Geometric, spectral and asymptotic properties of averaged products of projections in Banach spaces

by

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Abstract. According to the von Neumann–Halperin and Lapidus theorems, in a Hilbert space the iterates of products or, respectively, of convex combinations of ortho-projections are strongly convergent. We extend these results to the iterates of convex combinations of products of some projections in a complex Banach space. The latter is assumed uniformly convex or uniformly smooth for the orthoprojections, or reflexive for more special projections, in particular, for the hermitian ones. In all cases the proof of convergence is based on a known criterion in terms of the boundary spectrum.

1. Introduction and background

1.1. What this paper is about. Let H be a Hilbert space, and let $M_1, \ldots, M_N$ be closed subspaces of H. Denote by P_k the orthoprojection onto M_k, and let $T = P_1 \cdots P_N$. It was proved by von Neumann [29] for $N = 2$ and by Halperin [16] for any N that T^n converges strongly as $n \to \infty$ to the orthoprojection onto $M_1 \cap \cdots \cap M_N$. The same was proved by Lapidus [21] for $T = \sum_{k=1}^{N} \alpha_k P_k$ with $\alpha_k > 0$, $1 \le k \le N$, and $\sum_{k=1}^{N} \alpha_k = 1$. Some different proofs of these results were recently given in [20]. The von Neumann–Halperin and Lapidus theorems were generalized to uniformly convex Banach spaces by Bruck and Reich [7] and Reich [30], respectively. For a survey see [10, Chapter 9].

In the present paper we consider the situation when T is a convex combination of products of some projections in a complex Banach space. Assuming some concordance between its geometry (uniform convexity or uniform smoothness, or reflexivity) and a class of projections (orthoprojections, hermitian projections, etc.) we establish a spectral property of T which implies the strong convergence of T^n as $n \to \infty$. The necessary background is presented below.

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[21]

1.2. Spaces and operators. From now on we denote by X a complex Banach space and by $\mathcal{B}(X)$ the Banach algebra of bounded linear operators on X. The identity operator will be denoted by I.

Recall that a space X is said to be *uniformly convex* if for every $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that for any two vectors x and y with $\|x\| \leq 1$ and $\|y\| \leq 1$ the inequality $\|x+y\|/2 > 1-\delta$ implies $\|x-y\| < \varepsilon$. Accordingly, the nondecreasing function

$$\delta_X(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\}$$

is called the *modulus of convexity* of the space X. This classical definition, due to Clarkson [8], can be formally applied to all Banach spaces, so the uniformly convex spaces are just those which satisfy $\delta_X(\varepsilon) > 0$ for all ε. Every Hilbert space H is uniformly convex, its modulus of convexity is

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}.$$

For more information on the modulus of convexity see e.g. [5], [15] and the references therein.

A space X is called *uniformly smooth* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\|x+y\| + \|x-y\| < 2 + \varepsilon\|y\|$ holds for any two vectors x and y with $\|x\| = 1$ and $\|y\| \leq \delta$. A relevant modulus of smoothness was introduced by Day [9]. However, for the purposes of this paper we only need to know that all uniformly convex and all uniformly smooth spaces are reflexive and a space X is uniformly smooth if and only if its dual X^* is uniformly convex (see e.g. [23]).

Let H be a Hilbert space. An operator $T \in \mathcal{B}(H)$ is hermitian ($\equiv$ self-adjoint) if and only if $\|\exp(itT)\| = 1$ for all real t. In any Banach space X the latter property is a definition of a hermitian operator. (In [26] such operators were called *conservative*. This is just the case when T and $-T$ are *dissipative*, i.e. generate semigroups of contractions [25].)

Note that every real combination of pairwise commuting hermitian operators is hermitian as well. In particular, the operator $T - \alpha I$ is hermitian for any hermitian T and any real α.

For any operator $T \in \mathcal{B}(X)$ its spectrum is usually denoted by $\sigma(T)$. If T is hermitian then $\sigma(T) \subset \mathbb{R}$. If T is a contraction, i.e. $\|T\| \leq 1$, then $\sigma(T) \subset \overline{\mathbb{D}}$, where $\mathbb{D}$ is the open unit disk in the complex plane. The intersection of $\sigma(T)$ with the unit circle $\partial\mathbb{D}$ is called the *boundary spectrum* of the contraction T. Every point $\lambda \in \sigma(T) \cap \partial\mathbb{D}$ of the boundary spectrum is an approximate eigenvalue, i.e. there is a sequence of vectors x_k of norm 1 such that $Tx_k - \lambda x_k \to 0$. The boundary spectrum may be empty. This happens if and only if there is $n \geq 1$ such that T^n is a strict contraction (i.e. $\|T^n\| < 1$) or, equivalently, $\|T^n\| \to 0$ as $n \to \infty$.

