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Generalized-lush spaces and the Mazur–Ulam property

by

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To the memory of Edward W. Odell

Abstract. We introduce a new class of Banach spaces, called generalized-lush spaces (GL-spaces for short), which contains almost-CL-spaces, separable lush spaces (in particular, separable C-rich subspaces of $C(K)$), and even the two-dimensional space with hexagonal norm. We find that the space $C(K,E)$ of vector-valued continuous functions is a GL-space whenever E is, and show that the set of GL-spaces is stable under c_0-, l_1- and l_∞-sums. As an application, we prove that the Mazur–Ulam property holds for a larger class of Banach spaces, called local-GL-spaces, including all lush spaces and GL-spaces. Furthermore, we generalize the stability properties of GL-spaces to local-GL-spaces. From this, we can obtain many examples of Banach spaces having the Mazur–Ulam property.

1. Introduction. The classical Mazur–Ulam theorem states that every surjective isometry between normed spaces is a linear mapping up to translation. In 1972, Mankiewicz [M] extended this by showing that every surjective isometry between open connected subsets of normed spaces can be extended to a surjective affine isometry on the whole space. This result implies that the metric structure on the unit ball of a real normed space constrains the linear structure of the whole space. It is of interest to us whether this result can be extended to unit spheres. In 1987, Tingley [T] first studied isometries on the unit sphere and raised the isometric extension problem:

PROBLEM 1.1. *Let E and F be normed spaces with unit spheres S_E and S_F, respectively. If $T : S_E \to S_F$ is a surjective isometry, does there exist a linear isometry $\widetilde{T} : E \to F$ such that $\widetilde{T}|_{S_E} = T$?*

There are a number of publications on this topic and many positive answers on special spaces, for example, $l^p(\Gamma)$, $L^p(\mu)$ $(0 < p \leq \infty)$, $C(K)$, even the James spaces and the (modified) Tsirelson spaces (see [D1, D2, L, LZ, T1, T2, T3, T4] and the references therein).

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Recently in [CD], Cheng and Dong considered the extension question of isometries between unit spheres of Banach space and introduced the Mazur–Ulam property:

Definition 1.2. A Banach space E is said to have the *Mazur–Ulam property* (briefly MUP) provided that for every Banach space F, every surjective isometry T between the unit spheres of E and F is the restriction of a linear isometry between the two spaces.

Cheng and Dong attacked the problem for the class of CL-spaces admitting a smooth point and polyhedral spaces. Unfortunately their interesting attempt failed by a mistake at the very end of the proof (see also the introduction in [KMMP, TL]). In [KM], Kadets and Martín proved that finite-dimensional polyhedral Banach spaces have the MUP. Notice that the problem is still open even in two dimensions. In [TL], Tan and Liu proved that every almost-CL-space admitting a smooth point (in particular, every separable almost-CL-space) has the MUP.

Recall that R. Fullerton [F] first introduced the notion of CL-space. It was extended by Lima [L1, L2] who introduced almost-CL-space and gave examples of real CL-spaces which are $L_1(\mu)$ and their isometric preduals, in particular $C(K)$, where K is a compact Hausdorff space. The infinite-dimensional complex $L_1(\mu)$ spaces were proved by Martín and Payá [MP1] to be only almost-CL-spaces. Lush spaces were recently introduced in [BKMW] and have been extensively studied in [BKMM, KMMP, KMMS]. Such spaces are of importance to supply an example of a Banach space E with numerical index $n(E) < n(E^*)$. It thus gives a negative answer to a question which has been latent since the beginning of the theory of numerical indices in the seventies. Now, a natural and interesting question is: "Does every almost-CL-space, even every lush space, has the MUP?"

In this paper, we introduce a natural concept of generalized-lush spaces (GL-spaces for short), which contains almost-CL-spaces, separable lush spaces (in particular, separable C-rich subspaces of $C(K)$), and even the two-dimensional space with hexagonal norm. We show that the space $C(K,E)$ of vector-valued continuous functions is a generalized-lush space whenever E is, and show the stability of generalized-lush spaces under c_0-, l_1- and l_∞-sums. Then we prove that the Mazur–Ulam property holds for a larger class of Banach spaces than GL-spaces, called local-GL-spaces, including all lush spaces and GL-spaces. Furthermore, we show that $C(K,E)$ is a local-GL-space whenever E is, and stability under c_0-, l_1- and l_∞-sums also holds for local-GL-spaces.

Throughout this paper, all spaces considered are over the real field. For a Banach space E, B_E, S_E, E^* and $L(E)$ will stand for the unit ball of E, the unit sphere of E, the dual space and the Banach algebra of all bounded linear operators on E. A *slice* is a subset of B_E of the form

$$S(x^*, \alpha) = \{x \in B_E : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{E^*}$ and $0 < \alpha < 1$.

We recall here some basic concepts.

DEFINITION 1.3. Let E be a Banach space.

