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Lineability and spaceability on vector-measure spaces

by

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Abstract. It is proved that if X is infinite-dimensional, then there exists an infinite-dimensional space of X-valued measures which have infinite variation on sets of positive Lebesgue measure. In term of spaceability, it is also shown that $ca(\mathcal{B},\lambda,X) \setminus M_\sigma$, the measures with non-$\sigma$-finite variation, contains a closed subspace. Other considerations concern the space of vector measures whose range is neither closed nor convex. All of those results extend in some sense theorems of Muñoz Fernández et al. [Linear Algebra Appl. 428 (2008)].

1. Brief introduction and results. We begin by recalling the following relatively new concepts related to the "algebraic size" of subsets of Banach spaces.

DEFINITION 1.1 (Gurariy, 1991). A subset M of a Banach space is said to be

- *n-lineable* if $M \cup \{0\}$ contains an n-dimensional vector subspace;
- *lineable* if $M \cup \{0\}$ contains an infinite-dimensional vector subspace;
- *dense-lineable* if $M \cup \{0\}$ contains an infinite-dimensional dense vector subspace;
- *spaceable* if $M \cup \{0\}$ contains an infinite-dimensional closed vector subspace.

Let $I = [0,1]$ be the unit interval and let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of I. Also, let λ be the Lebesgue measure on I. For a Banach space X we let $ca(\mathcal{B},\lambda,X)$ stand for the space of all vector measures $\mu : \mathcal{B} \to X$ which are countably additive and absolutely continuous with respect to λ. Then $ca(\mathcal{B},\lambda,X)$ is a Banach space endowed with the norm

$$\|\mu\|_{ca} = \sup_{A\in\mathcal{B}} \|\mu(A)\|_X.$$

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Let us also recall that the *variation* of a vector measure is defined by

$$|\mu|(A) = \sup\Big\{\sum_{i=1}^{n} \|\mu(A_i)\| : A_i \text{ pairwise disjoint with } \bigcup_{i=1}^{n} A_i = A\Big\}.$$

We shall use the following notation:

$$cabv(\mathcal{B}, \lambda, X) = \{\mu \in ca(\mathcal{B}, \lambda, X) : |\mu| \text{ is finite}\}.$$

The set $cabv(\mathcal{B}, \lambda, X)$ endowed with the variation norm $|\cdot|$ (i.e., the norm is $|\mu|(I)$) is a Banach space.

In [MPPS] the following is proved.

THEOREM 1.2. *Let $(\mathcal{B}, \lambda)$ be the Lebesgue measure space on the unit interval, and let $1 \leq p < \infty$. Then the set of ℓ_p-valued measures with relatively compact range such that their variation measures take the value infinity on every non-null set is lineable in $ca(\mathcal{B}, \lambda, \ell_p)$.*

We will prove that the above result holds for any infinite-dimensional Banach space in place of ℓ_p.

Following [JK], let us denote by M_σ the subspace of $ca(\mathcal{B}, \lambda, X)$ of all measures μ such that $|\mu|$ is σ-finite. Let ρ be the metrizable vector topology on M_σ defined by the base $\{V_n : n \in \mathbb{N}\}$, where

$$V_n = \left\{\mu \in ca(\mathcal{B}, \lambda, X) : \begin{array}{l} \|\mu\| \leq 2^{-n} \text{ and there exists } E \in \mathcal{B} \text{ with} \\ \lambda(E) \leq 2^{-n} \text{ and } |\mu|(I \setminus E) \leq 2^{-n} \end{array}\right\}.$$

It is easy to see that (M_σ, ρ) is complete.

At this stage, we are ready to show the following.

THEOREM 1.3. *Let X be any infinite-dimensional Banach space and let $(\mathcal{B}, \lambda)$ be the Lebesgue measure space on the unit interval. Then the set of X-valued measures with relatively compact range such that their variation measures take the value infinity on every non-null set is lineable in $ca(\mathcal{B}, \lambda, X)$.*

Proof. Let $(A_n)_n$ be a sequence of Borel sets in I such that

- $I = \bigcup_n A_n$;
- $A_n \cap A_m = \emptyset$ for $n \neq m$;
- $\lambda(A_n) > 0$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let P_n be the subspace of $ca(\mathcal{B}, \lambda, X)$ of all simple measures μ of the kind

$$\mu(A) = \sum_{\text{finite}} \lambda(A \cap A_n) x_k, \quad A \in \mathcal{B}.$$