1.3. Classes of contractions. A contraction is called *primitive* if its boundary spectrum is at most the singleton $\{1\}$. *If the space X is reflexive then the iterates T^n of any primitive contraction $T \in \mathcal{B}(X)$ are strongly convergent.* This fact is the key to all convergence problems studied in the present paper. Actually, it is a purely logical combination of two known general results:

1) If the space X is reflexive then every contraction T with at most countable boundary spectrum is almost periodic, i.e. all orbits $(T^n x)_{n \geq 0}$ are precompact [32].
2) In any Banach space the iterates of any primitive almost periodic contraction are strongly convergent [18]. (See also [27] for a general theory of almost periodic operator semigroups.)

An alternative proof (see Section 4 of the present paper) uses the Katznelson–Tzafriri theorem [19]: *in any Banach space*

$$\lim_{n \to \infty} \|T^n - T^{n+1}\| = 0$$

for every primitive contraction T.

Note that all the results stated above for contractions are automatically true for any power bounded operator $T \in \mathcal{B}(X)$ since T is a contraction in an equivalent norm on X. On the other hand, even the weak convergence of T^n implies the power boundedness of T.

The following geometric condition was introduced by Halperin in [16]:

(H)   there is $K \geq 0$ such that $\|x - Tx\|^2 \leq K(\|x\|^2 - \|Tx\|^2)$ $(x \in X)$.

Under this condition (the same as (K) in [13]), T is a contraction, and all strict contractions satisfy (H). We denote by $K(T)$ the smallest value of K. In particular, $K(I) = 0$.

Halperin proved that in a Hilbert space the iterates of every (H)-contraction are strongly convergent. In fact, this is true in any reflexive Banach space. Indeed, from (H) it follows that

(S) $\|x_k\| \leq 1$, $\|Tx_k\| \to 1$ $\Rightarrow$ $x_k - Tx_k \to 0$ strongly.

However, *every (S)-contraction is primitive.* Indeed, let $\|Tx_k - \lambda x_k\| \to 0$ for a $\lambda \in \partial\mathbb{D}$ and a sequence of normalized vectors x_k. Then $\|Tx_k\| \to 1$, hence $\|Tx_k - x_k\| \to 0$ by condition (S). Therefore, $\lambda = 1$. As a result, *the iterates of every (S)-contraction in a reflexive Banach space are strongly convergent.*

In Hilbert space this was proved in [3], where the condition (S) appears together with its weak version

(W) $\|x_k\| \leq 1$, $\|Tx_k\| \to 1$ $\Rightarrow$ $x_k - Tx_k \to 0$ weakly,

and the corresponding convergence result. The latter was extended to the reflexive Banach space in [12].

Obviously, the condition (W) implies

(W′) $\|Tx\| = \|x\| \;\Rightarrow\; Tx = x.$

Conversely, (W′) implies (W) if the space is Hilbert (see [3]) or, more generally, if it is a reflexive Banach space with a weakly sequentially continuous duality map (see [12]).

Note that for strict contractions conditions (S) and (W) are formally fulfilled but empty in content.

In [11] Dye proved that in a Hilbert space condition (H) is equivalent to

(D) $\exists r \in (0,1) : \|T - rI\| \le 1 - r.$

Obviously, under condition (D) the operator T is a contraction. Consequently, $\|T - rI\| \ge 1 - r$, so finally $\|T - rI\| = 1 - r$.

Every (D)-*contraction is primitive.* Indeed, if $\lambda \in \sigma(T)$, then $\lambda - r \in \sigma(T - rI)$, so $|\lambda - r| \le \|T - rI\| \le 1 - r$, whence $\lambda = 1$ for $|\lambda| = 1$. Thus, *the iterates of every* (D)-*contraction in a reflexive Banach space are strongly convergent.*

1.4. Projections. Recall that a linear operator $P \in \mathcal{B}(X)$ is called a *projection* if $P^2 = P$ or equivalently $\mathrm{Ker}(P) = \mathrm{Ran}(I - P)$. Obviously, $\|P\| \ge 1$ if $P \ne 0$. A projection P is called an *orthoprojection* if it is a contraction, i.e. $\|P\| = 1$ or $P = 0$. In Hilbert space this definition is equivalent to the standard one: the subspaces $\mathrm{Ker}(P)$ and $\mathrm{Ran}(P)$ are mutually orthogonal. Equivalently, this means that P is hermitian. *In any Banach space every hermitian projection is an orthoprojection.* Indeed, for any projection P we have

(1.1) $\exp(itP) = (I - P) + e^{it}P.$

Hence,

$$P = \frac{1}{2\tau} \int\limits_{-\tau}^{\tau} \exp(itP)e^{-it}\,dt,$$

which yields $\|P\| \le 1$ if P is hermitian. However, if P is a hermitian projection then so is $I - P$, while for orthoprojections this is not true in general. Another specific feature of the non-Hilbert situation is that for some subspaces orthoprojections do not exist. We refer the reader to [2] and [6] for more details and references.