(i) E is said to be a *CL-space* if for every maximal convex set C of S_E, we have $B_E = \mathrm{co}(C \cup -C)$.
(ii) E is said to be an *almost-CL-space* if for every maximal convex set C of S_E, we have $B_E = \overline{\mathrm{co}}(C \cup -C)$.
(iii) E is said to be *lush* if for every $x, y \in S_E$ and every $\varepsilon > 0$, there exists a slice $S = S(x^*, \varepsilon)$ such that $x \in S$ and $\mathrm{dist}(y, \mathrm{aco}(S)) < \varepsilon$.

It is evident that (1)$\Rightarrow$(2)$\Rightarrow$(3), and none of the one-way implications can be reversed (see [MP1, Proposition 1] and [BKMW, Example 3.4]).

The numerical index of a Banach space E was first suggested by G. Lumer in 1968 (see [DMPW]); it is defined by

$$\begin{aligned} n(E) &= \inf\{v(T) : T \in L(E), \|T\| = 1\} \\ &= \max\{k \geq 0 : k\|T\| \leq v(T) \text{ for all } T \in L(E)\}, \end{aligned}$$

where $v(T)$ is the numerical radius of T given by

$$v(T) = \sup\{|x^*(T(x))| : x \in S_E,\ x^* \in S_{E^*},\ x^*(x) = 1\}.$$

More information and background on numerical indices can be found in the recent survey [KMP] and references therein.

2. Generalized-lush spaces. The aim of this section is to study generalized-lush spaces (GL-spaces for short). We present many examples and prove a stronger property for separable GL-spaces; we also show that GL-spaces have some stability properties.

DEFINITION 2.1. A Banach space E is said to be a *generalized-lush space* (GL-space) if for every $x \in S_E$ and every $\varepsilon > 0$ there exists a slice $S = S(x^*, \varepsilon)$ with $x^* \in S_{E^*}$ such that

$$x \in S \quad \text{and} \quad \mathrm{dist}(y, S) + \mathrm{dist}(y, -S) < 2 + \varepsilon$$

for all $y \in S_E$.

The following proposition for separable GL-spaces is based on an idea from [KMMP, Lemma 4.2], and it is of independent interest. Given a Banach space E, a subset $G \subset E^*$ is called *norming* if $\|x\| = \sup\{|x^*(x)| : x^* \in G\}$ for every $x \in E$.

Proposition 2.2. *Let E be a separable GL-space, and let $G \subset S_{E^*}$ be norming and symmetric. Then for every $\varepsilon > 0$ the set*

$$\{x^* \in G : \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon \text{ for all } y \in S_E, \text{ where } S = S(x^*, \varepsilon)\}$$

is a weak G_δ-dense subset of the weak* closure of G.*

Proof. Let $(y_n) \subset S_E$ be a sequence dense in S_E. Fix $0 < \varepsilon < 1$. Given $n \geq 1$, set

$$K_n = \{x^* \in G : \operatorname{dist}(y_n, S) + \operatorname{dist}(y_n, -S) < 2 + \varepsilon, \text{ where } S = S(x^*, \varepsilon)\}.$$

Then K_n is weak*-open and $\overline{K_n}^{\omega^*} = \overline{G}^{\omega^*}$. Indeed, if $x^* \in K_n$, then there exist $x_n \in S(x^*, \varepsilon)$ and $z_n \in -S(x^*, \varepsilon)$ such that

$$\|x_n - y_n\| + \|y_n - z_n\| < 2 + \varepsilon.$$

Let

$$U = \{y^* \in G : y^*(x_n) > 1 - \varepsilon \text{ and } y^*(-z_n) > 1 - \varepsilon\}.$$

Then it is easily checked that U is a weak* neighborhood of x^* in G satisfying $U \subset K_n$. Thus K_n is weak*-open.

To prove $\overline{K_n}^{\omega^*} = \overline{G}^{\omega^*}$, it is enough to show that $G \subset \overline{K_n}^{\omega^*}$. Since [FHHMPZ, Lemma 3.40] states that for every $x^* \in G$, the weak*-slices containing x^* form a neighborhood base of x^*, it suffices to prove that $S(x, \varepsilon_1) \cap K_n \neq \emptyset$ for all $\varepsilon_1 \in (0, \varepsilon)$. Since E is a GL-space, there is a slice $S = S(y^*, \varepsilon_1/3)$ such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(y_n, S) + \operatorname{dist}(y_n, -S) < 2 + \varepsilon_1.$$

Thus we may find $x_n' \in S$ and $z_n' \in -S$ such that

$$\|x_n' - y_n\| + \|y_n - z_n'\| < 2 + \varepsilon_1 \quad \text{and} \quad \|x + x_n' - z_n'\| > 3 - \varepsilon_1.$$

Note that G is norming and symmetric. Thus there is a $z^* \in G$ such that

$$z^*(x + x_n' - z_n') > 3 - \varepsilon_1.$$

This implies that $z^* \in S(x, \varepsilon_1) \cap K_n$.

Now set $K = \bigcap_{n \in \mathbb{N}} K_n$. Then by the Baire theorem, K is a weak* G_δ-dense subset of $\overline{G}^{\omega^*}$. This together with density of (y_n) in S_E gives the desired conclusion. ∎

As a consequence, we have a stronger characterization for separable GL-spaces which indicates that the x^* in the definition of GL-spaces can be chosen from $\operatorname{ext}(B_{E^*})$.

