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Isolated points of spectrum of k-quasi-$*$-class A operators

by

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Abstract. Let T be a bounded linear operator on a complex Hilbert space H. In this paper we introduce a new class, denoted $\mathcal{KQA}^*$, of operators satisfying $T^{*k}(|T^2|-|T^*|^2)T^k \geq 0$ where k is a natural number, and we prove basic structural properties of these operators. Using these results, we also show that if E is the Riesz idempotent for a non-zero isolated point μ of the spectrum of $T \in \mathcal{KQA}^*$, then E is self-adjoint and $EH = \ker(T-\mu) = \ker(T-\mu)^*$. Some spectral properties are also presented.

1. Introduction. Let $B(H)$ be the algebra of all bounded linear operators acting on an infinite-dimensional separable complex Hilbert space H. An operator $T \in B(H)$ is said to have the *single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to H$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$, called the *local resolvent set* of T at x, is defined to consist of all $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, which satisfies $(T-z)f(z) = x$. We denote by $\sigma_T(x)$ the complement of $\rho_T(x)$, called the *local spectrum* of T at x, and define the *local spectral subspace* of T, $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$, for each subset F of $\mathbb{C}$.

An operator $T \in B(H)$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

$$\text{Bishop's property } (\beta) \ \Rightarrow\ \text{Dunford's property (C)} \ \Rightarrow\ \text{SVEP}.$$

As an easy extension of normal operators, hyponormal operators have been studied by many mathematicians. Though there are many unsolved

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interesting problems for hyponormal operators (e.g., the invariant subspace problem), one of recent trends in operator theory is to study natural extensions of hyponormal operators. Below we introduce some of these non-hyponormal operators. Recall ([3, 14]) that $T \in B(H)$ is called *hyponormal* if $T^*T \geq TT^*$, *paranormal* (resp. $*$-*paranormal*) if $\|T^2x\| \geq \|Tx\|^2$ (resp. $\|T^2x\| \geq \|T^*x\|^2$) for all unit vectors $x \in H$. Following [14] and [21] we say that $T \in B(H)$ belongs to *class* A if $|T^2| \geq |T|^2$ where $T^*T = |T|^2$. Recently, B. P. Duggal, I. H. Jeon and I. H. Kim [12] considered the following new class of operators: we say that $T \in B(H)$ belongs to $*$-*class* A if $|T^2| \geq |T^*|^2$.

For brevity, we shall denote classes of hyponormal operators, paranormal operators, $*$-paranormal operators, class A operators, and $*$-class A operators by $\mathcal{H}$, $\mathcal{PN}$, $\mathcal{PN}^*$, $\mathcal{A}$, and $\mathcal{A}^*$, respectively. From [3] and [14], it is well known that

$$\mathcal{H} \subset \mathcal{A} \subset \mathcal{PN} \quad \text{and} \quad \mathcal{H} \subset \mathcal{A}^* \subset \mathcal{PN}^*.$$

Recently, the authors of [35] have extended $*$-class A operators to *quasi-*$*$-*class* A operators. An operator $T \in B(H)$ is said to be *quasi-*$*$-*class* A if $T^*|T^2|T \geq T^*|T^*|^2T$, and *quasi-*$*$-*paranormal* if $\|T^*Tx\|^2 \leq \|T^3x\|\,\|Tx\|$ for all $x \in H$. In [28], many results on quasi-$*$-paranormal operators were proved. In particular, quasi-$*$-paranormal operators have Bishop's property (β). If we denote the class of quasi-$*$-class A operators by $\mathcal{QA}^*$ and of quasi-$*$-paranormal operators by $\mathcal{QPN}^*$, we have

$$\mathcal{H} \subset \mathcal{A}^* \subset \mathcal{QA}^* \subset \mathcal{QPN}^*$$

(see Proposition 2.1). As a further generalization, we introduce the class of k-quasi-$*$-class A operators. An operator T is said to be a k-*quasi-*$*$-*class* A operator if

$$T^{*k}(|T^2| - |T^*|^2)T^k \geq 0,$$

where k is a natural number. 1-quasi-$*$-class A is quasi-$*$-class A.