For our purposes it is important to note that *all* (D)-*projections are orthoprojections.* Also note that *every hermitian projection P satisfies* (D) with $r = 1/2$, i.e. it is a *u-projection* in the sense of [14]. This immediately follows from (1.1) by taking $t = \pi$. Obviously, if P is a u-projection then so is $I - P$ and both are orthoprojections.

MAIN THEOREM. *Let $P_1, \ldots, P_N$ be some orthoprojections in a complex Banach space X, and let $\mathcal{S} = \mathcal{S}(P_1, \ldots, P_N)$ be the convex multiplicative semigroup generated by $P_1, \ldots, P_N$, i.e. the convex hull of the semigroup consisting of all products with factors from $\{P_1, \ldots, P_N\}$. Assume that one of the following conditions is satisfied:*

(i) *X is uniformly convex;*
(ii) *X is uniformly smooth;*
(iii) *X is reflexive and all P_k are of class* (D).

Then for every operator $T \in \mathcal{S}(P_1, \ldots, P_N)$ the iterates T^n converge strongly to an orthoprojection T^∞. In addition, if the P_k are of class (W$'$) *then*

$$(1.2) \qquad \operatorname{Ran}(T^\infty) = \bigcap_{k \in F_T} \operatorname{Ran}(P_k)$$

where F_T is the set of all indices k occurring in the decomposition of T as a member of $\mathcal{S}(P_1, \ldots, P_N)$. The formula (1.2) *is true in the class of all orthoprojections if the space X is uniformly convex or uniformly smooth and strictly convex.*

Recall that a Banach space is called *strictly convex* if all points of its unit sphere are extreme.

In the case (i) the strong convergence of T^n, where T is a product or convex combination of orthoprojections, was proved in [7] and in [30], respectively. The space X in these papers is real, but the results are automatically true for complex uniformly convex spaces by realification. On the other hand, there is an example of divergence in $\ell^\infty_{4,\mathbb{R}}$, i.e. in $\mathbb{R}^4$ endowed with the max-norm [7, p. 464]. Another related example is in [28]. In fact, there is an example even in $\ell^\infty_{2,\mathbb{R}}$, a fortiori, in $\ell^\infty_{2,\mathbb{C}}$. Namely, let

$$P_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$(P_1 P_2)^n = \begin{pmatrix} 0 & (-1)^{n+1} \\ 0 & (-1)^n \end{pmatrix},$$

so the iterates $(P_1 P_2)^n$ are divergent.

The space in our example is not strictly convex. An open question is about the existence of an example of divergence in a strictly convex space. Such a space would have to be infinite-dimensional since every finite-dimensional strictly convex space is uniformly convex.

For any Banach space X and its closed subspace M, we denote by $P_M(x)$, $x \in X$, the set of points in M whose distance to x is minimal. If X is reflexive then the set $P_M(x)$ is nonempty for every x. If, in addition, X is

strictly convex, then $P_M(x)$ is a singleton. In this situation $P_M(x)$ can be considered as a point in X and P_M as a mapping $X \to X$, a *nearest point projection* onto M. In general, this "projection" is nonlinear. However, in a Hilbert space, P_M coincides with the orthoprojection onto M.

For a strictly convex reflexive space X with $\dim X > 2$, Stiles proved in [31] that if $(P_M P_N)^n$ converges strongly to $P_{M \cap N}$ for every pair (M, N) of closed subspaces of X, then X is a Hilbert space. Thus, the von Neumann theorem cannot be extended to the nearest point projections in a non-Hilbert space. See however [30, Lemma 3.1] for a relation between linear nearest point projections and orthoprojections. This makes it possible to obtain a counterpart of the Main Theorem for linear nearest point projections. This observation was kindly communicated to us by S. Reich.

Note that the weak convergence of the iterates of a product or a convex combination of orthoprojections in a uniformly smooth space follows from [7] and [30] by duality.

1.5. Organization of the paper. The next section contains some information on the Apostol modulus $\varphi_T(\varepsilon)$ and its modification $\widetilde{\varphi}_T(\varepsilon)$ for a contraction T in a Banach space. In Section 3 we apply it to prove that the classes (H), (S) and (D) are multiplicative semigroups, furthermore, (S) and (D) are convex. This is an important ingredient of the proof of the Main Theorem. The latter is given in Section 4 after a proof of the convergence of the iterates of a primitive contraction in a reflexive Banach space. We conclude with an Appendix (Section 5) where we study some relations between the Apostol moduli and a geometric characteristic of the boundary spectrum. This yields a new look at a generalization of the Katznelson–Tzafriri theorem obtained by Allan and Ransford [1].

