problem 8.

In fact, Theorem 5.1 of [P1] allows one to describe $\mathcal{A}(c \mathbin{\widehat{\otimes}} c)$. First, let us observe that Theorem 5.1 of [P1] easily implies the following description of $\mathcal{A}(c_0 \mathbin{\widehat{\otimes}} c_0)$.

Theorem 3.1.

$$\mathcal{A}(c_0 \mathbin{\widehat{\otimes}} c_0) = \{\{\widehat{f}(n)\}_{n\ge 0} : f \in b^0_{1,\infty}\}.$$

Recall that $b^0_{1,\infty}$ is the closure of the polynomials in $B^0_{1,\infty}$ (see §2). Theorem 3.1, in turn, easily implies the following description of $\mathcal{A}(c \mathbin{\widehat{\otimes}} c)$.

Theorem 3.2.

$$\mathcal{A}(c \mathbin{\widehat{\otimes}} c) = \{\{\widehat{f}(n) + d\}_{n\ge 0} : f \in b^0_{1,\infty},\ d \in \mathbb{C}\}.$$

4. Problem 88. Recall that in Problem 88 of [SB], S. Mazur asked *whether a Hankel matrix $\{\gamma_{j+k}\}_{j,k\ge 0}$ in the injective tensor product $\ell^1 \mathbin{\check{\otimes}} \ell^1$ must satisfy the condition*

$$\sum_{k=0}^{\infty} (1+k)|\gamma_k| < \infty,$$

i.e., whether the sum of the moduli of the matrix entries must be finite.

As mentioned in the Introduction, a negative solution to Problem 88 follows immediately from the results of §5 of [P1]. A complete description of Hankel matrices in $\ell^1 \mathbin{\check{\otimes}} \ell^1$ is given by Theorem 5.2 of [P1]:

Theorem (5.2 of [P1]). *A Hankel matrix $\{\gamma_{j+k}\}_{j,k\ge 0}$ belongs to $\ell^1 \mathbin{\check{\otimes}} \ell^1$ if and only if the function f defined by*

$$f(z) = \sum_{n\ge 0} \gamma_n z^n$$

belongs to the Besov class $B^1_{\infty,1}$.

Let us obtain the best possible estimate on the moduli of the matrix entries of Hankel matrices in $\ell^1 \check{\otimes} \ell^1$.

Since $\|f * W_n\|_{L^2} \le \|f\|_{L^2} \|W_n\|_{L^1} \le \frac{3}{2}\|f\|_{L^2}$, it follows easily from (2.3) that if $f \in B^1_{\infty,1}$, then

$$\sum_{n=0}^{\infty} 2^n \Big(\sum_{k=2^n}^{2^{n+1}-1} |\widehat{f}(k)|^2 \Big)^{1/2} < \infty. \tag{4.1}$$

Let us show that this is the best possible estimate for the moduli of the Fourier coefficients of functions in $B^1_{\infty,1}$. To show this, we are going to use a version of the de Leeuw–Katznelson–Kahane theorem. It was proved in [dLKK] that if $\{\beta_n\}_{n\in\mathbb{Z}}$ is a sequence of nonnegative numbers in $\ell^2(\mathbb{Z})$, then there exists a continuous function f on $\mathbb{T}$ such that

$$|\widehat{f}(n)| \ge \beta_n, \qquad n \in \mathbb{Z}.$$

We refer the reader to [K1], [K2], and [N] for refinements of the de Leeuw–Katznelson–Kahane theorem and different proofs. We need the following version of the de Leeuw–Katznelson–Kahane theorem:

LEMMA 4.1. *There is a positive number K such that for arbitrary nonnegative numbers $\beta_0, \beta_1, \ldots, \beta_m$, there exists a polynomial f of degree m such that*

$$|\widehat{f}(j)| \ge \beta_j, \quad 0 \le j \le n, \quad \textit{and} \quad \|f\|_{L^\infty} \le K \Big(\sum_{j=0}^{n} \beta_j^2 \Big)^{1/2}.$$

Lemma 4.1 follows easily from the results of [K2].