Corollary 2.3. *Let E be a separable Banach space. Then E is a GL-space if and only if for every $x \in S_E$ and every $\varepsilon > 0$ there exists a slice $S = S(x^*, \varepsilon)$ with $x^* \in \operatorname{ext}(B_{E^*})$ such that*

$$x \in S \quad \text{and} \quad \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon$$

for all $y \in S_E$.

Now we have the following important examples.

EXAMPLE 2.4. *Every almost-CL-space is a GL-space.*

Proof. Let E be an almost-CL-space. For every $x \in S_E$ and $\varepsilon > 0$, there exists a maximal convex set C of S_E such that $x \in C$. Choose $f \in S_{E^*}$ such that $f(z) = 1$ for every $z \in C$, and set $S = S(f, \varepsilon)$. Then $C \subset S$. Since E is an almost-CL-space, it follows that $B_E = \overline{\mathrm{co}}(S \cup -S)$. So for every $y \in S_E$, there are $\lambda \in [0,1]$, $y_1 \in S$ and $y_2 \in -S$ such that

$$\|\lambda y_1 + (1-\lambda)y_2 - y\| < \varepsilon/2.$$

This leads to

$$\|y_1 - y\| + \|y_2 - y\| < 2 + \varepsilon,$$

which completes the proof. ∎

Since all $C(K)$ and all real $L_1(\mu)$ are CL-spaces (in particular, almost-CL-spaces), they are GL-spaces. Below, we exhibit a larger class of spaces which are GL-spaces, and they are not almost-CL-spaces in general (see [BKMW, Example 3.4]).

EXAMPLE 2.5. *Every separable lush space is a GL-space.*

Proof. Note that [KMMP, Theorem 4.3] implies that if E is a separable lush space, then there is a norming subset K of S_{E^*} such that

$$B_E = \overline{\mathrm{co}}(S(x^*, \varepsilon) \cup -S(x^*, \varepsilon))$$

for every $x^* \in K$ and every $\varepsilon > 0$. A similar analysis to the one in Example 2.4 yields the desired conclusion. ∎

Let K be a compact Hausdorff space. A closed subspace X of $C(K)$ is said to be *C-rich* if for every nonempty open subset U of K and every $\varepsilon > 0$, there is a positive function h with $\|h\| = 1$ and $\mathrm{supp}(h) \subset U$ such that $\mathrm{dist}(h, X) < \varepsilon$. This definition covers all finite-codimensional subspaces of $C[0,1]$ (see [BKMW, Proposition 2.5]) and all subspaces X of $C[0,1]$ such that $C[0,1]/X$ does not contain a copy of $C[0,1]$ (see [KP, Proposition 1.2 and Definition 2.1]). For more examples and results about C-rich subspaces we refer to [BKMM, KMMS, KMMP] and references therein. Notice that all C-rich subspaces of $C(K)$ have been proved in [BKMW, Theorem 2.4] to be lush. Therefore we get the following example.

EXAMPLE 2.6. *Every C-rich separable subspace of $C(K)$ is a GL-space.*

Observe that all the above examples of GL-spaces are Banach spaces with numerical index 1. We remark from the following examples that there may exist many GL-spaces whose numerical index is not 1. The two-dimensional space with hexagonal norm is an example.

Example 2.7. *The space* $E = (\mathbb{R}^2, \|\cdot\|)$ *whose norm is given by*

$$\|(\xi,\eta)\| = \max\{|\eta|, |\xi| + \tfrac{1}{2}|\eta|\} \quad \forall (\xi,\eta) \in E$$

has numerical index $1/2$ *and it is a GL-space.*

Proof. It is shown by [MM, Theorem 1] that E has numerical index $1/2$. To prove that E is a GL-space, given $x = (a,b) \in S_E$ and $\varepsilon > 0$, we divide the proof into two cases. By symmetry considerations, we assume that $a, b \geq 0$.

Case 1: $b = 1$. Define a functional $f \in S_{E^*}$ by $f(z) = \eta$ for all $z = (\xi,\eta) \in E$. Set $S = S(f,\varepsilon)$. Then $x \in S$, and for every $y = (c,d) \in S_E$, consider the two vectors

$$y_1 = (c,1) \quad \text{and} \quad y_2 = (c,-1).$$

We clearly have $y_1 \in S$ and $y_2 \in -S$, and moreover

$$\|y - y_1\| + \|y - y_2\| = 2 < 2 + \varepsilon.$$

Case 2: $b < 1$. We make the convention that $\operatorname{sign}(0) = 1$. Let $f \in S_{E^*}$ be defined by $f(z) = \xi + \eta/2$ for every $z = (\xi,\eta) \in E$. This guarantees that $x \in S = S(f,\varepsilon)$. For every $y = (c,d) \in S_E$, we set

$$\begin{cases} y_1 = (\operatorname{sign}(c), 0),\ y_2 = \operatorname{sign}(d)(1/2, 1) & \text{if } cd \leq 0; \\ y_1 = -(\operatorname{sign}(c), 0),\ y_2 = \operatorname{sign}(d)(1/2, 1) & \text{if } cd > 0 \text{ and } |d| = 1; \\ y_1 = y,\ y_2 = -y & \text{if } cd > 0 \text{ and } |d| < 1. \end{cases}$$

Then $y_1, y_2 \in S \cup -S$ satisfy

$$\|y - y_1\| + \|y - y_2\| = 2 < 2 + \varepsilon.$$

We thus complete the proof. ∎

By Example 2.7, Theorems 2.10, 2.11 below and [MP2, Proposition 1] which shows that the numerical index of the c_0-, l_1-, or l_∞-sum of Banach spaces is the infimum of the numerical indices of the summands, we may construct more examples of specific GL-spaces with numerical index $1/2$.