2. The Apostol modulus

2.1. Definitions and basic facts. For a contraction $T \in \mathcal{B}(X)$ we consider the *Apostol modulus*

$$\varphi_T(\varepsilon) = \sup\{\|x - Tx\| : \|x\| \leq 1, \, \|x\| - \|Tx\| \leq \varepsilon\}, \quad 0 < \varepsilon \leq 1.$$

This function was introduced and studied by Apostol [4] in the case of Hilbert space. For our purposes the following modification is convenient:

$$\widetilde{\varphi}_T(\varepsilon) = \sup\{\|x - Tx\| : \|x\| \leq 1, \, 1 - \|Tx\| \leq \varepsilon\}.$$

This definition is correct if and only if $\|T\| = 1$ since this is the only case when the set $\{x : \|x\| \leq 1, 1 - \|Tx\| \leq \varepsilon\}$ is nonempty for all ε. Thus, we will assume $\|T\| = 1$ anytime when dealing with $\widetilde{\varphi}_T(\varepsilon)$. On the other hand, in all further applications the case $\|T\| < 1$ is trivial.

Obviously, both functions $\varphi_T(\varepsilon)$ and $\widetilde{\varphi}_T(\varepsilon)$ are nondecreasing and

$$(2.1) \qquad 0 \le \widetilde{\varphi}_T(\varepsilon) \le \varphi_T(\varepsilon) \le \|I - T\| \le 2.$$

Actually, the most interesting information relates to their behavior as $\varepsilon \to 0$. Accordingly, we consider

$$\varphi_T^0 = \lim_{\varepsilon \to 0} \varphi_T(\varepsilon) = \inf_{\varepsilon > 0} \varphi_T(\varepsilon) \ge 0 \quad \text{and} \quad \widetilde{\varphi}_T^0 = \lim_{\varepsilon \to 0} \widetilde{\varphi}_T(\varepsilon) = \inf_{\varepsilon > 0} \widetilde{\varphi}_T(\varepsilon) \ge 0.$$

It turns out that these limit values coincide. In this sense the difference between the two versions of the Apostol modulus is not essential.

2.2. LEMMA. *If T a contraction of norm 1 and $T \ne I$ then $\varphi_T(\varepsilon) > 0$ for all ε and*

$$0 \le \varphi_T(\varepsilon) \le \widetilde{\varphi}_T \left(\frac{\|I - T\|\varepsilon}{\varphi_T(\varepsilon)} + 0 \right).$$

Proof. Assuming $\varphi_T(\varepsilon) = 0$ for an ε, we obtain $\|x - Tx\| = 0$ for all x with $\|x\| \le \varepsilon$, so $T = I$. Now let $T \ne I$. Take $q \in (0, 1)$ and find a vector x such that

$$\|x\| \le 1, \qquad \|x\| - \|Tx\| \le \varepsilon, \qquad \|x - Tx\| = q\theta$$

where $\theta = \varphi_T(\varepsilon) > 0$. Then for the normalized vector $z = x/\|x\|$ we have

$$1 - \|Tz\| \le \frac{\varepsilon}{\|x\|}, \qquad \|z - Tz\| = \frac{q\theta}{\|x\|} \ge q\theta,$$

whence

$$\widetilde{\varphi}_T \left(\frac{\varepsilon}{\|x\|} \right) \ge q\theta.$$

On the other hand,

$$\widetilde{\varphi}_T \left(\frac{\varepsilon}{\|x\|} \right) \le \widetilde{\varphi}_T \left(\frac{\|I - T\|\varepsilon}{q\theta} \right)$$

since $\|x\| \ge q\theta/\|I - T\|$. Thus,

$$q\theta \le \widetilde{\varphi}_T \left(\frac{\|I - T\|\varepsilon}{q\theta} \right).$$

It remains to substitute θ by $\varphi_T(\varepsilon)$ and pass to the limit as $q \to 1$. ∎

2.3. COROLLARY. *$\widetilde{\varphi}_T^0 = \varphi_T^0$ for all contractions T of norm 1.*

Proof. Since $\widetilde{\varphi}_I^0 = \varphi_I^0 = 0$, one can assume $T \ne I$ and apply Lemma 2.2. As $\varepsilon \to 0$ we get $\varphi_T^0 \le \widetilde{\varphi}_T^0$. The opposite inequality is trival. ∎

From now on we denote by ω_T the common value of φ_T^0 and $\widetilde{\varphi}_T^0$. For instance, $\omega_I = 0$. Accordingly, (2.1) can be extended to

$$(2.2) \qquad 0 \le \omega_T \le \widetilde{\varphi}_T(\varepsilon) \le \varphi_T(\varepsilon) \le \|I - T\| \le 2.$$

2.4. THEOREM. $\omega_T = 0$ *if and only if T is of class* (S).

Proof. "If". There is a sequence of vectors x_k such that $\|x_k\| \leq 1$, $1 - \|Tx_k\| \leq 1/k$ and $\widetilde{\varphi}_T(1/k) < 2\|x_k - Tx_k\|$. The latter norm tends to zero if T satisfies conditon (S).