Let $T \in B(H)$. Then $\ker T$ denotes the null space of T and $[\operatorname{ran} T]$ denotes the closure of $\operatorname{ran} T$, where $\operatorname{ran} T$ is the range of T. The operator T is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Let μ be an isolated point of $\sigma(T)$. Then the *Riesz idempotent* E of T with respect to μ is defined by

$$E := \frac{1}{2\pi i} \int_{\partial D} (\mu I - T)^{-1}\, d\mu,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. It is well known that $E^2 = E$, $ET = TE$, $\sigma(T|_{E(H)}) = \{\mu\}$ and $\ker(T - \mu I) \subseteq E(H)$. In [36], Stampfli showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = \ker(T - \mu)$. Recently, Jeon and Kim [21] and Uchiyama [38] obtained Stampfli's result for quasi-class A operators and paranormal operators. In general even if T is a paranormal operator, the Riesz idempotent E of T with respect to μ is not necessarily self-adjoint.

Recently the authors of [39] showed that every $*$-paranormal operator has Bishop's property (β). In this paper we give basic properties of k-quasi-$*$-class A operators. We show that every k-quasi-$*$-class A operator has Bishop's property (β). It is also shown that if E is the Riesz idempotent for a nonzero isolated point μ of the spectrum of a k-quasi-$*$-class A operator T, then E is self-adjoint and $EH = \ker(T-\mu) = \ker(T^* - \overline{\mu})$. Some spectral properties are also presented.

2. Main results. We begin with the following lemma which is the essence of this paper; it is a structure theorem for k-quasi-$*$-class A operators.

LEMMA 2.1. *Let $T \in B(H)$ be a k-quasi-$*$-class A operator, and suppose the range of T^k is not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } H = [\operatorname{ran} T^k] \oplus \ker T^{*k}.$$

Then T_1 is a $*$-class A operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let P be the orthogonal projection of H onto $[\operatorname{ran} T^k]$. Since T is k-quasi-$*$-class A, we have

$$P(T^{*2}T^2 - TT^*)P \geq 0, \qquad P(T^{*2}T^2)P - P(TT^*)P \geq 0.$$

Hence $T_1^{*2}T_1^2 - T_1T_1^* \geq 0$. This shows that T_1 is $*$-class A on $[\operatorname{ran} T^k]$.

Further, we have

$$\langle T_3^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{*k}(I-Px)\rangle = 0$$

for any $x = \binom{x_1}{x_2} \in H$. Thus $T_3^k = 0$.

We have $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup G$, where G is the union of certain holes in $\sigma(T)$ which are subsets of $\sigma(T_1) \cap \sigma(T_3)$ [19, Corollary 7]. Since $\sigma(T_1) \cap \sigma(T_3)$ has no interior points, we have

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}. \blacksquare$$

Let K be an infinite-dimensional separable Hilbert space. The above decomposition of k-quasi-$*$-class A operators motivates the following question: Is the operator matrix

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

acting on $H \oplus K$ a k-quasi-$*$-class A operator if A is $*$-class A and $C^k = 0$? We do not know the answer. However, for $k = 1$ we have

THEOREM 2.1. *Let T be an operator on $H \oplus K$ defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is $*$-class A, then T is 1-quasi-$*$-class A.

Proof. A simple calculation shows that

$$T^*(T^{*2}T^2 - TT^*)T = \begin{pmatrix} A^*(A^{*2}A^2 - AA^*)A & A^*(A^{*2}A^2 - AA^*)B \\ B^*(A^{*2}A^2 - AA^*)A & B^*(A^{*2}A^2 - AA^*)B \end{pmatrix}.$$

Let $u = x \oplus y \in H \oplus K$. Then

$$\begin{aligned} &\langle (T^*(T^{*2}T^2 - TT^*)T)u, u\rangle \\ &\quad = \langle A^*(A^{*2}A^2 - AA^*)Ax, x\rangle + \langle A^*(A^{*2}A^2 - AA^*)By, x\rangle \\ &\qquad + \langle B^*(A^{*2}A^2 - AA^*)Ax, y\rangle + \langle B^*(A^{*2}A^2 - AA^*)By, y\rangle \\ &\quad = \langle (A^{*2}A^2 - AA^*)(Ax + By), (Ax + By)\rangle \ge 0 \end{aligned}$$

because A is $*$-class A. This proves the result. ∎

THEOREM 2.2. *Let $T \in B(H)$ be a k-quasi-$*$-class A operator. Then T has Bishop's property (β), the single-valued extension property and Dunford property* (C).