Example 2.8. *The space* $E = (c_0, \|\cdot\|)$ *equipped with the norm*

$$\|x\| = \max\Big\{\sup_{k\in\mathbb{N}} |\xi_k|, |\xi_1| + \tfrac{1}{2}|\xi_2|\Big\} \quad \forall x = (\xi_k) \in E$$

is a GL-space with numerical index $1/2$.

Proof. It is actually the space $c_0 \oplus_\infty X$ where X is the hexagonal space as in Example 2.7. ∎

Observe that in the definition of GL-spaces we can take y to be in the unit ball instead of being in the unit sphere. With the help of this observation, one can check whether the space being considered is a GL-space in an easier way. We will use this later to get some stability properties of GL-spaces.

LEMMA 2.9. *If E is a GL-space, then for every $x \in S_E$ and every $\varepsilon > 0$ there exists a slice $S = S(x^*, \varepsilon)$ with $x^* \in S_{E^*}$ such that*

$$x \in S \quad \textit{and} \quad \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon$$

for all $y \in B_E$.

Proof. For every $x \in S_E$ and every $\varepsilon > 0$, let $S = S(x^*, \varepsilon)$ be such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(z, S) + \operatorname{dist}(z, -S) < 2 + \varepsilon$$

for all $z \in S_E$. Given $y \in B_E$, since the case $y = 0$ is trivial, we may assume that $y \neq 0$. Then there exist $u, -v \in S$ such that

$$\left\| u - \frac{y}{\|y\|} \right\| + \left\| v - \frac{y}{\|y\|} \right\| < 2 + \varepsilon.$$

The triangle inequality hence yields

$$\|u - y\| + \|v - y\| < 2 + \varepsilon\|y\| \le 2 + \varepsilon,$$

completing the proof. ■

Given a compact Hausdorff space K and a Banach space E, we denote by $C(K, E)$ the Banach space of all continuous functions from K into E, endowed with its natural supremum norm.

THEOREM 2.10. *Let K be a compact Hausdorff space and E a GL-space. Then $C(K, E)$ is a GL-space.*

Proof. Given $f \in S_{C(K,E)}$ and $\varepsilon > 0$, there exists a $t_0 \in K$ such that $\|f(t_0)\| = 1$. Since E is a GL-space, it follows from Lemma 2.9 that there is an $x^* \in S_E^*$ with $S_{x^*} = S(x^*, \varepsilon/2)$ such that

$$f(t_0) \in S_{x^*} \quad \text{and} \quad \operatorname{dist}(y, S_{x^*}) + \operatorname{dist}(y, -S_{x^*}) < 2 + \varepsilon/2$$

for all $y \in B_E$. Define a functional $f^* \in S_{C(K,E)^*}$ by $f^*(g) = x^*(g(t_0))$ for every $g \in C(K, E)$, and put $S = S(f^*, \varepsilon)$. For every $g \in S_{C(K,E)}$, we have $g(t_0) \in B_E$. Thus there are $y_1 \in S_{x^*}$ and $y_2 \in -S_{x^*}$ such that

$$\|g(t_0) - y_1\| + \|g(t_0) - y_2\| < 2 + \varepsilon/2.$$

Then we can define a continuous map $\phi : K \to [0, 1]$ by

$$\phi(t_0) = 1 \quad \text{and} \quad \phi(t) = 0 \quad \text{if } \|g(t) - g(t_0)\| \ge \varepsilon/4.$$

Consider $h_1 \in S$ and $h_2 \in -S$ given by

$$h_i(t) = \phi(t) y_i + \big(1 - \phi(t)\big) g(t) \quad (i = 1, 2) \quad \text{for every } t \in K.$$

Then it is trivial to see that

$$\|g - h_1\| + \|g - h_2\| < 2 + \varepsilon.$$

Hence $C(K, E)$ is a GL-space. ■

For more examples of GL-spaces, we need to discuss the stability of GL-spaces under c_0-, l_1- and l_∞-sums. Recall that the c_0-sum (resp. l_1-sum and l_∞-sum) of a family $\{E_\lambda : \lambda \in \Lambda\}$ of Banach spaces is denoted by $[\bigoplus_{\lambda\in\Lambda} E_\lambda]_{c_0}$ (resp. $[\bigoplus_{\lambda\in\Lambda} E_\lambda]_{l_1}$ and $[\bigoplus_{\lambda\in\Lambda} E_\lambda]_{l_\infty}$).

