STUDIA MATHEMATICA 208 (1) (2012)

Invariant measures for position dependent random maps with continuous random parameters

by

Tomoki Inoue (Matsuyama)

Abstract. We consider a family of transformations with a random parameter and study a random dynamical system in which one transformation is randomly selected from the family and applied on each iteration. The parameter space may be of cardinality continuum. Further, the selection of the transformation need not be independent of the position in the state space. We show the existence of absolutely continuous invariant measures for random maps on an interval under some conditions.

1. Introduction. We consider a family of transformations $\tau_t : X \to X$ ($t \in W$) and study a random dynamical system such that one transformation is randomly selected from the family $\{\tau_t : t \in W\}$ and then applied on each iteration.

In many papers on random maps, for example [Ba-G, G-Bo, P], the number of elements of W (the parameter space) is finite. In this paper, however, W may even be of cardinality continuum.

Further, in many papers on random maps, for example [M, P], the selection of τ_t is independent of the position $x \in X$. In this paper, τ_t is selected according to a probability density function $p(t, x)$.

The maps τ_t we mainly consider in this paper are piecewise monotone transformations on an interval. A simple example to which our theory can be applied is the following family of transformations:

Example 1.1. Define $\tau_t : [0, 1] \to [0, 1]$ ($t \in [1/2, 2]$) by

$$\tau_t(x) = \begin{cases} tx, & x \in [0, 1/2), \\ 2x - 1, & x \in [1/2, 1], \end{cases}$$

where t is randomly selected from $W = [1/2, 2]$ according to the probability density function $p(t, x) = 2/3$ for each $x \in [0, 1]$.

2010 *Mathematics Subject Classification*: Primary 37E05; Secondary 37A05.
Key words and phrases: random map, absolutely continuous invariant measure, Perron–Frobenius operator.

DOI: 10.4064/sm208-1-2

As in this example, the random map $T = \{\tau_t, p(t,x) : t \in W\}$ is determined by a family of transformations $\{\tau_t : t \in W\}$ and a probability density function $p(t,x)$.

We note that τ_t may not be expanding. In Example 1.1, τ_t is not expanding for $t \in [1/2, 1)$.

Further, in Example 1.1, the partition $\{[0,1/2),[1/2,1]\}$ does not vary with the random parameter t. But we do not assume such a condition in our main theorems. The partition of $[0,1]$ may vary with t, as in the following family of transformations:

Example 1.2. Define $\tau_t : [0,1] \to [0,1]$ by

$$\tau_t(x) = tx \quad \text{for } t \in (0,1]$$

and

$$\tau_t(x) = \begin{cases} tx, & x \in [0,1/t), \\ tx-1, & x \in [1/t,1], \end{cases} \quad \text{for } t \in (1,2),$$

where t is randomly selected from $W = (0,2)$ according to the probability density function $p(t,x) = \frac{3}{8}t^2$ for each $x \in [0,1]$.

If we replace $p(t,x) = \frac{3}{8}t^2$ by $p(t,x) = \frac{3}{8}t^2 x + \frac{1}{4}t^3(1-x)$, then $p(t,x)$ really depends on the position $x \in [0,1]$.

In Example 1.2, we note that $\tau_t'(x) \in (0,\epsilon)$ if $t \in (0,\epsilon)$ for $\epsilon > 0$. We can apply our theorems to such a random map if we choose a suitable probability density function $p(t,x)$.

In this paper, we prove the existence of an absolutely continuous invariant probability measure for a random map on an interval under some conditions. Of course, the previous examples of random maps satisfy the conditions which we will state in Section 5.

Our result (Theorem 5.2) is a generalization of the well known result of [R] (Theorem 1.3 below) as well as of the famous result of Lasota and Yorke [L-Y] for deterministic dynamical systems.

In Theorem 1.3, we assume that $\{I_i\}$ is a countable family of closed intervals with disjoint interiors and that $\mathrm{Leb}([0,1] \setminus \bigcup_i I_i) = 0$. For $\tau : [0,1] \to [0,1]$, we assume that the restriction of τ to $\mathrm{int}(I_i)$ is a C^1 and monotone function. (In this paper $\mathrm{int}(I)$ stands for the interior of an interval I.) We define

$$g(x) = \begin{cases} 1/|\tau'(x)|, & x \in \bigcup_i \mathrm{int}(I_i), \\ 0, & x \in [0,1] \setminus \bigcup_i \mathrm{int}(I_i). \end{cases}$$

The following theorem is well known:

Theorem 1.3 ([Bo-G, R]). *Let $\tau : [0,1] \to [0,1]$ be as above. Suppose that:*

(a^1) $\inf_x |\tau'(x)| > 1$ *wherever $\tau'(x)$ is defined;*

(b^1) *$g(x)$ is of bounded variation.*

Then τ has an invariant probability measure which is absolutely continuous with respect to Lebesgue measure.

Our result concerns position dependent random maps, and is a generalization of the result of [Ba-G] and [G-Bo] (Theorem 1.4 below), hence also a generalization of Theorem 1 in [P] for position independent random maps.

In Theorem 1.4, we assume that $\{I_i\}$ is a finite family of closed intervals with disjoint interiors and $\mathrm{Leb}([0,1] \setminus \bigcup_i I_i) = 0$. For $\tau_k : [0,1] \to [0,1]$ ($k = 1, \ldots, K$), we assume that the restriction of τ_k to $\mathrm{int}(I_i)$ is a C^1 and monotone function. Moreover, we assume that $\{\tilde{p}_k(x)\}_{k=1}^K$ is a set of position dependent measurable probabilities, that is, $\sum_{k=1}^K \tilde{p}_k(x) = 1$ and $\tilde{p}_k(x) \geq 0$ for $k = 1, \ldots, K$. We define

$$\tilde{g}_k(x) = \begin{cases} \tilde{p}_k(x)/|\tau_k'(x)|, & x \in \bigcup_i \mathrm{int}(I_i), \\ 0, & x \in [0,1] \setminus \bigcup_i \mathrm{int}(I_i). \end{cases}$$

In the position dependent random map $T = \{\tau_k, \tilde{p}_k(x)\}$, τ_k is selected with probability $\tilde{p}_k(x)$.

The following theorem is well known:

THEOREM 1.4 ([Ba-G]). *For $k = 1, \ldots, K$, let $\tau_k : [0,1] \to [0,1]$ be as above. Suppose that the random map $T = \{\tau_k, \tilde{p}_k(x)\}$ satisfies the following conditions:*

(a^2) *$\sup_x \sum_{k=1}^K \tilde{g}_k(x) < 1$;*
(b^2) *$\tilde{g}_k(x)$ is of bounded variation for each k.*

Then the random map $T = \{\tau_k, \tilde{p}_k(x)\}$ has an invariant probability measure which is absolutely continuous with respect to Lebesgue measure.

The definitions of the random map $T = \{\tau_k, \tilde{p}_k(x)\}$ and its invariant measure are found in Section 2 as well as in [Ba-G].

The paper is organized as follows: In Section 2, we formulate the definition of a random map T as a Markov process. Further, we introduce the corresponding Perron–Frobenius operator P_T. In Section 3, we state some basic assumptions and give a representation of P_T under those assumptions. In Section 4, we prove some basic properties of P_T. In Section 5, we give our main theorem on the existence of an absolutely continuous invariant measure for T and give a key inequality which implies the quasi-compactness of P_T as well as our main theorems. In Section 6, we give some basic estimates to prove the key inequality. In Section 7, we prove the key inequality.