Proof. From the introduction, it suffices to prove that T has Bishop's property (β). If the range of T^k is dense, then T is a $*$-class A operator, and hence has Bishop's property (β) by [12]. So, we assume that the range of T^k is not dense. Suppose $(T - z)f_n(z) \to 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3 - z)f_{n2}(z) \end{pmatrix} \to 0.$$

Since T_3 is nilpotent, it has Bishop's property (β). Hence $f_{n2}(z) \to 0$ uniformly on every compact subset of D. Then $(T_1 - z)f_{n1}(z) \to 0$. Since T_1 is a $*$-class A operator, it has Bishop's property (β) [12]. Hence $f_{n1}(z) \to 0$ uniformly on every compact subset of D. Thus T has Bishop's property (β). ∎

T is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

LEMMA 2.2. *Let $T \in B(H)$ be a k-quasi-$*$-class A operator. Then T is isoloid.*

Proof. Suppose T has a representation given in Lemma 2.1. Let z be an isolated point in $\sigma(T)$. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, z is an isolated point in $\sigma(T_1)$ or $z = 0$. If z is an isolated point in $\sigma(T_1)$, then $z \in \sigma_p(T_1)$. Assume that $z = 0$ and $z \notin \sigma(T_1)$. Then for $x \in \ker T_3$, $-T_1^{-1}T_2x \oplus x \in \ker T$. This completes the proof. ■

The following theorems are structural results.

THEOREM 2.3. *Let $T \in B(H)$ be a k-quasi-$*$-class A operator, and let M be a closed T-invariant subspace of H. Then the restriction $T|_M$ of T to M is a k-quasi-$*$-class A operator.*

Proof. Let

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \quad \text{on } H = M \oplus M^{\perp}.$$

Since T is quasi-$*$-class A, we have

$$T^{*2}T^2 - TT^* \geq 0.$$

Hence

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*k} \left[\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*2} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{2} - \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*} \right] \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{k} \geq 0.$$

Therefore

$$\begin{pmatrix} A^{*k}(A^{*2}A^2 - (AA^* + CC^*))A^k & E \\ F & G \end{pmatrix} \geq 0$$

for some operators E, F and G. Hence

$$A^{*k}(A^{*2}A^2 - AA^*)A^k \geq A^{*k}(CC^*)A^k \geq 0.$$

This implies that $A = T|_M$ is k-quasi-$*$-class A. ■

THEOREM 2.4. *Let M be a closed non-trivial invariant subspace for a k-quasi-$*$-class A operator T. If $T|_M$ is an injective normal operator, then M reduces T.*

Proof. Suppose that P is an orthogonal projection of H onto $[\operatorname{ran} T^k]$. Since T is a k-quasi-$*$-class A operator, we have $P(T^{*2}T^2 - TT^*)P \geq 0$. Since by assumption $T|_M$ is an injective normal operator, we have $E \leq P$ for the orthogonal projection E of H onto M and $[\operatorname{ran} T^k|_M] = M$ because $T|_M$ has dense range. Therefore $M \subseteq [\operatorname{ran} T^k]$ and hence $E(T^{*2}T^2 - TT^*)E \geq 0$. Let

$$T = \begin{pmatrix} T|_M & A \\ 0 & B \end{pmatrix} \quad \text{on } H = M \oplus M^{\perp}.$$

Then we have

$$TT^* = \begin{pmatrix} T|_M T^*|_M + AA^* & AB^* \\ B^*A & BB^* \end{pmatrix}$$

and

$$T^{*2}T^2 = \begin{pmatrix} T^{*2}|_M T^2|_M & E \\ F & G \end{pmatrix}$$

for some operators E, F and G. Thus

$$\begin{pmatrix} T|_M T^*|_M + AA^* & 0 \\ 0 & 0 \end{pmatrix} = ETT^*E = E|T^*|^2E \le E(T^{*2}T^2)^{1/2}E$$
$$\le (ET^{*2}T^2E)^{1/2} = \begin{pmatrix} T^{*2}|_M T^2|_M & 0 \\ 0 & 0 \end{pmatrix}^{1/2}.$$