THEOREM 2.11. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, and let $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_F$ where $F = c_0$, l_∞ or l_1. Then E is a GL-space if and only if each E_λ is a GL-space.*

Proof. Note that $E^* = [\bigoplus_{\lambda\in\Lambda} E_\lambda^*]_{l_1}$ if $F = c_0$ and $E^* = [\bigoplus_{\lambda\in\Lambda} E_\lambda^*]_{l_\infty}$ if $F = l_1$. This fact will be used without comment in the following proof.

In the c_0-sum case, we first show the "if" part. Fix $x = (x_\lambda) \in S_E$ and $\varepsilon > 0$. We may find a λ_0 such that $\|x_{\lambda_0}\| = 1$. Since E_{λ_0} is a GL-space, by Lemma 2.9 there is a slice $S_{\lambda_0} = S(x^*_{\lambda_0}, \varepsilon) \subset B_{E_{\lambda_0}}$ with $x^*_{\lambda_0} \in S_{E^*_{\lambda_0}}$ such that

$$x_{\lambda_0} \in S_{\lambda_0} \quad \text{and} \quad \operatorname{dist}(z, S_{\lambda_0}) + \operatorname{dist}(z, -S_{\lambda_0}) < 2 + \varepsilon$$

for all $z \in B_{E_{\lambda_0}}$. Choose $x^* = (x^*_\lambda) \in S_{E^*}$ with $x^*_\lambda = 0$ for all $\lambda \neq \lambda_0$, and let $S = S(x^*, \varepsilon)$. Then $x \in S$, and it is easy to see from the definition of E that

$$\operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon \tag{2.1}$$

for all $y \in S_E$. Thus E is a GL-space.

Now we deal with the "only if" part. For every $\lambda \in \Lambda$, fix $x_\lambda \in S_{E_\lambda}$ and $\varepsilon > 0$. Take $x = (x_\delta) \in S_E$ with $x_\delta = 0$ for all $\delta \neq \lambda$. Then $x \in S_E$, and thus there exists an $x^* = (x^*_\delta) \in S_{E^*}$ with $S = S(x^*, \varepsilon/2)$ such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon/2 \tag{2.2}$$

for all $y \in S_E$. Note that $x_\lambda \in S_\lambda = S(x^*_\lambda/\|x^*_\lambda\|, \varepsilon)$. To show that E_λ is a GL-space, it remains to check that for all $y_\lambda \in S_{E_\lambda}$,

$$\operatorname{dist}(y_\lambda, S_\lambda) + \operatorname{dist}(y_\lambda, -S_\lambda) < 2 + \varepsilon.$$

Now given $y_\lambda \in S_{E_\lambda}$, consider $y = (y_\delta) \in S_E$ with $y_\delta = 0$ for all $\delta \neq \lambda$. By (2.2), there are $u = (u_\delta) \in S$ and $v = (v_\delta) \in -S$ such that

$$\|y - u\| + \|y - v\| < 2 + \varepsilon/2.$$

The definition of E thus gives

$$\|y_\lambda - u_\lambda\| + \|y_\lambda - v_\lambda\| < 2 + \varepsilon/2.$$

Observe that $\|x^*_\lambda\| \geq x^*_\lambda(x_\lambda) > 1 - \varepsilon/2$, and therefore $\sum_{\delta\neq\lambda} \|x^*_\delta\| < \varepsilon/2$. So

$$x^*_\lambda(u_\lambda) > 1 - \varepsilon/2 - \sum_{\delta\neq\lambda} \|x^*_\delta\| > 1 - \varepsilon.$$

Similarly, $x^*_\lambda(-v_\lambda) > 1 - \varepsilon$. Hence E_λ is a GL-space.

In the l_∞-sum case, the "if" part follows from a slight modification of the c_0-case. For the "only if" part, the proof of the c_0-sum also works since if $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_\infty$, then for every λ_0, we may write $E = E_{\lambda_0} \oplus_\infty Z$ for a suitable Z.

In the l_1-sum case, let us prove the "if" part. Given $x = (x_\lambda) \in S_E$ and $\varepsilon > 0$, for each λ with $x_\lambda \neq 0$, there is a corresponding slice $S_\lambda = S(x^*_\lambda, \varepsilon)$ with $x^*_\lambda \in S_{E^*_\lambda}$ such that

$$x^*_\lambda(x_\lambda) > (1-\varepsilon)\|x_\lambda\| \quad \text{and} \quad \mathrm{dist}(z_\lambda, S_\lambda) + \mathrm{dist}(z_\lambda, -S_\lambda) < 2 + \varepsilon$$

for all $z_\lambda \in S_{E_\lambda}$. Then $x^* = (x^*_\lambda) \in S_{E^*}$ with $x^*_\lambda = 0$ whenever $x_\lambda = 0$, and the required slice satisfying (2.1) is $S(x^*, \varepsilon)$. Therefore E is a GL-space.