THEOREM 4.2. *Let $\{\alpha_k\}_{k\ge 0}$ be a sequence of nonnegative numbers such that*

$$\sum_{n=0}^{\infty} 2^n \Big(\sum_{k=2^n}^{2^{n+1}-1} \alpha_k^2 \Big)^{1/2} < \infty. \tag{4.2}$$

Then there exists a function φ in the space $B^1_{\infty,1}$ such that $|\widehat{\varphi}(k)| \ge \alpha_k$ for $k \ge 0$.

Proof. By Lemma 4.1, there exists $K > 0$ and a sequence of polynomials f_n, $n \ge 0$, such that

$$f_0(z) = \widehat{f}_0(0) + \widehat{f}_0(1) z, \qquad f_n(z) = \sum_{k=2^n}^{2^{n+1}-1} \widehat{f}_n(k) z^k \quad \text{for } n \ge 1,$$

$$|\widehat{f}_0(k)| \ge \alpha_k \quad \text{for } k = 0, 1, \quad |\widehat{f}_n(k)| \ge \alpha_k \quad \text{for } n \ge 1,\ 2^n \le k \le 2^{n+1} - 1,$$

and

$$\|f_0\|_{L^\infty} \le K(\alpha_0^2+\alpha_1^2)^{1/2}, \qquad \|f_n\|_{L^\infty} \le K\Big(\sum_{k=2^n}^{2^{n+1}-1} \alpha_k^2\Big)^{1/2} \quad \text{for } n \ge 1.$$

We can now define the function φ by

$$\varphi = \sum_{n\ge 0} f_n.$$

Obviously, $|\widehat{\varphi}(k)| \ge \alpha_k$ for $k \ge 0$. Let us show that $\varphi \in B^1_{\infty,1}$. We have

$$\begin{aligned}
\sum_{n\ge1} 2^n \|\varphi * W_n\|_{L^\infty} &= \sum_{n\ge1} 2^n \|(f_{n-1}+f_n+f_{n+1}) * W_n\|_{L^\infty} \\
&\le \sum_{n\ge1} 2^n \|(f_{n-1}+f_n+f_{n+1})\|_{L^\infty}\|W_n\|_{L^1} \\
&\le 3\sum_{n\ge1} 2^n \|f_n\|_{L^\infty}\|W_n\|_{L^1} \le 3\cdot\frac{3}{2}\sum_{n\ge1} 2^n\|f_n\|_{L^\infty} \\
&\le \frac{9}{2}K\sum_{n=0}^{\infty} 2^n\Big(\sum_{k=2^n}^{2^{n+1}-1}\alpha_k^2\Big)^{1/2} < \infty. \ \blacksquare
\end{aligned}$$

In [PS] the following problem was considered. Let Ψ be the function on $(0,\infty)$ defined by

$$\Psi(t) = \begin{cases} \frac{3}{2}t - 1, & 0 < t \le 2, \\ t, & t > 2. \end{cases}$$

Let $\{\gamma_{j+k}\}_{j,k\ge0}$ be a Hankel matrix. The following result was proved in [PS] (Theorem 6.7):

(i) *if* $\beta < \Psi(t)$, *then*

$$\sum_{k\ge0} |\gamma_k|^t (1+k)^\beta < \infty \quad \textit{whenever } \{\gamma_{j+k}\}_{j,k\ge0} \in \ell^1 \check{\otimes} \ell^1;$$

(ii) *if* $\beta > \Psi(t)$, *then*

$$\sum_{k\ge0} |\gamma_k|^t (1+k)^\beta = \infty \quad \textit{for some } \{\gamma_{j+k}\}_{j,k\ge0} \in \ell^1 \check{\otimes} \ell^1;$$

(iii) *if* $\beta = \Psi(t)$ *and* $4/3 \le t < \infty$, *then*

$$\sum_{k\ge0} |\gamma_k|^t (1+k)^\beta < \infty \quad \textit{whenever } \{\gamma_{j+k}\}_{j,k\ge0} \in \ell^1 \check{\otimes} \ell^1.$$

In [PS] the problem is raised of finding whether

$$\sum_{k\ge0} |\gamma_k|^t (1+k)^{\Psi(t)}$$

has to be finite for $t \in (0, 4/3)$ whenever $\{\gamma_{j+k}\}_{j,k\ge0} \in \ell^1 \check{\otimes} \ell^1$.