By the Dvoretzky–Rogers trick, it is not hard to show that

$$\overline{P_n}^{ca(\mathcal{B},\lambda,X)} \not\subseteq M_\sigma \quad (\text{see [JK]}).$$

Therefore, if we pick $\mu_n \in \overline{P_n}^{ca(\mathcal{B},\lambda,X)} \setminus M_\sigma$, we find that

- all the μ_n's have relatively compact range and their variation measures take the value infinity on every non-null set (see [JK, Theorem 2]),
- all linear combinations of the μ_n's have relatively compact range and their variation measures take the value infinity on every non-null set, and
- the μ_n's are linearly independent (because they have disjoint supports). ∎

Now, we would like to deal with the following question.

QUESTION. Is $ca(\mathcal{B},\lambda,X) \setminus M_\sigma$ spaceable?

In [KT] (see also [D]) the following remarkable result is proved.

THEOREM 1.4. *Let Z_n $(n \in \mathbb{N})$ be Banach spaces and X a Fréchet space. Let $T_n : Z_n \to X$ be continuous linear operators and Y the linear span of $\bigcup_n T_n(Z_n)$. If Y is not closed in X, then the complement $X \setminus Y$ is spaceable.*

Before going on, let us recall some standard concepts. For a sequence of Banach spaces $(X_n, \|\cdot\|_n)$ such that all X_n's are (isomorphic to) a closed subspace of a bigger Banach space $\mathcal{X}$, consider

$$\Big(\bigoplus_{n\in\mathbb{N}} X_n\Big)_c = \Big\{x_n \in X_n : \lim_{n\to\infty} x_n \text{ exists in } \mathcal{X}\Big\},$$

endowed with the norm

$$\|(x_n)_n\| = \sup_n \|x_n\|_n.$$

Then $(\bigoplus_{n\in\mathbb{N}} X_n)_c$ is a Banach space.

We are ready to state the main theorem of this note.

THEOREM 1.5. *If X is infinite-dimensional, then $ca(\mathcal{B},\lambda,X) \setminus M_\sigma$ is spaceable.*

Proof. Let us fix a sequence $(A_n)_n \subseteq \mathcal{B}$ such that

- $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$,
- $\lambda(A_{n+1} \setminus A_n) > 0$ for each $n \in \mathbb{N}$,
- $\bigcup_{n\in\mathbb{N}} A_n = I$.

Let $\Sigma_n = \{E \cap A_n : E \in \mathcal{B}\}$ be the σ-algebra generated by A_n. Since, for each $n \in \mathbb{N}$, we can see $(cabv(\Sigma_n,\lambda,X), \|\cdot\|_{ca})$ as a closed subspace of $(cabv(\mathcal{B},\lambda,X), \|\cdot\|_{ca})$ (via the natural map that associates to each $\mu \in (cabv(\Sigma_n,\lambda,X), \|\cdot\|_{ca})$ the measure that is equal to μ on Σ_n and zero outside A_n), we can consider the Banach space

$$\Big(\bigoplus_{n\in\mathbb{N}} (cabv(\Sigma_n,\lambda,X,\|\cdot\|_{ca})\Big)_c.$$

Let us define

$$\mathcal{M} = \Big\{(\mu_n)_n \in \Big(\bigoplus_{n\in\mathbb{N}}(cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca})\Big)_c : \mu_{n+1}|_{\Sigma_n} = \mu_n\Big\}.$$

Let us show that $\mathcal{M}$ is a closed subspace of $(\bigoplus_{n\in\mathbb{N}}(cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca}))_c$.

Let $(\overline{\mu}^p)_p \subseteq \mathcal{M}$ (where $\overline{\mu}^p = (\mu_n^p)_n$ for each $p \in \mathbb{N}$) be a sequence such that

$$\lim_{p\to\infty} = \overline{\mu} = (\mu_n)_n \in \Big(\bigoplus_{n\in\mathbb{N}}(cabv(\Sigma_n, \lambda, X), \|\cdot\|_{ca})\Big)_c;$$

explicitly,

$$\sup_n \sup_A \|\mu_n^p(A) - \mu_n(A)\| \xrightarrow{p\to\infty} 0.$$

Let $A \in \Sigma_n$. Since $\mu_{n+1}^p(A) = \mu_n^p(A)$ we have

$$\|\mu_{n+1}(A) - \mu_n(A)\| \le \|\mu_{n+1}^p(A) - \mu_{n+1}(A)\| + \|\mu_n^p(A) - \mu_n(A)\| \xrightarrow{p\to\infty} 0.$$

Namely, $\overline{\mu} \in \mathcal{M}$. Therefore, $\mathcal{M}$ is a Banach space.