2. Random maps and invariant measures. In this section, first, we define random maps in a general setting. Later, we will define invariant measures for random maps.

Let $(W, \mathcal{B}, \nu)$ be a σ-finite measure space. We use W as a parameter space. Let $(X, \mathcal{A}, m)$ be a σ-finite measure space. We use X as a state space. Let $\tau_t : X \to X$ $(t \in W)$ be a nonsingular transformation, which means that $m(\tau_t^{-1} A) = 0$ if $m(A) = 0$ for any $A \in \mathcal{A}$. Assume that $\tau_t(x)$ is a measurable function of t for each $x \in X$.

Let $p : W \times X \to [0, \infty)$ be a measurable function which is a probability density function of $t \in W$ for each $x \in X$, that is, $\int_W p(t,x)\,\nu(dt) = 1$ for $x \in X$. We sometimes write $p_t(x)$ instead of $p(t,x)$.

The random map $T = \{\tau_t,\, p_t(x) : t \in W\}$ is defined as a Markov process with the transition function

$$\mathbf{P}(x, A) := \int_W p(t,x)\, 1_A(\tau_t(x))\, \nu(dt), \tag{2.1}$$

where $A \in \mathcal{A}$ and 1_A is the indicator function of A. The transition function $\mathbf{P}$ induces an operator $\mathbf{P}_*$ on measures on X defined by

$$\begin{aligned} \mathbf{P}_*\mu(A) &:= \int_X \mathbf{P}(x, A)\, \mu(dx) \\ &= \int_X \int_W p(t,x)\, 1_A(\tau_t(x))\, \nu(dt)\, \mu(dx) \qquad \text{for } A \in \mathcal{A}. \end{aligned}$$

If $\mathbf{P}_*\mu = \mu$, then μ is called an *invariant measure* for $T = \{\tau_t,\, p(t,x) : t \in W\}$.

Remark 2.1. If W is a finite set $\{1, \dots, K\}$, then (2.1) can be represented by

$$\mathbf{P}(x, A) = \sum_{k=1}^{K} \tilde{p}_k(x)\, 1_A(\tau_k(x)),$$

where $\tilde{p}_k(x) = p(k,x)\nu(\{k\})$. In this case $\mathbf{P}_*\mu = \mu$ means

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} \tilde{p}_k(x)\, \mu(dx) \qquad \text{for } A \in \mathcal{A}.$$

This is the definition of an invariant measure for $T = \{\tau_k, \tilde{p}_k(x)\}$ in Theorem 1.4. In addition, if $\tilde{p}_k(x) = \hat{p}_k$ (constant), then $\mathbf{P}_*\mu = \mu$ means $\mu(A) = \sum_{k=1}^{K} \hat{p}_k \mu(\tau_k^{-1}(A))$ for $A \in \mathcal{A}$. Of course, if W consists of only one element, then $\mathbf{P}_*\mu = \mu$ is the usual definition of an invariant measure for a deterministic transformation.

If μ has a density f, then $\mathbf{P}_*\mu$ has a density, which we denote $P_T f$. So, $P_T : L^1(m) \to L^1(m)$ is the operator satisfying

$$(2.2)\qquad \mathbf{P}_*\mu(A) = \int_A P_T f(x)\, m(dx) = \int_X \int_W p(t,x)\, 1_A(\tau_t(x))\, \nu(dt)\, f(x)\, m(dx) \quad \text{for } A \in \mathcal{A}.$$

REMARK 2.2. Assume that W consists of only one element t. Then T is a deterministic transformation, that is, $T = \tau_t$. In this situation, if μ has a density f, then

$$\mathbf{P}_*\mu(A) = \int_A P_T f(x)\, m(dx) = \int_{T^{-1}A} f(x)\, m(dx) \quad \text{for } A \in \mathcal{A},$$

which means that P_T is the Perron–Frobenius operator corresponding to $T = \tau_t$ and m. (For the Perron–Frobenius operator, [Bo-G] and [L-M] are good references.)

By this remark, for a random map $T = \{\tau_t,\, p(t,x) : t \in W\}$, it is natural to call the operator $P_T : L^1(m) \to L^1(m)$ defined by (2.2) the *Perron–Frobenius operator* corresponding to T and m. Let $P_{\tau_t} : L^1(m) \to L^1(m)$ be the Perron–Frobenius operator corresponding to τ_t and m. Then the Fubini theorem implies that

$$(2.3)\qquad (P_T f)(x) = \int_W P_{\tau_t}(p_t f)(x)\, \nu(dt).$$

We will use this equality later.

3. One-dimensional random maps. From now on, let $X = [0,1]$ and let m be the Lebesgue measure. The other symbols are as in the previous section. So, τ_t is a map from $[0,1]$ into itself for each $t \in W$, $\tau_t(x)$ is a measurable function of t for each $x \in [0,1]$, and $p : W \times X \to [0,\infty)$ is a measurable function such that $\int_W p(t,x)\ \nu(dt) = 1$ for $x \in [0,1]$.

Let Λ be a countable or finite set and let $\Lambda_t \subseteq \Lambda$ for each $t \in W$. We use Λ as a set of indices of subintervals of $[0,1]$. For each $t \in W$, we assume that $\{I_{t,i}\}_{i\in\Lambda_t}$ is a family of closed intervals such that $\mathrm{int}(I_{t,i}) \cap \mathrm{int}(I_{t,j}) = \emptyset$ $(i \neq j)$ and $m([0,1] \setminus \bigcup_{i\in\Lambda_t} I_{t,i}) = 0$.

REMARK 3.1. In Example 1.2, we may consider $\Lambda = \{1,2\}$,

$$\Lambda_t = \{1,2\}, \quad I_{t,1} = [0,1/t] \quad \text{and} \quad I_{t,2} = [1/t,1] \quad \text{for } t \in (1,2),$$
$$\Lambda_t = \{1\} \quad \text{and} \quad I_{t,1} = [0,1] \quad \text{for } t \in (0,1].$$

To avoid using the subscript t on Λ_t, we set $I_{t,i} = \emptyset$ for $i \in \Lambda \setminus \Lambda_t$. For convenience, we consider the empty set as a closed interval.

For a random map $T = \{\tau_t, p(t,x), \{I_{t,i}\}_{i\in\Lambda} : t \in W\}$, we make two basic assumptions. The first is:

(A1) The restriction of τ_t to $\mathrm{int}(I_{t,i})$ is a C^1 and monotone function for each $i \in \Lambda$ and $t \in W$.

Let $\tau_{t,i}$ be the restriction of τ_t to $\mathrm{int}(I_{t,i})$ for each $t \in W$ and $i \in \varLambda$. Put

$$\phi_{t,i}(x) := \begin{cases} \tau_{t,i}^{-1}(x), & x \in \tau_{t,i}(\mathrm{int}(I_{t,i})), \\ 0, & x \in [0,1] \setminus \tau_{t,i}(\mathrm{int}(I_{t,i})), \end{cases}$$

for each $t \in W$ and $i \in \varLambda$. We note that $\phi_{t,i}(x) = 0$ if $i \in \varLambda \setminus \varLambda_t$.

The second assumption is:

(A2) For each $x \in X$ and $i \in \varLambda$, $w_{x,i}(t) := \phi_{t,i}(x)$ is a measurable function of t.