For the "only if" part, fix $x_\lambda \in S_{E_\lambda}$ and $0 < \varepsilon < 1/2$. Then $x = (x_\delta) \in S_E$ where $x_\delta = 0$ for all $\delta \neq \lambda$. Since E is a GL-space, there is an $x^* = (x^*_\delta) \in S_{E^*}$ with $S = S(x^*, \varepsilon/4)$ such that

$$x \in S \quad \text{and} \quad \mathrm{dist}(y, S) + \mathrm{dist}(y, -S) < 2 + \varepsilon/4$$

for all $y \in S_E$. We shall prove that the slice $S_\lambda = S(x^*_\lambda/\|x^*_\lambda\|, \varepsilon)$ is as desired, namely $x_\lambda \in S_\lambda$ and $\mathrm{dist}(y_\lambda, S_\lambda) + \mathrm{dist}(y_\lambda, -S_\lambda) < 2 + \varepsilon$ for all $y_\lambda \in S_{E_\lambda}$.

It is easily checked that $x_\lambda \in S_\lambda$. For every $y_\lambda \in S_{E_\lambda}$, since $y = (y_\delta)$ is in S_E where $y_\delta = 0$ for all $\delta \neq \lambda$, there are $u = (u_\delta) \in S$ and $v = (v_\delta) \in -S$ such that

$$\|y - u\| + \|y - v\| < 2 + \varepsilon/4. \tag{2.3}$$

It follows from the definition of E that

$$\begin{aligned}\|y - u\| + \|y - v\| &= \|y_\lambda - u_\lambda\| + \sum_{\delta\neq\lambda} \|u_\delta\| + \|y_\lambda - v_\lambda\| + \sum_{\delta\neq\lambda} \|v_\delta\| \\ &> \|y_\lambda - u_\lambda\| + 1 - \varepsilon/4 - \|u_\lambda\| + \|y_\lambda - v_\lambda\| + 1 - \varepsilon/4 - \|v_\lambda\| \\ &= \|y_\lambda - u_\lambda\| - \|u_\lambda\| + \|y_\lambda - v_\lambda\| - \|v_\lambda\| + 2 - \varepsilon/2.\end{aligned} \tag{2.4}$$

We deduce from (2.3) and (2.4) that

$$\|u_\lambda\| > 1/2 - \varepsilon/2 \quad \text{and} \quad \|v_\lambda\| > 1/2 - \varepsilon/2.$$

Hence

$$x^*_\lambda(u_\lambda) > 1 - \varepsilon/4 - \sum_{\delta\neq\lambda} \|u_\delta\| \geq 1 - \varepsilon/4 - 1 + \|u_\lambda\| \geq (1-\varepsilon)\|u_\lambda\|,$$

and similarly

$$x^*_\lambda(-v_\lambda) > (1-\varepsilon)\|v_\lambda\|.$$

So $w_\lambda = u_\lambda/\|u_\lambda\|$ and $t_\lambda = -v_\lambda/\|v_\lambda\|$ are in S_λ. The desired estimate

$$\|y_\lambda - w_\lambda\| + \|y_\lambda + t_\lambda\| < 2 + \varepsilon$$

which follows from (2.4) completes the proof. ∎

3. The Mazur–Ulam property for local-GL-spaces. The main aim of this section is to prove that a larger class of Banach spaces have the Mazur–Ulam property. We begin with a basic lemma.

LEMMA 3.1. *Let E be a GL-space. Then for every $x \in S_E$ and $\varepsilon > 0$ there exists a functional $f \in S_{E^*}$ with $x \in S(f, \varepsilon)$ such that for all $y \in S_E$ there are $y_1, y_2 \in S_E \cap S(f, \varepsilon)$ satisfying*

$$\|y - y_1\| + \|y + y_2\| < 2 + \varepsilon.$$

Proof. Since E is a GL-space, there exists a functional $f \in S_{E^*}$ with $S = S(f, \varepsilon/3)$ such that

$$x \in S \quad \text{and} \quad \operatorname{dist}(y, S) + \operatorname{dist}(y, -S) < 2 + \varepsilon/3$$

for all $y \in S_E$. Therefore, there exist $y_1, y_2 \in S$ such that

$$\|y - y_1\| + \|y + y_2\| < 2 + \varepsilon/3.$$

It is clear that $x \in S \subset S(f, \varepsilon)$. Let $\tilde{y}_i = y_i/\|y_i\|$ for $i = 1, 2$. Then

$$f(\tilde{y}_i) = f(y_i)/\|y_i\| > 1 - \varepsilon$$

and $\tilde{y}_i \in S_E \cap S(f, \varepsilon)$. Since $\|y_i - \tilde{y}_i\| = \big|\|y_i\| - 1\big| < \varepsilon/3$, we have

$$\|y - \tilde{y}_1\| + \|y + \tilde{y}_2\| < 2 + \varepsilon. \ \blacksquare$$

Now we give a proposition which is the key step to proving Theorem 3.8.