"Only if". Let $\|x_k\| \leq 1$ and $\|Tx_k\| \to 1$. Without loss of generality one can assume $\|Tx_k\| < 1$, otherwise we change x_k to $q_k x_k$ where all $q_k \in (0,1)$ and $q_k \to 1$ as $k \to \infty$. Since $\omega_T = 0$ we have $\widetilde{\varphi}_T(1 - \|Tx_k\|) \to 0$, whence $\|x_k - Tx_k\| \to 0$ by the obvious inequality

$$\|x\| - \|Tx\| \leq \widetilde{\varphi}_T(1 - \|Tx\|) \quad (\|x\| \leq 1, \|Tx\| < 1). \quad \blacksquare$$

2.5. REMARK. Theorem 2.4 remains in force for $\|T\| < 1$ if we set $\omega_T = 0$ in this case. The latter definition is natural. Indeed, if $\|T\| < 1$ then

$$\varphi_T(\varepsilon) \leq \frac{\|I - T\|\varepsilon}{1 - \|T\|},$$

whence $\varphi_T^0 = 0$. (Recall that $\widetilde{\varphi}_T^0$ is not defined for $\|T\| < 1$.)

2.6. REMARK. Let T be an isometry. Then $\varphi_T(\varepsilon) = \|I - T\|$ for all ε, hence $\omega_T = \|I - T\|$, therefore $\omega_T > 0$ if $T \neq I$.

2.7. The Apostol modulus for orthoprojections. If P is an orthoprojection, so that $\|P\| \leq 1$, then

$$(2.3) \qquad \|Px\| = \frac{1}{2}\|P(x + Px)\| \leq \frac{1}{2}\|x + Px\| \leq \|x\|.$$

Now let $\|x\| \leq 1$, and let $1 - \|Px\| \leq \varepsilon$. Then $\|Px\| \leq 1$ and $\frac{1}{2}\|x + Px\| \geq 1 - \varepsilon$. Hence, $\|x - Px\| \leq \beta_X(\varepsilon)$ where

$$\beta_X(\varepsilon) = \sup\{\|x - y\| : \|x\| \leq 1, \|y\| \leq 1, \|x + y\|/2 \geq 1 - \varepsilon\}.$$

This results in the inequality

$$(2.4) \qquad\qquad \widetilde{\varphi}_P(\varepsilon) \leq \beta_X(\varepsilon).$$

The function β_X was introduced and investigated in [5]. It is closely related to the modulus of convexity. In particular, $\lim_{\varepsilon \to 0} \beta_X(\varepsilon) = 0$ if the space X is uniformly convex, otherwise this limit is the supremum of those ε for which $\delta_X(\varepsilon) = 0$. The latter quantity (or 0 if X is uniformly convex) is called the *characteristic of convexity* of the space X (see [15]).

2.8. PROPOSITION. *If P is an orthoprojection in a uniformly convex space then $\omega_P = 0$.*

Proof. This follows from (2.4) by passing to the limit as $\varepsilon \to 0$. $\quad \blacksquare$

2.9. COROLLARY. *Every orthoprojection in a uniformly convex space is of class* (S).

2.10. REMARK. This corollary can be obtained directly from (2.3). In this way Proposition 2.8 follows from Theorem 2.4.

The uniform convexity of X is not necessary for the existence of (S)-orthoprojections. For instance, if a projection P in X is such that $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in X$ (an *L-projection* [17]) then P is an orthoprojection and $\omega_P = 0$. Indeed, either $P = I$ or $\varphi_P(\varepsilon) = \varepsilon$ for all ε. In this situation X may not be uniformly convex. An example is $X = \ell^1$ where any coordinate projection is an *L-projection*.

2.11. REMARK. From (2.3) it follows that *every orthoprojection in a strictly convex space is of class* (W′).

3. Structure properties of the classes (H)**,** (S) **and** (D)**.** In this section we prove the following theorem.

3.1. THEOREM. *In any Banach space the sets of contractions of classes* (H)*,* (S) *and* (D) *are multiplicative semigroups. In addition, they are convex in the cases* (S) *and* (D)*.*

This theorem is an immediate consequence of the lemmas proven below.