Let $f \in L^1(X, m)$. Under assumptions (A1) and (A2), we are going to find a representation of $P_T f$.

Put

$$\phi_{t,i}^*(x) := \begin{cases} \phi_{t,i}'(x), & x \in \tau_{t,i}(\mathrm{int}(I_{t,i})), \\ 0, & x \in [0,1] \setminus \tau_{t,i}(\mathrm{int}(I_{t,i})). \end{cases}$$

By a change of variable we obtain

$$\begin{aligned} \text{(3.1)} \qquad \mathbf{P}_*\mu(A) &= \int_A P_T f(x)\, m(dx) \\ &= \int_W \int_X p(t,x)\, 1_A(\tau_t(x)) f(x)\, m(dx)\, \nu(dt) \\ &= \int_W \int_A \sum_{i \in \varLambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) \phi_{t,i}^*(x)\, m(dx)\, \nu(dt) \qquad \text{for } A \in \mathcal{A}. \end{aligned}$$

Remark 3.2. Assumption (A2) ensures that $p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) \phi_{t,i}^*(x)$ is a measurable function of t for each x and i.

Since the equality (3.1) holds for any measurable set A, we obtain

$$(P_T f)(x) = \int_W \sum_{i \in \varLambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) |\phi_{t,i}^*(x)|\, \nu(dt) \tag{3.2}$$

for m-a.e. x, or

$$(P_T f)(x) = \int_W \sum_{i \in \varLambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) |\phi_{t,i}'(x)|\, 1_{\tau_t(\mathrm{int}(I_{t,i}))}(x)\, \nu(dt) \tag{3.3}$$

for m-a.e. x.

Remark 3.3. We note that

$$\sum_{i \in \varLambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) |\phi_{t,i}'(x)|\, 1_{\tau_t(\mathrm{int}(I_{t,i}))}(x) = P_{\tau_t}(p_t f)(x) \tag{3.4}$$

for m-a.e. x, where P_{τ_t} is the Perron–Frobenius operator for τ_t.

4. Properties of the Perron–Frobenius operator. In this section, we summarize the properties of the Perron–Frobenius operator P_T corresponding to a random map T. The following lemma gives the basic properties of P_T.

Lemma 4.1. *Let $T = \{\tau_t, p(t,x) : t \in W\}$ be a random map defined in Section 2, let $P_T : L^1(m) \to L^1(m)$ be the corresponding Perron–Frobenius operator, and let $f \in L^1(m)$. Then*

(i) *P_T is linear;*
(ii) *$P_T f \geq 0$ if $f \geq 0$;*
(iii) *$\int_X P_T f\, dm = \int_X f\, dm$;*
(iv) *$\|P_T f\|_{L^1(m)} \leq \|f\|_{L^1(m)}$.*

The proof is analogous to the proof for a deterministic transformation.

The following lemma is important in proving the main result. It follows from (2.2).

Lemma 4.2. *Let $T = \{\tau_t, p(t,x) : t \in W\}$ be a random map defined in Section 2, let $P_T : L^1(m) \to L^1(m)$ be the corresponding Perron–Frobenius operator, and let f be a probability density function on the measure space $(X, \mathcal{A}, m)$. Set $\mu(A) = \int_A f(x)\, m(dx)$ for $A \in \mathcal{A}$. Then $P_T f = f$ m-a.e. if and only if μ is an invariant probability measure for T.*

Now, we consider the Perron–Frobenius operator corresponding to the composition of random maps.

Lemma 4.3. *Let $T = \{\tau_t, p(t,x), \{I_{t,i}\}_{i \in \Lambda(T)} : t \in W\}$ and $S = \{\varsigma_s, q(s,x), \{\tilde{I}_{s,j}\}_{j \in \Lambda(S)} : s \in W\}$ be random maps as in Section 3, and let $P_T : L^1(m) \to L^1(m)$ be the corresponding Perron–Frobenius operator. Then*

$$P_{T \circ S} = P_T P_S,$$

where

$$T \circ S = \{\tau_t \circ \varsigma_s, p(t, \varsigma_s(x)) q(s,x), \{\mathrm{cl}(\psi_{s,j}(\mathrm{int}(I_{t,i}) \cap \varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))))\}_{(i,j) \in \Lambda(T) \times \Lambda(S)} : t, s \in W\}$$

and $\psi_{s,j} = \varsigma_{s,j}^{-1}$. In particular, $P_{T^K} = P_T^K$.

Proof. Let $f \in L^1(m)$. Put $\phi_{t,i} = \tau_{t,i}^{-1}$.By (3.3) we have

$$P_T(P_S f)(x) = \int_W \sum_{i \in \Lambda(T)} p(t, \phi_{t,i}(x)) (P_S f)(\phi_{t,i}(x)) \cdot |\phi'_{t,i}(x)|\, 1_{\tau_t(\mathrm{int}(I_{t,i}))}(x)\, \nu(dt)$$

and

$$(P_S f)(\phi_{t,i}(x)) = \int_W \sum_{j \in \Lambda(S)} q(s, \psi_{s,j}(\phi_{t,i}(x))) f(\psi_{s,j}(\phi_{t,i}(x))) \cdot |\psi'_{s,j}(\phi_{t,i}(x))|\, 1_{\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))}(\phi_{t,i}(x))\, \nu(ds).$$

Thus,

$$
\begin{aligned}
(4.1)\qquad & P_T(P_S f)(x)\\
&= \int_W \sum_{i\in\Lambda(T)} p(t,\phi_{t,i}(x)) \int_W \sum_{j\in\Lambda(S)} q(s,\psi_{s,j}(\phi_{t,i}(x)))f(\psi_{s,j}(\phi_{t,i}(x)))\\
&\quad \cdot |\psi'_{s,j}(\phi_{t,i}(x))|\, 1_{\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))}(\phi_{t,i}(x))\,\nu(ds)|\phi'_{t,i}(x)|\, 1_{\tau_t(\mathrm{int}(I_{t,i}))}(x)\nu(dt)\\
&= \int_W \int_W \sum_{i\in\Lambda(T)} \sum_{j\in\Lambda(S)} p(t,\phi_{t,i}(x))q(s,\psi_{s,j}(\phi_{t,i}(x)))f(\psi_{s,j}(\phi_{t,i}(x)))\\
&\quad \cdot |(\psi_{s,j}\circ\phi_{t,i})'(x)|\, 1_{\tau_t(\mathrm{int}(I_{t,i}))}(x)\, 1_{\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))}(\phi_{t,i}(x))\,\nu(ds)\,\nu(dt).
\end{aligned}
$$