PROPOSITION 3.2. *Let E, F be Banach spaces, and let $T : S_E \to S_F$ be an isometry* (*not necessarily surjective*). *If E is a GL-space, then*

$$\|T(x) - \lambda T(y)\| \geq \|x - \lambda y\| \quad \text{for all } x, y \in S_E \text{ and } \lambda \geq 0.$$

Proof. Given $x, y \in S_E$ with $x \neq y$ and $\lambda > 0$, set

$$z = \frac{x - \lambda y}{\|x - \lambda y\|}.$$

By Lemma 3.1, given $\varepsilon > 0$, there exists $f \in S_{E^*}$ with $S = S(f, \varepsilon)$ such that $z \in S$ and there exist $x_1, y_1 \in S_E \cap S$ and $x_2, y_2 \in S_E \cap -S$ such that

$$\|x - x_1\| + \|x - x_2\| < 2 + \varepsilon \quad \text{and} \quad \|y - y_1\| + \|y - y_2\| < 2 + \varepsilon.$$

Then

$$2 - 2\varepsilon < f(x_1) - f(x) + f(x) - f(x_2) \leq \|x - x_1\| + \|x - x_2\| < 2 + \varepsilon.$$

This implies that

$$f(x_1) - f(x) \geq \|x - x_1\| - 3\varepsilon. \tag{3.1}$$

A similar analysis gives

$$f(y) - f(y_2) \geq \|y - y_2\| - 3\varepsilon. \tag{3.2}$$

Then there exists a functional $g \in S_{F^*}$ such that

$$g(T(x_1)) - g(T(y_2)) = \|T(x_1) - T(y_2)\| = \|x_1 - y_2\| > 2 - 2\varepsilon.$$

It follows that

$$g(T(x_1)) > 1 - 2\varepsilon \quad \text{and} \quad g(T(y_2)) < -1 + 2\varepsilon.$$

Thus by (3.1) and (3.2), we have

$$\begin{aligned} f(x) &\le f(x_1) - \|x - x_1\| + 3\varepsilon \le 1 - \|T(x) - T(x_1)\| + 3\varepsilon \\ &\le 1 - \big(g(T(x_1)) - g(T(x))\big) + 3\varepsilon \le g(T(x)) + 5\varepsilon \end{aligned}$$

and

$$\begin{aligned} f(y) &\ge f(y_2) + \|y - y_2\| - 3\varepsilon \ge -1 + \|T(y) - T(y_2)\| - 3\varepsilon \\ &\ge -1 + \big(g(T(y)) - g(T(y_2))\big) - 3\varepsilon \ge g(T(y)) - 5\varepsilon. \end{aligned}$$

As a consequence,

$$\begin{aligned} \|x - \lambda y\|(1-\varepsilon) < f(x - \lambda y) &\le g(T(x)) + 5\varepsilon - \lambda g(T(y)) + 5\lambda\varepsilon \\ &\le \|T(x) - \lambda T(y)\| + (5 + 5\lambda)\varepsilon. \end{aligned}$$

Since ε can be arbitrarily small, the proof is complete. ∎

THEOREM 3.3. *Every GL-space E has the MUP.*

Proof. Let F be a Banach space, and let $T : S_E \to S_F$ be a surjective isometry. We need to show that T can be extended to a linear surjective isometry from E onto F. We first claim that for all $x, y \in S_E$ and $\lambda \ge 0$.

$$\|T(x) - \lambda T(y)\| = \|x - \lambda y\|. \tag{3.3}$$

Otherwise by Proposition 3.2, there exist $\lambda_0 > 0$ and $x_0, y_0 \in S_E$ such that

$$\|T(x_0) - \lambda_0 T(y_0)\| > \|x_0 - \lambda_0 y_0\|. \tag{3.4}$$

Multiplying by $1/\lambda_0$ if necessary, we may assume that $\lambda_0 < 1$. Since $\|\lambda_0 T(y_0)\| = \lambda_0 < 1$, there exists $T(v) \in S_F$ with $v \in S_E$ such that $\lambda_0 T(y_0)$ belongs to the segment $(T(x_0), T(v))$ of B_F. By (3.4) and Proposition 3.2 we have

$$\begin{aligned} \|v - x_0\| = \|T(v) - T(x_0)\| &= \|T(v) - \lambda_0 T(y_0)\| + \|\lambda_0 T(y_0) - T(x_0)\| \\ &> \|v - \lambda_0 y_0\| + \|\lambda_0 y_0 - x_0\| \ge \|v - x_0\|, \end{aligned}$$

a contradiction.

Now we may define the required extension $\widetilde{T}$ of T by

$$\widetilde{T}(x) = \begin{cases} \|x\| T(x/\|x\|) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easily seen from (3.3) that $\widetilde{T} : E \to F$ is a surjective isometry whose restriction to the unit sphere S_E is just T. The Mazur–Ulam theorem hence shows that $\widetilde{T}$ is linear as desired. The proof is complete. ∎

Note that the technique in the proof of Theorem 3.3 is still valid in a more general case. We now state a result that will be of use later.

PROPOSITION 3.4. *Let E, F be Banach spaces, and let $T : S_E \to S_F$ be a surjective isometry such that*

$$\|T(x) - \lambda T(y)\| \ge \|x - \lambda y\| \qquad \textit{for all } x, y \in S_E \textit{ and } \lambda \ge 0.$$

Then there exists a linear isometry $\widetilde{T} : E \to F$ such that $T = \widetilde{T}|_{S_E}$.

Now we introduce a class of spaces called local-GL-spaces (including GL-spaces and lush spaces) which have the MUP. This definition is a weakening of the definition of GL-space.

DEFINITION 3.5. A Banach space E is said to be a *local-GL-space* if for every separable subspace $X \subset E$, there is a GL-subspace $Y \subset E$ such that $X \subset Y \subset E$.