It is easy to deduce Theorem 6.7 of [PS] from (4.1) and the above Theorem 4.2. Moreover, using (4.1) and Theorem 4.2, we can solve the problem posed in [PS] and settle the case $t \in (0, 4/3)$.

THEOREM 4.3. *If* $1 \le t < 4/3$, *then*

$$\sum_{k\ge 0} |\gamma_k|^t (1+k)^{3t/2-1} < \infty \quad \textit{whenever } \{\gamma_{j+k}\}_{j,k\ge 0} \in \ell^1 \check{\otimes} \ell^1.$$

If $0 < t < 1$, *then*

$$\sum_{k\ge 0} |\gamma_k|^t (1+k)^{3t/2-1} = \infty \quad \textit{for some } \{\gamma_{j+k}\}_{j,k\ge 0} \in \ell^1 \check{\otimes} \ell^1.$$

Proof. Suppose that $1 \le t < 2$. By Hölder's inequality, we have

$$\begin{aligned}\sum_{k\ge 1} |\gamma_k|^t (1+k)^{3t/2-1} &\le \text{const} \sum_{n\ge 0} 2^{n(3t/2-1)} \sum_{k=2^n}^{2^{n+1}} |\gamma_k|^t \\ &\le \text{const} \sum_{n\ge 0} 2^{3nt/2} 2^{-n} \Big(\sum_{k=2^n}^{2^{n+1}} |\gamma_k|^2 \Big)^{t/2} 2^{n(1-t/2)} \\ &= \text{const} \sum_{n\ge 0} 2^{nt} \Big(\sum_{k=2^n}^{2^{n+1}} |\gamma_k|^2 \Big)^{t/2}.\end{aligned}$$

Since $t \ge 1$, the ℓ^t norm of a sequence does not exceed its ℓ^1 norm, and so

$$\sum_{n\ge 0} 2^{nt} \Big(\sum_{k=2^n}^{2^{n+1}} |\gamma_k|^2 \Big)^{t/2} \le \Big(\sum_{n\ge 0} 2^n \Big(\sum_{k=2^n}^{2^{n+1}} |\gamma_k|^2 \Big)^{1/2} \Big)^t.$$

The result now follows from Theorem 5.2 of [P1] and (4.1).

Suppose now that $0 < t < 1$. It follows from Theorem 4.2 that it suffices to find a sequence $\{\alpha_k\}_{k\ge 0}$ of nonnegative numbers that satisfies (4.2) and

$$\sum_{k\ge 0} \alpha_k^t (1+k)^{3t/2-1} = \infty.$$

Let $\{\delta_n\}_{n\ge 0}$ be a sequence of positive numbers such that $\{2^{3n/2}\delta_n\}_{n\ge 0} \in \ell^1$, but $\{2^{3n/2}\delta_n\}_{n\ge 0} \notin \ell^t$. Put

$$\alpha_0 = 0, \qquad \alpha_k = \delta_n \quad \text{if } 2^n \le k \le 2^{n+1} - 1.$$

We have

$$\sum_{n\ge 0} 2^n \Big(\sum_{k=2^n}^{2^{n+1}-1} \alpha_k^2 \Big)^{1/2} = \sum_{n\ge 0} 2^{3n/2} \delta_n < \infty.$$

However,

$$\sum_{k\ge 0}\alpha_k^t(1+k)^{3t/2-1}\ge \text{const}\sum_{n\ge 0}2^{n(3t/2-1)}\sum_{k=2^n}^{2^{n+1}}\alpha_k^t$$
$$=\text{const}\sum_{n\ge 0}2^{n(3t/2-1)}2^n\delta_n^t=\text{const}\sum_{n\ge 0}2^{3nt/2}\delta_n^t=\infty.\ \blacksquare$$

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Received March 20, 2007
Revised version March 27, 2007 (6109)