Let us define

$$T : \mathcal{M} \to (ca(\mathcal{B}, \lambda, X),\ \|\cdot\|_{ca})$$

by

$$T((\mu_n)_n)(A) = \lim_{n\to\infty} \mu_n(A \cap A_n) \quad \forall A \in \mathcal{B}.$$

Let us prove that T is a continuous linear operator such that $T(\mathcal{M}) = M_\sigma$.

First, let us note that T is well defined. Indeed, let $(E_k)_k \subseteq \mathcal{B}$ be a disjoint sequence of sets. Then

$$\begin{aligned} T((\mu_n)_n)\Big(\bigcup_k E_k\Big) &= \lim_{n\to\infty} \mu_n\Big(\Big(\bigcup_k E_k\Big) \cap A_n\Big) = \lim_{n\to\infty} \mu_n\Big(\bigcup_k (E_k \cap A_n)\Big) \\ &= \lim_{n\to\infty} \sum_k \mu_n(E_k \cap A_n) = \sum_k \lim_{n\to\infty} \mu_n(E_k \cap A_n) \\ &= \sum_k T((\mu_n)_n)(E_k), \end{aligned}$$

since we have convergence with respect to the semivariation norm $\|\cdot\|_{ca}$. Moreover, it is evident that $T((\mu_n)_n)$ is λ-continuous.

Linearity follows directly from the definition.

For continuity,

$$\begin{aligned} \|T((\mu_n)_n)\|_{ca} &= \sup_{A\in\mathcal{B}} \|T((\mu_n)_n)(A)\| = \sup_{A\in\mathcal{B}} \|\lim_{n\to\infty} \mu_n(A \cap A_n)\| \\ &\le \sup_{A\in\mathcal{B}} \lim_{n\to\infty} \|\mu_n(A \cap A_n)\| = \sup_{n\in\mathbb{N}} \sup_{A\in\mathcal{B}} \|\mu_n(A \cap A_n)\| \\ &= \|(\mu_n)_n\|_{\mathcal{M}}. \end{aligned}$$

From the equality $T(\mathcal{M}) = M_\sigma$, let us first note that $T((\mu_n)_n)$ is a measure of σ-finite variation. Indeed, by construction, for each $s \in \mathbb{N}$,

$$\begin{aligned} |T((\mu_n)_n)|(A_s) &\le \lim_{n\to\infty} |\mu_n|(A_s) \\ \text{(by the definition on } \mathcal{M}) \quad &= |\mu_s|(A_s) \\ &< \infty. \end{aligned}$$

Moreover, if $\mu \in M_\sigma$, since $|\mu|$ is σ-finite, consider an increasing sequence $(C_n)_n$ such that

$$\bigcup_n C_n = I \quad \text{and} \quad |\mu|(C_n) < \infty \quad \text{for all } n \in \mathbb{N};$$

now, take $\mu_n \in cabv(\Sigma_n, \lambda, x)$ defined by $\mu_n(A \cap A_n) = \mu(A \cap A_n \cap C_n)$. Then, by construction, $(\mu_n)_n \in \mathcal{M}$ and we have

$$T((\mu_n)_n) = \mu.$$

It was already observed in [JK] that M_σ, with respect to the complete metric ρ, is not closed in $(ca(\mathcal{B}, \lambda, X), \|\cdot\|_{ca})$. Since the topology generated by ρ is stronger than the norm topology of $\|\cdot\|_{ca}$, it follows that $(M_\sigma, \|\cdot\|_{ca})$ is not closed in $(ca(\mathcal{B}, \lambda, X), \|\cdot\|_{ca})$ either. The proof is concluded by simply applying Theorem 1.4 above. ∎

Let us recall the following definition (see [H]).