This implies that $T|_M T^*|_M + AA^* \le T|_M T^*|_M$. Since $T|_M$ is normal and AA^* is positive, it follows that $A = 0$. Hence M reduces T. ∎

Remark 2.1. In Theorem 2.4 we cannot drop the injectivity condition. Without it, M may not reduce T. Indeed, take any nilpotent operator T with $T^{k-1} \neq 0 = T^k$. Then $T|_{[\operatorname{ran} T^{k-1}]} = 0$ is normal. If $[\operatorname{ran} T^{k-1}]$ reduces T, then $T^*T^{k-1}H \subset [\operatorname{ran} T^{k-1}]$. Hence $T^{*k-1}T^{k-1}H \subset [\operatorname{ran} T^{k-1}]$ and $\ker T^{k-1} = \ker T^{*k-1}T^{k-1} \supset \ker T^{*k-1}$. Since $T^{*k} = T^{*k-1}T^* = 0$, we have $T^{k-1}T^* = 0$. Hence $T^{k-1}T^{*k-1} = 0$, and hence $T^{k-1} = 0$. This is a contradiction.

Theorem 2.5. *Let T be k-quasi-$*$-class A. If $\lambda \neq 0$ and $(T-\lambda)x = 0$, then $(T-\lambda)^*x = 0$.*

Proof. We may assume $x \neq 0$. Let $M = \operatorname{span}\{x\}$ and

$$T = \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix} \quad \text{on } M \oplus M^\perp,$$

and let P be the orthogonal projection from H onto M. Then $T|_M = \lambda$ and $T|_M$ is an injective normal operator. This implies that M reduces T by Theorem 2.4. Hence $A = 0$. ∎

Proposition 2.1. *If $T \in B(H)$ is quasi-$*$-class A, then it is quasi-$*$-paranormal.*

Proof. Since T is quasi-$*$-class A, we have $T^*|T^*|^2T \le T^*|T^2|T$. Let $x \in H$. Then

$$\|T^*T\|^2 = \langle T^*Tx, T^*Tx\rangle = \langle T^*|T^*|^2Tx, x\rangle$$
$$\le \langle T^*|T^2|Tx, x\rangle \le \big\||T^2|Tx\big\|\,\|Tx\| = \|T^3x\|\,\|Tx\|.$$

Therefore $\|T^*Tx\|^2 \le \|T^3x\|\,\|Tx\|$. Hence T is quasi-$*$-paranormal. ∎

Theorem 2.6. *Let $T \in B(H)$ be a quasi-$*$-paranormal operator. Then it is normaloid, i.e. $\|T\| = r(T)$ (the spectral radius of T).*

Proof. It suffices to show

$$\|T^{2m}\| = \|T\|^{2m} \tag{$*$}$$

for $m = 1, 2, \ldots$. We argue by induction. First we prove $(*)$ for $m = 1$. Since T is quasi $*$-paranormal,

$$\|T\|^4 = \|T^*T\|^2 \le \|T^3\|\,\|T\| \le \|T^2\|\,\|T\|^2 \le \|T\|^4.$$

Hence $\|T\|^2 = \|T^2\|$. Now assume that $(*)$ is true for $m = k$. Since

$$\|T^3x\|^2 + \lambda^2\|Tx\|^2 \ge 2\lambda\|T^*Tx\|^2,$$

we have

$$\begin{aligned}
&\|T^{2(k+1)}x\| + \lambda^2\|T^{2k}x\| \ge 2\lambda\|T^*T^{2k}x\|^2\\
&\qquad\Rightarrow\ \|T^{2(k+1)}\|^2 + \lambda^2\|T^{2k}\|^2 \ge 2\lambda\|T^*T^{2k}\|^2\\
&\qquad\Rightarrow\ \|T\|^{2(2k-1)}[\|T^{2(k+1)}\|^2 + \lambda^2\|T^{2k}\|^2] \ge 2\lambda\|T\|^{2(2k-1)}\|T^*T^{2k}\|^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ge 2\lambda\|T^{*2k}T^{2k}\|^2\\
&\qquad\Rightarrow\ \|T\|^{2(2k-1)}[\|T^{2(k+1)}\|^2 + \lambda^2\|T^{2k}\|^2] \ge 2\lambda\|T^{2k}\|^4.
\end{aligned}$$