3.2. LEMMA. *Let A and B be two contractions satisfying condition* (H)*. Then the product AB also satisfies* (H) *and*

$$K(AB) \leq 2\max(K(A), K(B)).$$

Proof. We have

$$\|x - ABx\|^2 \leq (\|x - Bx\| + \|Bx - ABx\|)^2 \leq 2(\|x - Bx\|^2 + \|Bx - ABx\|^2),$$

whence

$$\|x - ABx\|^2 \leq 2K(B)(\|x\|^2 - \|Bx\|^2) + 2K(A)(\|Bx\|^2 - \|ABx\|^2)$$
$$\leq 2\max(K(A), K(B))(\|x\|^2 - \|ABx\|^2). \ \blacksquare$$

Thus, *the set of* (H)*-contractions is a multiplicative semigroup.*

3.3. REMARK. If T is an (H)-contraction then

$$\varphi_T(\varepsilon) \leq \sqrt{2K(T)\varepsilon}.$$

Indeed, if $\|x\| \leq 1$ and $\|x\| - \|Tx\| \leq \varepsilon$, then

$$\|x - Tx\|^2 \leq K(T)(\|x\|^2 - \|Tx\|^2) \leq 2K(T)(\|x\| - \|Tx\|) \leq 2K(T)\varepsilon.$$

In particular, if P is an orthoprojection in a Hilbert space H then

$$\|x - Px\|^2 = \|x\|^2 - \|Px\|^2.$$

Thus, P satisfies (H) with constant $K(P) = 1$. Hence, $\varphi_P(\varepsilon) \leq \sqrt{2\varepsilon}$.

3.4. Lemma.

(i) *Let A and B be some contractions of norm 1. Then either $\|AB\| < 1$ or*

$$\widetilde{\varphi}_{AB}(\varepsilon) \leq \widetilde{\varphi}_A(\widetilde{\varphi}_B(\varepsilon) + \varepsilon) + \widetilde{\varphi}_B(\varepsilon).$$

(ii) *Let*

$$T = \sum_{k=1}^{N} \alpha_k A_k$$

be a convex combination of contractions A_k of norm 1, and let all $\alpha_k > 0$. Then either $\|T\| < 1$ or

$$\widetilde{\varphi}_T(\varepsilon) \leq \sum_{k=1}^{N} \alpha_k \widetilde{\varphi}_{A_k}(\alpha_k^{-1}\varepsilon).$$

Proof. (i) Let $\|AB\| = 1$. Then $\|A\| = \|B\| = 1$, so the functions $\widetilde{\varphi}_A, \widetilde{\varphi}_B$ are well defined along with $\widetilde{\varphi}_{AB}$. Take any vector x such that $\|x\| \leq 1$ and $1 - \|ABx\| \leq \varepsilon$. Then

$$\|x - ABx\| \leq \|x - Ax\| + \|Ax - ABx\| \leq \|x - Ax\| + \|x - Bx\|.$$

Thus,

$$\|x - ABx\| \leq \widetilde{\varphi}_A(1 - \|Ax\|) + \widetilde{\varphi}_B(1 - \|Bx\|).$$

Let us estimate $1 - \|Ax\|$ and $1 - \|Bx\|$. We have

$$1 - \|Bx\| \leq 1 - \|ABx\| \leq \varepsilon$$

and then

$$1 - \|Ax\| \leq 1 + \|A(x - Bx)\| - \|ABx\| \leq \|x - Bx\| + (1 - \|ABx\|).$$

Thus,

$$1 - \|Ax\| \leq \widetilde{\varphi}_B(\varepsilon) + \varepsilon.$$

As a result,

$$\|x - ABx\| \leq \widetilde{\varphi}_A(\widetilde{\varphi}_B(\varepsilon) + \varepsilon) + \widetilde{\varphi}_B(\varepsilon).$$

(ii) Let $\|T\| = 1$. Then all $\|A_k\|$ are 1, so the functions $\widetilde{\varphi}_{A_k}$ are well defined along with $\widetilde{\varphi}_T$. Take x such that $\|x\| \leq 1$, $1 - \|Tx\| \leq \varepsilon$, i.e.

$$1 - \left\|\sum_{k=1}^{N} \alpha_k A_k x\right\| \leq \varepsilon.$$

A fortiori,

$$\sum_{k=1}^{N} \alpha_k(1 - \|A_k x\|) \leq \varepsilon,$$

whence $1 - \|A_k x\| \le \alpha_k^{-1} \varepsilon$ for every k. Hence,

$$\|x - Tx\| \le \sum_{k=1}^{N} \alpha_k \|x - A_k x\| \le \sum_k \alpha_k \widetilde{\varphi}(\alpha_k^{-1} \varepsilon). \ \blacksquare$$

As a consequence, if $\omega_A = \omega_B = 0$ then ω_{AB} are 0, and if all $\omega_{A_k} = 0$ then $\omega_T = 0$. By Theorem 2.4, *the set of* (S)-*contractions is a convex multiplicative semigroup.*

Now for a contraction T we consider the set

$$R(T) = \{r \in (0,1) : \|T - rI\| \le 1 - r\}.$$

By definition, T is a (D)-contraction if and only if $R(T) \ne \emptyset$.