EXAMPLE 3.6. *GL-spaces are local-GL-spaces.*

The equivalent definition of lush space [BKMM, Theorem 4.2] proves the following.

EXAMPLE 3.7. *Lush spaces are local-GL-spaces.*

We now present the main result of this section.

THEOREM 3.8. *Every local-GL-space has the MUP.*

Proof. Let E be a local-GL-space, F a Banach space and $T : S_E \to S_F$ a surjective isometry. We next show that T can be extended to a linear surjective isometry from E onto F. By Proposition 3.4, it is enough to show that

$$\|T(x) - \lambda T(y)\| \geq \|x - \lambda y\| \tag{3.5}$$

for every $x, y \in S_E$ and $\lambda > 0$. Now, fix $x, y \in S_E$ and $\lambda > 0$. Let $X = \operatorname{span}(x, y)$ and consider a GL-space $Y \subset E$ such that $X \subset Y$. We consider T as an isometry from S_Y into S_F. As Y is a GL-space, Proposition 3.2 gives (3.5), as desired. ■

We emphasize two evident consequences of the above theorem.

COROLLARY 3.9. *Every lush space has the MUP.*

COROLLARY 3.10. *Every C-rich subspace of $C(K)$ has the MUP.*

By the following properties, we can get more examples of spaces having the MUP.

PROPOSITION 3.11. *If E is a local-GL-space, then $C(K, E)$ is a local-GL-space.*

Proof. Let X be a separable subspace of $C(K, E)$. We shall prove that the set

$$E_X = \bigcup_{t \in K} \{f(t) : f \in X\}$$

is a separable subset of E. Indeed, let $\{f_n\}$ be a dense sequence of X. Given $n, m \geq 1$ and $s \in K$, set $V_{s,m,n} = \{t \in K : \|f_n(t) - f_n(s)\| < 1/m\}$. The compactness of K implies that there is a finite subset $\{s_i^{m,n} : i = 1, \dots, k_{m,n}\}$ of K such that $K = \bigcup_{i=1}^{k_{m,n}} V_{s_i^{m,n},m,n}$. Then it is an elementary check that the set

$$M = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{f_n(s_i^{m,n}) : i = 1, \dots, k_{m,n}\}$$

is a dense subset of E_X. It follows that $N_X = \overline{\operatorname{span}\{E_X\}}$ is a separable subspace of E. Note that E is a local-GL-space. So we may find a GL-space M_X such that $N_X \subset M_X \subset E$.

Let $Y = C(K, M_X)$. Then $X \subset Y$, and Theorem 2.10 shows that Y is a GL-space. This completes the proof. ∎

Corollary 3.12. *Let E be a local-GL-space and K be a compact Hausdorff space. Then $C(K, E)$ has the MUP.*

The proof of Theorem 2.11 can be adapted to yield a characterization of the c_0-, l_1-sums of lush spaces in both real and complex cases, which is a special case of the results in [P]. We next give an analogue for local-GL-spaces. The proof of this result is routine based on Theorem 2.11.

Proposition 3.13. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces, and let $E = [\bigoplus_{\lambda\in\Lambda} E_\lambda]_F$ where $F = c_0$, l_∞ or l_1. Then E is a local-GL-space if and only if E_λ is a local-GL-space for every $\lambda \in \Lambda$.*

Proof. Let P_λ be the projection of E onto E_λ, and let I_λ be the injection of E_λ into E.

We first show the "if" part. Fix a separable subspace X of E. Then $P_\lambda(X) \subset E_\lambda$ is separable. Since E_λ is a local-GL-space, there is a GL-space $Y_\lambda \subset E_\lambda$ such that $P_\lambda(X) \subset Y_\lambda$. Then $Y = [\bigoplus_{\lambda\in\Lambda} Y_\lambda]_F$ containing X is a subspace of E. Moreover it follows from Theorem 2.11 that Y is a GL-space, and hence E is a local-GL-space.

Now let us deal with the "only if" part. Given $\lambda \in \Lambda$, let X_λ be a separable subspace of E_λ. Since E is a local-GL-space, there is a GL-space Y such that $I_\lambda(X_\lambda) \subset Y \subset E$. Note from Theorem 2.11 that $Y_\lambda = P_\lambda(Y)$ is a GL-space such that $X_\lambda \subset Y_\lambda \subset E_\lambda$. Thus E_λ is a local-GL-space. ∎

As an immediate consequence of the proposition above, we obtain:

Corollary 3.14. *Let $\{E_\lambda : \lambda \in \Lambda\}$ be a family of local-GL-spaces. Then the space $E = [\bigoplus_\lambda E_\lambda]_F$, where $F = c_0$, l_1 or l_∞, has the MUP.*

Throughout this paper, we can see that the geometric properties, isometric extension, and even the numerical index on unit spheres have harmonious inner relationship and may provide a possible way to solve the isometric extension problem in more general cases. Note that there exist examples of Banach spaces with numerical index 1 which are not lush spaces (see [KMMS, Remark 4.2]). Then the first natural question to ask is the following:

PROBLEM 3.15. *Does every Banach space with numerical index* 1 *have the MUP?*

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