Definition 1.6. Let (Ω, Σ) be a measurable space, λ a positive measure on Σ, and X an infinite-dimensional Banach space. A measure $\mu \in ca(\lambda, X)$ is said to be *injective* when for each $\phi, \psi \in L_\infty(\lambda)$ the following condition holds:

$$\text{if} \quad \int \phi \, d\mu = \int \psi \, d\mu \quad \text{then} \quad \phi = \psi \quad \lambda\text{-a.e.}$$

In [MPPS], using an elegant construction, the authors were able to show the following

Theorem 1.7. *Let λ be the Lebesgue measure on the Borel sets in $[0, 1]$ and X an infinite-dimensional Banach space. Then the set of injective measures is lineable in $ca(\lambda, X)$.*

Here we are interested in the spaceability of the set of injective measures. In [W], A. Wilansky proved the following general criterion for spaceability:

Theorem 1.8. *Let E be a Banach space. If F is a closed infinite-codimensional vector subspace of a Banach space E, then $E \setminus F$ is spaceable.*

We will use this criterion to prove the following.

Theorem 1.9. *Let λ be the Lebesgue measure on the Borel sets in $[0, 1]$, and X be an infinite-dimensional Banach space. Then the set of injective measures is spaceable in $ca(\lambda, X)$.*

Before providing the proof, we need the following lemma.

LEMMA 1.10. *The space*

$$\mathcal{NI} = \{\mu \in ca(\lambda, X) : \mu \text{ is not injective}\}$$

is a closed subspace of $ca(\lambda, X)$.

Proof. We will provide two different proofs:

First proof. To show the closedness it is enough to note the following: a measure $\mu \in ca(\lambda, X)$ is injective if and only if the integral operator associated to μ,

$$T_\mu : L_\infty(\lambda) \to X, \quad T_\mu(f) = \int f\, d\mu,$$

is injective.

Suppose that $(\mu_n)_n \subseteq \mathcal{NI}$ converges to $\mu \in ca(\lambda, X)$, and μ is injective. Then,

$$L_\infty(\lambda)^* = \overline{T_\mu^*(X^*)}^{\text{weak}^*}$$

Since $(\mu_n)_n$ converges to μ, we have

$$\overline{T_\mu^*(X^*)}^{\text{weak}^*} \subseteq \bigcup_{n\in\mathbb{N}} \overline{T_{\mu_n}^*(X^*)}^{\text{weak}^*}.$$

Thus, there must exist $\overline{n} \in \mathbb{N}$ such that

$$\text{weak}^*\text{-int}(T_{\mu_{\overline{n}}}^*(X^*)) \neq \emptyset.$$

Since $T_{\mu_{\overline{n}}}^*(X^*)$ is a vector subspace, that would imply

$$\overline{T_{\mu_{\overline{n}}}^*(X^*)}^{\text{weak}^*} = L_\infty^*(\lambda),$$

contradicting the fact that $\mu_{\overline{n}} \in \mathcal{NI}$.

Second proof. It is well known that $\mu \in \mathcal{NI}$ if and only if for each $B \in \mathcal{B}$, $\{\mu(A \cap B) : A \in \mathcal{B}\}$ is convex and weakly compact (see [K]). However, the limit of a sequence of non-empty convex closed sets in the Hausdorff metric is still a non-empty convex closed (see [KT, 4.3.11]). Moreover, since $\mu_n \xrightarrow{n\to\infty} \mu$ implies $\{\mu_n(A \cap B) : A \in \mathcal{B}\} \xrightarrow{n\to\infty} \{\mu(A \cap B) : A \in \mathcal{B}\}$ in the Hausdorff metric, we deduce that if each $\{\mu_n(A \cap B) : A \in \mathcal{B}\}$ is convex, weakly compact, and

$$\mu_n \xrightarrow{n\to\infty} \mu,$$

then $\{\mu(A \cap B) : A \in \mathcal{B}\}$ is convex and weakly compact too.

From the above, it follows that $\mathcal{NI}$ is a closed subspace of $ca(\lambda, X)$. ∎

Proof of Theorem 1.9. From Lemma 1.10, we know that $\mathcal{NI}$ is a closed subspace of $ca(\lambda, X)$ of infinite codimension. To show that the quotient $ca(\lambda, X)/\mathcal{NI}$ is infinite-dimensional, it is sufficient to use a similar construction to the proof of [MPPS, Theorem 2.4]. Then Theorem 1.8 applies. ∎

Since it is well known that every injective measure has range neither closed nor convex, we finally obtain the following corollary.

COROLLARY 1.11. *Let λ be the Lebesgue measure on the Borel sets in $[0,1]$, and X an infinite-dimensional Banach space. Then the set of measures whose range is neither closed nor convex is spaceable in $ca(\lambda, X)$.*

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