3.5. LEMMA. *For any contractions A and B, if $r \in R(A)$ and $s \in R(B)$ then $rs \in R(AB)$ and $\alpha r + \beta s \in R(\alpha A + \beta B)$ with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$.*

Proof. First, we have

$$\|AB - rsI\| = \|A(B - sI) + s(A - rI)\|$$
$$\le \|B - sI\| + s\|A - rI\| \le 1 - rs.$$

Secondly,

$$\|(\alpha A + \beta B) - (\alpha r + \beta s)I\| \le \alpha \|A - rI\| + \beta \|B - sI\|$$
$$\le \alpha(1 - r) + \beta(1 - s) = 1 - (\alpha r + \beta s). \ \blacksquare$$

Thus, *the set of* (D)-*contractions is a convex multiplicative semigroup.* The proof of Theorem 3.1 is complete.

4. Proof of the Main Theorem. The following general result is a key lemma in the proof of our Main Theorem.

4.1. THEOREM. *If X is a reflexive space and T is a primitive contraction in X then the iterates T^n converge strongly. The limit operator T^∞ coincides with the orthoprojection E_T onto the subspace $L = \text{Ker}(I - T)$ along the closure $M = \overline{\text{Ran}(I - T)}$. The convergence is uniform if and only if $\text{Ran}(I - T)$ is closed.*

Proof. By the classical ergodic theorem [24], the Cesàro means of $(T^n)_{n \ge 0}$ converge strongly to the projection E_T onto L along M. A part of this statement is that X is the direct sum $L \oplus M$. Let $x = u + v$ where $u \in L$, i.e. $Tu = u$, and $v \in M$, i.e. $v = \lim_{k \to \infty} (z_k - Tz_k)$ for a sequence $(z_k)_{k \ge 0}$. Given $\varepsilon > 0$, we fix k such that $\|v - (z_k - Tz_k)\| < \varepsilon$. Then $\|T^n v - (T^n - T^{n+1}) z_k\| < \varepsilon$ for all n. Hence, $\|T^n v\| < \varepsilon + \|T^n - T^{n+1}\| \|z_k\| < 2\varepsilon$ for large n by the Katznelson–Tzafriri theorem [19]. Thus, $\lim_{n \to \infty} T^n v = 0$. As a result, $\lim_{n \to \infty} T^n x = u = E_T x$, i.e. $T^\infty = E_T$. The latter is an orthoprojection since T is a contraction.

Now suppose that $\operatorname{Ran}(I - T)$ is closed, i.e. $M = \operatorname{Ran}(I - T)$. The operator $I - T$ acts bijectively on the invariant subspace M. Since M is closed, the inverse operator $S = ((I-T)|M)^{-1}$ is bounded. Since $(T|M)^n = (T^n - T^{n+1})S$, we obtain $\|(T|M)^n\| \to 0$ by the Katznelson–Tzafriri theorem again. Conversely, if T^n converges uniformly then the same is true for the Cesàro means, and then $\operatorname{Ran}(I - T)$ is closed [22]. ■

An alternative proof is merely a logical combination of two results proved in [32] and [18] as we indicated in the Introduction.

4.2. Proof of the Main Theorem. Let $T \in \mathcal{S}(P_1, \ldots, P_N)$ where $P_1, \ldots, P_N$ are some orthoprojections in a Banach space X. Obviously, T is a contraction. By Theorem 4.1 it suffices to show that T is primitive in all cases (i)–(iii). Recall that all contractions of classes (S) and (D) are primitive. (See Section 1.)

(i) The space X is uniformly convex. Then by Corollary 2.9 all P_k are of class (S). By Theorem 3.1 so is T. Therefore, T is primitive.

(ii) The space X is uniformly smooth. Then X^* is uniformly convex and $T^* \in \mathcal{S}(P_1^*, \ldots, P_N^*)$. All P_k^* are orthoprojections since $\|A^*\| = \|A\|$ for any operator A. Therefore, T^* is primitive like T in (i). Then T is also primitive since $\sigma(A) = \sigma(A^*)$ for any operator A and $T = T^{**}$ by reflexivity of X.

(iii) The space X is reflexive. Since all P_k are of class (D), such is also T by Theorem 3.1. Thus, T is primitive again.