On the other hand, in a way similar to obtaining (3.2), we have

$$
\begin{aligned}
(4.2)\qquad & P_{T\circ S} f(x)\\
&= \int_W \int_W \sum_{i\in\Lambda(T)} \sum_{j\in\Lambda(S)} p(t,\phi_{t,i}(x))q(s,\psi_{s,j}(\phi_{t,i}(x)))f(\psi_{s,j}(\phi_{t,i}(x)))\\
&\quad \cdot |(\psi_{s,j}\circ\phi_{t,i})'(x)|\, 1_{\tau_t(\varsigma_s(\psi_{s,j}(\mathrm{int}(I_{t,i})\cap\varsigma_s(\mathrm{int}(\tilde{I}_{s,j})))))}(x)\,\nu(ds)\,\nu(dt).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\tau_t(\varsigma_s(\psi_{s,j}(\mathrm{int}(I_{t,i})\cap\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))))) = \tau_t(\mathrm{int}(I_{t,i})\cap\varsigma_s(\mathrm{int}(\tilde{I}_{s,j})))\\
&\quad = \phi_{t,i}^{-1}(\mathrm{int}(I_{t,i})\cap\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))) \;=\; \phi_{t,i}^{-1}(\mathrm{int}(I_{t,i}))\cap\phi_{t,i}^{-1}(\varsigma_s(\mathrm{int}(\tilde{I}_{s,j})))\\
&\quad = \tau_t(\mathrm{int}(I_{t,i}))\cap\phi_{t,i}^{-1}(\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))),
\end{aligned}
$$

it follows from (4.2) that

$$
\begin{aligned}
(4.3)\qquad & P_{T\circ S} f(x)\\
&= \int_W \int_W \sum_{i\in\Lambda(T)} \sum_{j\in\Lambda(S)} p(t,\phi_{t,i}(x))q(s,\psi_{s,j}(\phi_{t,i}(x)))f(\psi_{s,j}(\phi_{t,i}(x)))\\
&\quad \cdot |(\psi_{s,j}\circ\phi_{t,i})'(x)|\, 1_{\tau_t(\mathrm{int}(I_{t,i}))}(x)\, 1_{\varsigma_s(\mathrm{int}(\tilde{I}_{s,j}))}(\phi_{t,i}(x))\,\nu(ds)\,\nu(dt).
\end{aligned}
$$

Therefore, by (4.1) and (4.3) we obtain the lemma. ∎

5. Existence of an absolutely continuous invariant measure. In the setting of Section 3, we give a sufficient condition for the existence of an absolutely continuous invariant measure for a random map $T = \{\tau_t, p(t,x), \{I_{t,i}\}_{i\in\Lambda} : t\in W\}$.

For $t\in W$ and $x\in[0,1]$, put

$$
g(t,x) = \begin{cases} p(t,x)/|\tau_t'(x)|, & x\in\bigcup_i \mathrm{int}(I_{t,i}),\\ 0, & x\in[0,1]\setminus\bigcup_i \mathrm{int}(I_{t,i}). \end{cases}
$$

We denote by $\bigvee_I f$ the total variation of f on I. Further, we assume the following conditions:

(a) $\sup_{x\in[0,1]} \int_W g(t,x)\,\nu(dt) < \alpha < 1$;

(b) There exists a constant M such that $\bigvee_{[0,1]} g(t,\cdot) < M$ for a.s. $t \in W$, that is, there exists a ν-measurable set $W_0 \subset W$ such that $\int_{W_0} p(t,x)\,\nu(dt) = 1$ and $\bigvee_{[0,1]} g(t,\cdot) < M$ for all $t \in W_0$.

Remark 5.1. If $\inf_{x\in[0,1]} |\tau_t'(x)| > 1$, condition (a) is automatically satisfied. Even if $\inf_{x\in[0,1]} |\tau_t'(x)| < 1$, condition (a) is satisfied by choosing a suitable probability density function $p(t,x)$. For example, in Examples 1.1 and 1.2, condition (a) is satisfied, while $\inf_{x\in[0,1]} |\tau_t'(x)| < 1$. Moreover, we allow $\inf_{t\in W} \sup_{x\in[0,1]} |\tau_t'(x)| = 0$ if we choose a suitable probability density function $p(t,x)$. See Example 1.2.

Now we give our main theorem.

Theorem 5.2. *Let $T = \{\tau_t,\, p(t,x) : t \in W\}$ be a random map as in Section* 3. *Assume that the random map T satisfies conditions* (a) *and* (b) *above. Then T has an invariant probability measure which is absolutely continuous with respect to Lebesgue measure.*

Remark 5.3. We have assumed that T satisfies conditions (a) and (b). However, it is enough to assume that some iterate T^n satisfies conditions corresponding to (a) and (b).

Remark 5.4. If W consists of only one element, conditions (a) and (b) coincide with (a^1) and (b^1) of Theorem 1.3. Hence, Theorem 1.3 is a corollary of Theorem 5.2.

Remark 5.5. Assume that W is a finite set $\{1,\ldots,K\}$ and $\{I_i\} = \{I_{t,i}\}$ for $t \in W$. Set $\tilde p_k(x) = p_k(x)\nu(\{k\})$. Then conditions (a) and (b) imply (a^2) and (b^2) of Theorem 1.4. Hence, Theorem 1.4 is a corollary of Theorem 5.2.

Theorem 5.2 can be obtained from Lemma 4.2 if we show the existence of a fixed point of the Perron–Frobenius operator P_T.

Let K be a constant. To show the existence of a fixed point of P_T, we will prove that for any function $f \ge 0$ on $[0,1]$ of bounded variation with $\int_0^1 f(x)\,m(dx) = 1$,

$$\bigvee_{[0,1]} P_T^K f < \gamma \bigvee_{[0,1]} f + \beta, \tag{5.1}$$

where $0 < \gamma < 1$ and $\beta > 0$. Using a standard technique of [Bo-G] or [L-M], the existence of a fixed point of P_T follows from (5.1). Furthermore, P_T can be shown to be quasi-compact and constrictive (see [Bo-G] and [L-M]). So, we can also obtain the following theorem.

Theorem 5.6. *Let $T = \{\tau_t; p(t,x) : t \in W\}$ be a random map as in Theorem* 5.2 *and let $P_T : L^1(m) \to L^1(m)$ be the corresponding Perron–Frobenius operator. Then there exists a positive integer r, a sequence of* *probability density functions $f_1, \dots, f_r$, a sequence of bounded linear functionals $\eta_1, \dots, \eta_r$ and an operator $Q : L^1(m) \to L^1(m)$ such that*

$$P_T f = \sum_{k=1}^{r} \eta_k(f) f_k + Qf \quad \text{for any } f \in L^1(m),$$

where:

(1) *$f_i \cdot f_j = 0$ for all $i \neq j$;*
(2) *$P_T f_k = f_{Perm(k)}$, where Perm is a permutation of $1, \dots, r$;*
(3) *$\lim_{n\to\infty} \|P_T^n Q f\|_{L^1(m)} = 0$ for any $f \in L^1(m)$.*

Remark 5.7. For random maps T in Examples 1.1 and 1.2, the r in this theorem is 1, which will be shown in another paper.

6. Basic estimates. Let T be a random map as in Section 3. We give some basic estimates to prove the key inequality (5.1).

Put

$$F_{t,i}(x) := |\phi_{t,i}^*(x)| p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) \quad \text{on } [0,1]$$

for $t \in W$ and $i \in \Lambda$.