Since $(*)$ is true for $m = k$, we find

$$\|T^{2(k+1)}\|^2 + \lambda^2\|T\|^{4k} \ge 2\lambda\|T\|^{4k+2}.$$

Let $\lambda = \|T\|^2$. Then the last inequality gives

$$\|T^{2(k+1)}\|^2 + \|T\|^4\|T\|^{4k} \ge 2\|T\|^{4k+4}.$$

Hence

$$2\|T\|^{4k+4} \ge \|T^{2(k+1)}\|^2 + \|T\|^{4k+4} \ge 2\|T\|^{4k+4}.$$

Clearly $\|T\|^{2(k+1)} = \|T^{2(k+1)}\|$. This proves the result. ∎

Remark 2.2. For $k > 1$, a nilpotent operator is k-quasi-$*$-class A. This shows that operators in this class need not be normaloid. However, it is obvious that for $k = 1$, this is not true. But for $k = 1$, operators of this class are normaloid. Indeed, a quasi-$*$-class A operator is quasi-$*$-paranormal by Proposition 2.1 and a quasi-$*$-paranormal operator is normaloid by Theorem 2.6. Hence a quasi-$*$-class A operator is normaloid.

Corollary 2.1. *A $*$-paranormal operator T is normaloid. In particular a $*$-class A operator is normaloid.*

Theorem 2.7. *Let A be a k-quasi-$*$-class A operator and λ be a non-zero isolated point of $\sigma(A)$. Then the Riesz idempotent E for λ is self-adjoint and*

$$EH = \ker(A - \lambda) = \ker\,(A - \lambda)^*.$$

Proof. If A is k-quasi-$*$-class A, then λ is an eigenvalue of A and $EH = \ker(A - \lambda)$ by Lemma 2.2. Since $\ker(A - \lambda) \subset \ker\,(A - \lambda)^*$ by Theorem 2.5, it suffices to show that $\ker\,(A - \lambda)^* \subset \ker(A - \lambda)$. Since $\ker(A - \lambda)$ is a reducing subspace of A by Theorem 2.5 and the restriction of a k-quasi-$*$-class A operator to its reducing subspace is also a k-quasi-$*$-class A operator by Theorem 2.3, A can be written as

$$A = \lambda \oplus A_1 \qquad \text{on } H = \ker(A-\lambda) \oplus (\ker(A-\lambda))^{\perp},$$

where A_1 is k-quasi-$*$-class A with $\ker(A_1 - \lambda) = \{0\}$. Since

$$\lambda \in \sigma(A) = \{\lambda\} \cup \sigma(A_1)$$

is isolated, only two cases occur: either $\lambda \notin \sigma(A_1)$, or λ is an isolated point of $\sigma(A_1)$ and this contradicts the fact that $\ker(A_1 - \lambda) = \{0\}$. Since A_1 is invertible as an operator on $(\ker(A-\lambda))^{\perp}$, we have $\ker(A-\lambda) = \ker\,(A-\lambda)^*$.

Next, we show that E is self-adjoint. Since

$$EH = \ker(A-\lambda) = \ker\,(A-\lambda)^*,$$

we have

$$((z-A)^*)^{-1}E = \overline{(z-\lambda)^{-1}}\,E.$$

Therefore

$$\begin{aligned} E^*E &= -\frac{1}{2\pi i}\int_{\partial D} ((z-A)^*)^{-1}E\,d\overline{z} = -\frac{1}{2\pi i}\int_{\partial D} \overline{(z-A)^{-1}}\,E\,d\overline{z} \\ &= \overline{\left(\frac{1}{2\pi i}\int_{\partial D}(z-A)^{-1}\,dz\right)}E = E. \end{aligned}$$

This completes the proof. ∎

COROLLARY 2.2. *Let $A \in B(H)$ be quasi-$*$-class A and λ be a non-zero isolated point of $\sigma(A)$. Then the Riesz idempotent E for λ is self-adjoint and*

$$EH = \ker(A-\lambda) = \ker\,(A-\lambda)^*.$$

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