To complete the proof of the Main Theorem we note that the subspace $\operatorname{Ran}(T^\infty)$ coincides with the subspace $\operatorname{Ker}(I-T)$ of fixed points of the operator T. Thus, it suffices to refer to the following lemma and Remark 2.11. ■

4.3. Lemma.

(i) *Let A and B be some (W')-contractions. Then*

$$\operatorname{Ker}(I - AB) = \operatorname{Ker}(I - A) \cap \operatorname{Ker}(I - B).$$

(ii) *Let $T = \sum_{k=1}^{N} \alpha_k A_k$ be a convex combination of N (W')-contractions with all $\alpha_k > 0$. Then*

$$\operatorname{Ker}(I - T) = \bigcap_k \operatorname{Ker}(I - A_k).$$

Proof. In both cases the inclusion of the right-hand side into the left-hand side is trivial. The proofs of the converse inclusions are as follows.

(i) For $x \in \operatorname{Ker}(I - AB)$ we have

$$\|x\| = \|ABx\| \le \|Bx\| \le \|x\|.$$

Therefore, $\|Bx\| = \|x\|$, whence $Bx = x$ and then $Ax = x$ by condition (W').

(ii) For $x \in \mathrm{Ker}(I - T)$ we have

$$\|x\| \leq \sum_{k=1}^{N} \alpha_k \|A_k x\| \leq \sum_{k=1}^{N} \alpha_k \|x\| = \|x\|.$$

Thus, $\|A_k x\| = \|x\|$, hence $A_k x = x$ for every k. ∎

4.4. REMARK. The same argument shows that *the contractions of class* (W′) *constitute a convex multiplicative semigroup.*

5. Appendix: The amplitude of the boundary spectrum. Let T be a contraction in a Banach space X, and let the boundary spectrum of T be nonempty. We call the quantity

$$a_T = \max\{|\lambda - 1| : \lambda \in \sigma(T), |\lambda| = 1\}$$

the *amplitude of the boundary spectrum* of T. Obviously, $0 \leq a_T \leq 2$, and $a_T = 0$ if and only if the contraction T is primitive. In view of Theorem 2.4, the fact that (S)-contractions are primitive is a particular case of the following inequality.

5.1. PROPOSITION. $a_T \leq \omega_T$.

Proof. Let $\lambda \in \sigma(T)$ with $|\lambda| = 1$. Then for every $\varepsilon > 0$ there exists a vector x of norm 1 such that $\|Tx - \lambda x\| \leq \varepsilon$. Hence, $1 - \|Tx\| \leq \varepsilon$ and

$$|\lambda - 1| \leq \|x - Tx\| + \|Tx - \lambda x\| \leq \widetilde{\varphi}_T(\varepsilon) + \varepsilon.$$

The result follows as $\varepsilon \to 0$. ∎

5.2. COROLLARY. *If $a_T = 2$ then $\omega_T = 2$ and $\widetilde{\varphi}_T(\varepsilon) = \varphi_T(\varepsilon) = 2$ for all ε. Also, $\|I - T\| = 2$ in this case.*

Proof. We have $\omega_T \geq 2$. Now everything follows from (2.2). ∎

Obviously, $a_T = 2$ if and only if $-1 \in \sigma(T)$. Therefore, *if $-1 \in \sigma(T)$ then $\omega_T = 2$.*

5.3. PROPOSITION. *If the space X is uniformly convex and $\omega_T = 2$ then $a_T = 2$.*

Proof. We have $\widetilde{\varphi}_T(\varepsilon) = 2$ for every $\varepsilon \in (0, 1)$. By definition, there is a vector $x = x(\varepsilon)$ of norm 1 such that $\|x - Tx\| \geq 2 - 2\varepsilon$. Hence, $\|x + Tx\| \leq \beta_X(\varepsilon)$ where β_X is defined in Section 2. Since X is uniformly convex, we have $\lim_{\varepsilon \to 0} \beta_X(\varepsilon) = 0$. A fortiori, $\lim_{\varepsilon \to 0} \|x(\varepsilon) + Tx(\varepsilon)\| = 0$. This means that $-1 \in \sigma(T)$. ∎

The amplitude a_T is the maximal deviation of the boundary spectrum of T from the point 1 in the metric of the complex plane. Alternatively, one can use the metric of the unit circle. This "intrinsic" amplitude is

$$\tau_T = 2 \arcsin(a_T/2).$$

In [1] Allan and Ransford obtained the following quantitative version of the Katznelson–Tzafriri theorem:

$$\limsup_{n\to\infty} \|T^n - T^{n+1}\| \le 2\tan(\tau_T/2), \quad \tau_T < \pi.$$

In terms of the amplitude a_T this means that

$$\limsup_{n\to\infty} \|T^n - T^{n+1}\| \le \frac{2a_T}{\sqrt{4 - a_T^2}}, \quad a_T < 2.$$

Combining this result with Proposition 5.1 we obtain

5.4. THEOREM. *Let T be a contraction acting on the complex Banach space X. If $\omega_T < 2$ then*

$$\limsup_{n\to\infty} \|T^n - T^{n+1}\| \le \frac{2\omega_T}{\sqrt{4 - \omega_T^2}}.$$

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