Lemma 6.1. *Let T be a random map as in Section 3 and let $P_T : L^1(m) \to L^1(m)$ be the corresponding Perron–Frobenius operator. Then*

$$\bigvee_{[0,1]} P_T f \le \int_W \sum_{i\in\Lambda} \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})))} F_{t,i}\, \nu(dt).$$

Further, let $J_1, \dots, J_s$ be closed intervals such that

$$[0,1] = \bigcup_{j=1}^{s} J_j, \quad \mathrm{int}(J_i) \cap \mathrm{int}(J_j) = \emptyset \quad (i \neq j).$$

Then

$$\bigvee_{[0,1]} P_T f \le \sum_{j=1}^{s} \int_W \sum_{i\in\Lambda} \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})\cap J_j))} F_{t,i}\, \nu(dt).$$

Proof. By (2.3) it is easy to see that

$$\bigvee_{[0,1]} P_T f \le \int_W \bigvee_{[0,1]} P_{\tau_t}(p_t f)\, \nu(dt). \tag{6.1}$$

By (3.4) we have

$$\bigvee_{[0,1]} P_{\tau_t}(p_t f) \le \sum_{i\in\Lambda} \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})))} F_{t,i}. \tag{6.2}$$

By (6.1) and (6.2) we obtain the first half of the lemma. The rest is obtained similarly. ∎

In Lemma 6.1, the sequence of intervals $J_1, \dots, J_s$ is independent of the random map T. But, in some lemmas in the rest of this section it does depend on T. More precisely, the intervals $J_1, \dots, J_s$ will appear in condition (a1). To state it, we prepare some notation. Let $\tilde{\alpha}$ satisfy

$$\sup_{x\in[0,1]} \int_W g(t,x)\,\nu(dt) < \tilde{\alpha} < \alpha$$

and put $\alpha_0 := 2\tilde{\alpha}$. We will show the following lemma.

LEMMA 6.2. *Let T be a random map as in Section* 3*. If condition* (b) *is satisfied, then condition* (a) *implies the following condition* (a1)*:*

(a1) *For any fixed $\kappa > 0$ there exists a positive integer s, a sequence of closed intervals $J_1, \dots, J_s$ and a sequence of measurable functions $\alpha_1(t), \dots, \alpha_s(t)$ such that*

$$[0,1] = \bigcup_{j=1}^{s} J_j, \quad \mathrm{int}(J_i) \cap \mathrm{int}(J_j) = \emptyset \quad (i \neq j),$$

$$\int_W \alpha_j(t)\ \nu(dt) < \alpha_0 \quad \textit{and} \quad \nu(W \setminus A_j) < \kappa \quad \textit{for } j = 1, \dots, s,$$

where $A_j = \{t \in W : g(t,x) \le \alpha_j(t) \text{ for all } x \in J_j\}$.

Before proving the lemma, we give a remark.

REMARK 6.3. If $\int_W \sup_{x\in[0,1]} g(t,x)\,\nu(dt) < \alpha$, then (a1) is satisfied with $s = 1$, $J_1 = [0,1]$ and $\alpha_1(t) = \sup_{x\in[0,1]} g(t,x)$. In the next section, we will use the lemma under the condition $\alpha < 1/8$. So, the readers who are only interested in the case $\int_W \sup_{x\in[0,1]} g(t,x)\,\nu(dt) < 1/8$ do not need to read the proof of Lemma 6.2.

Proof of Lemma 6.2. By (a) we can choose a constant $k_0 > 1$ such that

$$\sup_{x\in[0,1]} \int_W k_0 g(t,x)\,\nu(dt) < \tilde{\alpha}.$$

By (b), for a.s. $t \in W$, $g(t,x)$ can be redefined on a countable set of x in $[0,1]$ to become an upper semicontinuous function of x, say $\bar{g}(t,x)$. Note that

$$(6.3) \qquad \begin{aligned} &\sup_{x\in[0,1]} \int_W \bar{g}(t,x)\,\nu(dt) \\ &\quad \le \sup_{x\in[0,1]} \int_W \lim_{y\to x+0} g(t,y)\,\nu(dt) + \sup_{x\in[0,1]} \int_W \lim_{y\to x-0} g(t,y)\,\nu(dt) \\ &\quad \le 2 \sup_{x\in[0,1]} \int_W g(t,x)\,\nu(dt). \end{aligned}$$

By upper semicontinuity, for any $x \in [0,1]$ and a.s. $t \in W$ there exists an $\varepsilon = \varepsilon(t,x)$ such that

$$\sup_{y \in U(x,\varepsilon)} g(t,y) \le \sup_{y \in U(x,\varepsilon)} \bar{g}(t,y) \le k_0 \bar{g}(t,x),$$

where $U(x,\varepsilon) = (x-\varepsilon, x+\varepsilon) \cap [0,1]$. So, for $x \in [0,1]$ and $\kappa > 0$, there exists an $\tilde{\varepsilon} = \tilde{\varepsilon}(x,\kappa)$ such that

$$\nu\Big(W \setminus \Big\{t \in W : \sup_{y \in U(x,\tilde{\varepsilon})} g(t,y) \le k_0 \bar{g}(t,x)\Big\}\Big) < \kappa.$$

Since $\{U(x,\tilde{\varepsilon}) : x \in [0,1]\}$ is an open covering of $[0,1]$, there exist $U(x_1,\tilde{\varepsilon}), \ldots, U(x_s,\tilde{\varepsilon})$ such that $\bigcup_{j=1}^{s} U(x_j,\tilde{\varepsilon}) = [0,1]$. Let s be the smallest integer with this property.

If $s = 1$, we put $J_1 = [0,1]$.

In the case $s \ge 2$, without loss of generality, we may assume $x_1 < \cdots < x_s$. Choose $y_j \in U(x_j,\tilde{\varepsilon}) \cap U(x_{j+1},\tilde{\varepsilon})$ for $j = 1, \ldots, s-1$. Put $J_1 = [0,y_1]$, $J_2 = [y_1,y_2], \ldots, J_s = [y_{s-1},1]$.

Then, in any case, $J_j \subset U(x_j,\tilde{\varepsilon})$ for $j = 1, \ldots, s$. Thus, the closed intervals $J_1, \ldots, J_s$ satisfy

$$[0,1] = \bigcup_{j=1}^{s} J_j, \quad \operatorname{int}(J_i) \cap \operatorname{int}(J_j) = \emptyset \quad (i \ne j),$$

$$\nu\Big(W \setminus \Big\{t \in W : \sup_{x \in J_j} g(t,x) \le k_0 \bar{g}(t,x_j)\Big\}\Big) < \kappa \quad \text{for } j = 1, \ldots, s.$$

Put $\alpha_j(t) := k_0 \bar{g}(t,x_j)$ for $j = 1, \ldots, s$. Then by (6.3) we have

$$\int_W \alpha_j(t)\,\nu(dt) \le \sup_{x \in [0,1]} \int_W k_0 \bar{g}(t,x)\,\nu(dt) < 2\tilde{\alpha} = \alpha_0 \quad \text{for } j = 1, \ldots, s.$$

Hence we obtain the lemma. ∎

Now, we are going to estimate $\sum_{i \in \Lambda} \bigvee_{\mathrm{cl}(\tau_t(\operatorname{int}(I_{t,i}) \cap J_j))} F_{t,i}$ for each closed interval J_j under condition (a1).

For an interval $I \subset [0,1]$, put

$$\Delta(t,I) := \sup\Big\{\sum_{k=0}^{n-1} f(x_{k+1})|g(t,x_{k+1}) - g(t,x_k)| : \inf I = x_0 < x_1 < \cdots < x_n = \sup I\Big\}.$$

We use this notation in Lemmas 6.4, 6.8 and 6.9.

Lemma 6.4. *Let T be a random map as in Section* 3. *Assume that* (a1) *is satisfied. If $t \in A_j$, then*

$$\sum_{i \in \Lambda} \bigvee_{\mathrm{cl}(\tau_t(\operatorname{int}(I_{t,i}) \cap J_j))} F_{t,i} \le \Delta(t,J_j) + \alpha_j(t) \bigvee_{J_j} f$$

for each $j = 1, \ldots, s$.

Further, assume that condition (b) *is satisfied instead of* (a1). *Then*

$$\sum_{i\in\Lambda}\ \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})\cap J))} F_{t,i} \le \Delta(t,J) + M\bigvee_J f$$

for any interval $J\subset[0,1]$ *and a.s.* $t\in W$.

Proof. For $y_k, y_{k+1}\in \mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})))$ we have

$$\begin{aligned}
&|F_{t,i}(y_{k+1}) - F_{t,i}(y_k)|\\
&\le ||\phi^*_{t,i}(y_{k+1})|p_t(\phi_{t,i}(y_{k+1}))f(\phi_{t,i}(y_{k+1})) - |\phi^*_{t,i}(y_k)|p_t(\phi_{t,i}(y_k))f(\phi_{t,i}(y_{k+1}))|\\
&\quad + ||\phi^*_{t,i}(y_k)|p_t(\phi_{t,i}(y_k))f(\phi_{t,i}(y_{k+1})) - |\phi^*_{t,i}(y_k)|p_t(\phi_{t,i}(y_k))f(\phi_{t,i}(y_k))|.
\end{aligned}$$

If $t\in A_j$, we have

$$\sup_{y\in\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})\cap J_j))} |\phi^*_{t,i}(y)|p_t(\phi_{t,i}(y)) = \sup_{x\in I_{t,i}\cap J_j} g(t,x) \le \alpha_j(t).$$

Thus we obtain

$$\bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})\cap J_j))} F_{t,i} \le \Delta(t, I_{t,i}\cap J_j) + \alpha_j(t)\bigvee_{I_{t,i}\cap J_j} f \quad \text{for } t\in A_j.$$

By summing up, we obtain the first half of the lemma.

Assume that condition (b) is satisfied. Then $g(t,x)\le M$ for a.s. $t\in W$ and $x\in[0,1]$, and we can argue. ∎

Let J_j be as in condition (a1) and let δ be a constant such that

$$0<\delta\le \frac{1}{2}\min\{m(J_j) : j=1,\dots,s\}. \tag{6.4}$$

Put

$$G_{\delta,j} := \Big\{t\in W : \bigvee_{[c,c+2\delta]} g(t,\cdot)\le 2\alpha_j(t) \text{ for all } c\in[\inf J_j, \sup J_j - 2\delta]\Big\} \tag{6.5}$$

for each $j=1,\dots,s$.

REMARK 6.5. Let T satisfy (a1). If $t\in A_j$ and $x_0\in J_j$, then

$$\Big|\lim_{x\to x_0+0} g(t,x) - g(t,x_0)\Big| \le \alpha_j(t), \qquad \Big|\lim_{x\to x_0-0} g(t,x) - g(t,x_0)\Big| \le \alpha_j(t).$$

So, if T satisfies (b), then there exists a finite partition $\inf J_j = c_{t,0} < c_{t,1} < c_{t,2} < \dots < c_{t,N_t} = \sup J_j$ such that

$$\bigvee_{[c_{t,i-1},c_{t,i}]} g(t,\cdot) < 2\alpha_j(t) \quad \text{for each } i=1,\dots,N_t.$$

REMARK 6.6. If t satisfies $0 < m(I_{t,i}\cap J_j) < 2\delta$ for some $i\in\Lambda$, it sometimes occurs that $t\notin G_{\delta,j}$.

REMARK 6.7. If W is finite, there exists $\delta > 0$ such that $\nu(W \setminus G_{\delta,j}) = 0$. But, if W is infinite, this is not necessarily true.

Keeping these remarks in mind, we will show the following lemma.

LEMMA 6.8. *Let T be a random map as in Section 3. Assume that conditions (a1) and (b) are satisfied. Let δ be a constant as in (6.4) and let $G_{\delta,j}$ be defined by (6.5). If $t \in G_{\delta,j} \cap A_j$, then*

$$\varDelta(t, J_j) \le 2\alpha_j(t)\Big(\bigvee_{J_j} f + \frac{1}{\delta}\int_{J_j} f(x)\, m(dx)\Big)$$

for each $j = 1, \ldots, s$.

Proof. If $t \in G_{\delta,j} \cap A_j$, there exists a finite partition $\inf J_j = c_{t,0} < c_{t,1} < c_{t,2} < \cdots < c_{t,N_t} = \sup J_j$ such that

$$c_{t,i} - c_{t,i-1} \ge \delta \quad \text{and} \quad \bigvee_{[c_{t,i-1}, c_{t,i}]} g(t,\cdot) < 2\alpha_j(t) \quad \text{for each } i = 1, \ldots, N_t.$$

So, if $t \in G_{\delta,j} \cap A_j$, we have

$$\begin{aligned}
\varDelta(t, [c_{t,i-1}, c_{t,i}]) &\le 2\alpha_j(t) \sup\{f(x) : x \in [c_{t,i-1}, c_{t,i}]\} \\
&\le 2\alpha_j(t)\Big(\bigvee_{[c_{t,i-1}, c_{t,i}]} f + \frac{1}{c_{t,i} - c_{t,i-1}} \int_{c_{t,i-1}}^{c_{t,i}} f(x)\, m(dx)\Big) \\
&\le 2\alpha_j(t)\Big(\bigvee_{[c_{t,i-1}, c_{t,i}]} f + \frac{1}{\delta} \int_{c_{t,i-1}}^{c_{t,i}} f(x)\, m(dx)\Big),
\end{aligned}$$

and the lemma follows. ∎

In Lemma 6.8 we have assumed that $t \in G_{\delta,j} \cap A_j$. In the next lemma we do not make this assumption.

LEMMA 6.9. *Let T be a random map as in Section 3. Assume that condition (b) is satisfied. For a.s. $t \in W$ and for any interval $J \subset [0,1]$,*

$$\varDelta(t, J) \le M\Big(\bigvee_{J} f + \frac{1}{m(J)}\int_{J} f(x)\, m(dx)\Big).$$

Proof. Since

$$\sup_{x \in J} f(x) \le \bigvee_{J} f + \frac{1}{m(J)}\int_{J} f(x)\, m(dx),$$

the lemma follows. ∎

Now, we give an estimate of $\int_W \sum_{i \in \varLambda} \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i}) \cap J_j))} F_{t,i}\, \nu(dt)$.

LEMMA 6.10. *Let T be a random map as in Section 3. Assume that conditions (a1) and (b) are satisfied. Let δ be a constant as in (6.4) and let*

$G_{\delta,j}$ *be defined by* (6.5). *Then*

$$\int\limits_W \sum_{i\in\Lambda} \bigvee_{cl(\tau_t(\mathrm{int}(I_{t,i})\cap J_j))} F_{t,i}\,\nu(dt)$$
$$\le (3\alpha_0 + 2M\nu(W\setminus(G_{\delta,j}\cap A_j)))\Big(\bigvee_{J_j} f + \frac{1}{\delta}\int\limits_{J_j} f(x)\,m(dx)\Big)$$

for each $j = 1,\ldots,s$.

Proof. By Lemmas 6.8 and 6.9, we obtain

$$\Delta(t,J_j) \le (2\alpha_j(t) + M\,1_{W\setminus(G_{\delta,j}\cap A_j)}(t))\Big(\bigvee_{J_j} f + \frac{1}{\delta}\int\limits_{J_j} f(x)\,m(dx)\Big) \tag{6.6}$$

for a.s. $t\in W$ and each $j=1,\ldots,s$. It follows from (6.6) and Lemma 6.4 that

$$\sum_{i\in\Lambda} \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})\cap J_j))} F_{t,i}$$
$$\le (3\alpha_j(t) + 2M\,1_{W\setminus(G_{\delta,j}\cap A_j)}(t))\Big(\bigvee_{J_j} f + \frac{1}{\delta}\int\limits_{J_j} f(x)\,m(dx)\Big)$$

for a.s. $t\in W$ and each $j=1,\ldots,s$. Thus, by the inequality $\int_W \alpha_j(t)\,\nu(dt) < \alpha_0$ in (a1), we obtain the lemma. ∎

7. Proof of the key inequality. As we have seen in Section 5, Theorems 5.2 and 5.6 follow from the key inequality (5.1). So, to complete the proof of Theorems 5.2 and 5.6 we are going to show (5.1).

Let $T = \{\tau_t; p(t,x) : t\in W\}$ be a random map as in Section 3. Since $P_T^K = P_{T^K}$ by Lemma 4.3, it is sufficient to show that

$$\bigvee_{[0,1]} P_{T^K} f < \gamma \bigvee_{[0,1]} f + \beta.$$

First, we are going to check that the random map

$$T^K = \{\tau_{t_K}\circ\cdots\circ\tau_{t_1}, p(t_1,x)\cdots p(t_K,\tau_{t_{K-1}}\circ\cdots\circ\tau_{t_1}(x)) : (t_1,\ldots,t_K)\in W^K\}$$

satisfies essentially the same assumptions as in Section 3.

Since $\tau_t(x)$ is a measurable function of t for each $x\in[0,1]$, $\tau_{t_K}\circ\cdots\circ\tau_{t_1}(x)$ is also a measurable function of $(t_1,\ldots,t_K)$ for each $x\in[0,1]$. Since $p: W\times X\to[0,\infty)$ is a measurable function such that $\int_W p(t,x)\,\nu(dt) = 1$ for $x\in[0,1]$, $p_K : W^K\times X\to[0,\infty)$ defined by

$$p_K(t_1,\ldots,t_K,x) = p(t_1,x)\cdots p(t_K,\tau_{t_{K-1}}\circ\cdots\circ\tau_{t_1}(x))$$

is a measurable function such that

$$\int_{W^K} p(t_1,x)\cdots p(t_K,\tau_{t_{K-1}}\circ\cdots\circ\tau_{t_1}(x))\,\nu(dt_1)\cdots\nu(dt_K)=1$$

for $x\in[0,1]$.

Since Λ is at most countable, so is Λ^K. For $(i_1,\ldots,i_k)\in\Lambda^k$ ($k=1,\ldots,K$), put

$$\begin{aligned} I_{t_1,t_2,(i_1,i_2)} &:= \mathrm{cl}(\phi_{t_1,i_1}(\mathrm{int}(I_{t_2,i_2})\cap\tau_{t_1}(\mathrm{int}(I_{t_1,i_1})))),\\ I_{t_1,\ldots,t_k,(i_1,\ldots,i_k)} &:= \mathrm{cl}(\phi_{t_1,i_1}\circ\cdots\circ\phi_{t_{k-1},i_{k-1}}(\mathrm{int}(I_{t_k,i_k})\\ &\qquad \cap\tau_{t_{k-1}}\circ\cdots\circ\tau_{t_1}(\mathrm{int}(I_{t_1,\ldots,t_{k-1},(i_1,\ldots,i_{k-1})}))))). \end{aligned}$$

For simplicity, $i\in\Lambda^k$ means $i=(i_1,\ldots,i_k)\in\Lambda^k$. Since $\{I_{t,i}\}_{i\in\Lambda}$ is a family of closed intervals such that $\mathrm{int}(I_{t,i})\cap\mathrm{int}(I_{t,j})=\emptyset$ ($i\neq j$) and $m([0,1]\setminus\bigcup_{i\in\Lambda}I_{t,i})=0$, the family of closed intervals $\{I_{t_1,\ldots,t_K,i}: i\in\Lambda^K\}$ satisfies

$$\begin{gathered} \mathrm{int}(I_{t_1,\ldots,t_K,i})\cap\mathrm{int}(I_{t_1,\ldots,t_K,j})=\emptyset\quad (i\neq j),\\ m\Big([0,1]\setminus\bigcup_{i\in\Lambda^K}I_{t_1,\ldots,t_K,i}\Big)=0. \end{gathered}$$

Assumption (A1) implies

(A*1) the restriction of $\tau_{t_K}\circ\cdots\circ\tau_{t_1}$ to $\mathrm{int}(I_{t_1,\ldots,t_K,i})$ is a C^1 and monotone function for each $i\in\Lambda^K$ and $(t_1,\ldots,t_K)\in W^K$.

Define $\tau_{t_1,\ldots,t_k}:=\tau_{t_k}\circ\cdots\circ\tau_{t_1}$ and let $\tau_{t_1,\ldots,t_k,(i_1,\ldots,i_k)}$ be the restriction of $\tau_{t_1,\ldots,t_k}$ to $\mathrm{int}(I_{t_1,\ldots,t_k,(i_1,\ldots,i_k)})$ for $k=1,\ldots,K$. For each $(t_1,\ldots,t_K)\in W^K$ and each $i\in\Lambda^K$, put

$$\phi_{t_1,\ldots,t_K,i}(x):=\begin{cases}\tau^{-1}_{t_1,\ldots,t_K,i}(x), & x\in\tau_{t_1,\ldots,t_K,i}(\mathrm{int}(I_{t_1,\ldots,t_K,i})),\\ 0, & x\in[0,1]\setminus\tau_{t_1,\ldots,t_K,i}(\mathrm{int}(I_{t_1,\ldots,t_K,i})).\end{cases}$$

Further, for each $x\in[0,1]$ and $i\in\Lambda^K$, put $w_{x,i}(t_1,\ldots,t_K):=\phi_{t_1,\ldots,t_K,i}(x)$ on W^K. Then assumption (A2) implies:

(A*2) $w_{x,i}(t_1,\ldots,t_K)$ is a measurable function of $(t_1,\ldots,t_K)$ for each $x\in[0,1]$ and $i\in\Lambda^K$.

Now, for $k=1,\ldots,K$, put

$$g(t_1,\ldots,t_k,x):=\begin{cases}p(t_1,x)\cdots p(t_k,\tau_{t_{k-1}}\circ\cdots\circ\tau_{t_1}(x))/|\tau'_{t_1,\ldots,t_k}(x)|\\ \qquad\text{on } W^k\times\bigcup_{i\in\Lambda^k}\mathrm{int}(I_{t_1,\ldots,t_k,i}),\\ 0\quad\text{on } W^k\times([0,1]\setminus\bigcup_{i\in\Lambda^k}\mathrm{int}(I_{t_1,\ldots,t_k,i})).\end{cases}$$

We are going to show the following lemma.

LEMMA 7.1. *Let $g(t,x)$ satisfy conditions* (a) *and* (b) *of Section* 5. *Then:*

(a*) $\sup_{x\in[0,1]}\int_{W^K}g(t_1,\ldots,t_K,x)\,\nu(dt_1)\cdots\nu(dt_K)<\alpha^K$;

(b*) *there exists a constant* M_K *such that* $\bigvee_{[0,1]} g(t_1,\ldots,t_K,\cdot) < M_K$ *for a.s.* $(t_1,\ldots,t_K) \in W^K$, *where* M_K *depends on* K *but is independent of the choice of* $(t_1,\ldots,t_K)$.

Proof. (a*) is easily obtained. We are going to show (b*).

It is easy to see that

$$\bigvee_{[0,1]} g(t_1,t_2,\cdot) = \bigvee_{[0,1]} g(t_1,\cdot)g(t_2,\tau_{t_1}(\cdot)) \tag{7.1}$$
$$\le \sup_{x\in[0,1]} g(t_2,\tau_{t_1}(x)) \bigvee_{[0,1]} g(t_1,\cdot) + \sum_i \sup_{x\in I_{t_1,i}} g(t_1,x) \bigvee_{I_{t_1,i}} g(t_2,\tau_{t_1}(\cdot)).$$

From

$$\bigvee_{I_{t_1,i}} g(t_2,\tau_{t_1}(\cdot)) \le \bigvee_{[0,1]} g(t_2,\cdot) < M$$

we have

$$\sum_i \sup_{x\in I_{t_1,i}} g(t_1,x) \bigvee_{I_{t_1,i}} g(t_2,\tau_{t_1}(\cdot)) \le M \sum_i \sup_{x\in I_{t_1,i}} g(t_1,x) \le M^2.$$

By this inequality and (7.1) we obtain

$$\bigvee_{[0,1]} g(t_1,t_2,\cdot) \le 2M^2.$$

Similarly, for $k = 2,3,\ldots$, we have

$$\bigvee_{[0,1]} g(t_1,\ldots,t_k,\cdot) = \bigvee_{[0,1]} g(t_1,\ldots,t_{k-1},\cdot)g(t_k,\tau_{t_1,\ldots,t_{k-1}}(\cdot))$$
$$\le \sup_{x\in[0,1]} g(t_k,\tau_{t_1,\ldots,t_{k-1}}(x)) \bigvee_{[0,1]} g(t_1,\ldots,t_{k-1},\cdot)$$
$$+ \sum_i \sup_{x\in I_{t_1,\ldots,t_{k-1},i}} g(t_1,\ldots,t_{k-1},x) \bigvee_{I_{t_1,\ldots,t_{k-1},i}} g(t_k,\tau_{t_1,\ldots,t_{k-1}}(\cdot)).$$

Hence, as we have seen above,

$$\bigvee_{[0,1]} g(t_1,\ldots,t_k,\cdot) \le 2M \bigvee_{[0,1]} g(t_1,\ldots,t_{k-1},\cdot).$$

Therefore,

$$\bigvee_{[0,1]} g(t_1,\ldots,t_K,\cdot) \le 2^{K-1}M^K,$$

which implies the lemma. ∎

Now, we are in a position to show the following lemma.

LEMMA 7.2. *Let* T *be a random map satisfying the assumption of Theorem* 5.2 *and let* $P_T : L^1(m) \to L^1(m)$ *be the corresponding Perron–Frobenius* *operator. Let K be a constant such that $8\alpha^K < 1$. Let $f \geq 0$ be a function of bounded variation with $\int_0^1 f\,dm = 1$. Then there exist constants $\gamma < 1$ and $\beta > 0$ such that*

$$\bigvee_{[0,1]} P_{T^K} f < \gamma \bigvee_{[0,1]} f + \beta.$$

Proof. Since $\alpha < 1$, we can choose a constant K with $\alpha^K < 1/8$. As we have seen before, the random map T^K satisfies essentially the same assumptions as in Section 3. Further, by Lemma 7.1, T^K satisfies conditions (a*) and (b*). So, for simplicity, we assume conditions (b) and (a) with $\alpha < 1/8$ and prove the lemma with $K = 1$. By Lemma 6.2 we may assume conditions (b) and (a1) with $\alpha < 1/8$.

Set $\kappa = \frac{1}{16M}$ in (a1), that is,

$$\nu(W \setminus A_j) < \frac{1}{16M} \quad \text{for } j = 1, \ldots, s. \tag{7.2}$$

Let δ be a constant as in (6.4) and let $G_{\delta,j}$ be defined by (6.5). Since $\nu(W \setminus G_{\delta,j}) \to 0$ as $\delta \to 0$, we can choose a constant $\delta_0 > 0$ such that

$$\nu(W \setminus G_{\delta_0,j}) < \frac{1}{16M} \quad \text{for } j = 1, \ldots, s. \tag{7.3}$$

It follows from (7.2) and (7.3) that

$$\nu(W \setminus (G_{\delta_0,j} \cap A_j)) < \frac{1}{8M} \quad \text{for } j = 1, \ldots, s.$$

Hence, by Lemma 6.10 we obtain

$$\int_W \sum_{i\in\Lambda} \bigvee_{\mathrm{cl}(\tau_t(\mathrm{int}(I_{t,i})\cap J_j))} F_{t,i}\,\nu(dt) \leq \left(3\alpha_0 + \frac{1}{4}\right)\left(\bigvee_{J_j} f + \frac{1}{\delta_0}\int_{J_j} f(x)\,m(dx)\right)$$

for $j = 1, \ldots, s$. Therefore, by Lemma 6.1,

$$\bigvee_{[0,1]} P_T f \leq \left(3\alpha_0 + \frac{1}{4}\right) \bigvee_{[0,1]} f + \left(3\alpha_0 + \frac{1}{4}\right)\frac{1}{\delta_0}.$$

Since we assume $\alpha < 1/8$, we have $\alpha_0 < 1/4$ and $3\alpha_0 + 1/4 < 1$. Hence, we obtain the lemma for $K = 1$ under conditions (b) and (a) with $\alpha < 1/8$.

In the general case we can prove the lemma by a minor modification. ∎

Proof of the key inequality. The key inequality follows from Lemmas 4.3 and 7.2. ∎

References

[Ba-G] W. Bahsoun and P. Góra, *Position dependent random maps in one and higher dimensions*, Studia Math. 166 (2005), 271–286.

[Bo-G] A. Boyarsky and P. Góra, *Laws of Chaos*, Birkhäuser, Boston, 1997.

[G-Bo] P. Góra and A. Boyarsky, *Absolutely continuous invariant measures for random maps with position dependent probabilities*, J. Math. Anal. Appl. 278 (2003), 225–242.

[L-M] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise. Stochastic Aspects of Dynamics*, Appl. Math. Sci. 97, Springer, New York, 1994.

[L-Y] A. Lasota and J. A. Yorke, *On the existence of invariant densities for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973), 481–488.

[M] T. Morita, *Random iteration of one-dimensional transformations*, Osaka J. Math. 22 (1985), 489–518.

[P] S. Pelikan, *Invariant densities for random maps of the interval*, Trans. Amer. Math. Soc. 281 (1984), 813–825.

[R] M. Rychlik, *Bounded variation and invariant measures*, Studia Math. 76 (1983), 69–80.

Tomoki Inoue
Division of Applied Mathematics
Department of Electrical and Electronic Engineering
Ehime University
Matsuyama 790-8577, Japan
E-mail: inoue.tomoki.mz@ehime-u.ac.jp

Received May 25, 2011
Revised version December 12, 2011 (7202)