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Continuous rearrangements of the Haar system in H_p for $0 < p < \infty$

by

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Abstract. We prove three theorems on linear operators $T_{\tau,p} : H_p(\mathcal{B}) \to H_p$ induced by rearrangement of a subsequence of a Haar system. We find a sufficient and necessary condition for $T_{\tau,p}$ to be continuous for $0 < p < \infty$.

1. Introduction. Denote by $\mathcal{D}$ the collection of all dyadic intervals in $[0,1]$. The Lebesgue measure on $[0,1]$, the cardinality of the set or the absolute value, depending on the context, will be denoted by the same $|\cdot|$. With each interval $I \in \mathcal{D}$, $I = [k/2^n, (k+1)/2^n)$, we associate the Haar function $h_{I,p}$,

$$h_{I,p}(t) = \begin{cases} 2^{n/p} & \text{if } 2k/2^{n+1} \le t < (2k+1)/2^{n+1}, \\ -2^{n/p} & \text{if } (2k+1)/2^{n+1} \le t < 2(k+1)/2^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

We define H_p as the space of all distributions $f = \sum a_{I,p} h_{I,p}$ for which

$$\|f\|_{H_p} = \Big[\int_0^1 \Big(\sum_{I \in \mathcal{D}} |a_{I,p} h_{I,p}(t)|^2 \Big)^{p/2} \Big]^{1/p} < \infty.$$

For $1 \le p < \infty$, $\|\cdot\|_{H_p}$ is actually a norm. When $0 < p < 1$ the above expression defines a quasi-norm. It is known ([W1]) that H_p spaces are isomorphic to classical Hardy spaces of analytic functions on the unit disc. Suppose $\mathcal{B} \subset \mathcal{D}$. Then $H_p(\mathcal{B})$ denotes the closed linear span of $\{h_{I,p} : I \in \mathcal{B}\}$ in H_p. For a one-to-one map $\tau : \mathcal{B} \to \mathcal{D}$ it is of interest to consider the operators $T_{\tau,p} : H_p(\mathcal{B}) \to H_p$, given by

$$T_{\tau,p}(h_{I,p}) = h_{\tau(I),p} \quad (I \in \mathcal{B}).$$

After [Mu] such operators will be called *rearrangements of the Haar system* (or subsystem), for short rearrangements in H_p. In this paper we describe

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the continuous rearrangements $T_{\tau,p} : H_p(\mathcal{B}) \to H_p$ for $0 < p < \infty$. This allows us for example to characterize the isomorphisms (isomorphic rearrangements) of $H_p(\mathcal{B})$ induced by τ. We restrict the discussion to H_p spaces for $0 < p < \infty$. However, for $p > 1$, $H_p = L_p$ with equivalent norms, so in this case the results presented here apply to L_p. The operators $T_{\tau,p}$ in L_p, $1 < p < \infty$, for τ length preserving were investigated by Semyonov [Sem]. The operators $T_{\tau,\infty}$ in BMO and $T_{\tau,p}$ in H_p for $1 \le p < 2$ and $2 < p < \infty$, for arbitrary injection τ, were thoroughly studied by Müller [Mu]. Geiss et al. [GMP] described extrapolation of rearrangement operators in H_p for $0 < p < 2$, namely they showed that for $0 < s < p < 2$ and $0 < \theta < 1$ satisfying $1/p = (1-\theta)/s + \theta/2$ there exists a constant $c > 0$, depending only on s and p, such that

$$\|T_{\tau,s} : H_s \to H_s\|^{1-\theta} \le \|T_{\tau,p} : H_p \to H_p\|$$

(the reverse inequality is rather standard and follows by interpolation). Thus results from [Mu] were extended in [GMP] to the case $0 < p < 1$.

For $\mathcal{L} \subset \mathcal{D}$ and $I \in \mathcal{D}$ we use $\mathcal{L} \cap I$ to denote the family of all intervals from the family $\mathcal{L}$ contained in I; $\mathcal{Q}(I)$ denotes $\mathcal{D} \cap I$. All intervals from $\mathcal{D}$ of length 2^{-m} will be denoted by $\mathcal{D}_m$. In other words, $\mathcal{D}_m = \mathcal{D} \cap \{I \subset [0,1] : |I| = 2^{-m}\}$. For $\mathcal{L} \subset \mathcal{D}$, the set of all maximal intervals in $\mathcal{L}$ with respect to inclusion will be denoted by $\mathfrak{max}(\mathcal{L})$. After [Mu] we say that $\mathcal{L} \subset \mathcal{D}$ satisfies the *M-Carleson condition* if

$$\sup_{J\in\mathcal{D}} \frac{1}{|J|} \sum_{I\in\mathcal{L}\cap J} |I| \le M. \tag{1}$$

We use $[\![\mathcal{L}]\!]$ to denote the infimum of the constants M that satisfy (1) and we call it the *Carleson constant* of the family $\mathcal{L}$. If there exists $N < \infty$ such that $[\![\tau^{-1}(\mathcal{L})]\!] \le N[\![\mathcal{L}]\!]$ for each $\mathcal{L} \subset \tau(\mathcal{B})$, we say that τ^{-1} *preserves the Carleson constant*, and we denote by $[\![\tau^{-1}]\!]$ the infimum of such constants N. Similarly, we say that τ *preserves the Carleson constant* if there exists $N < \infty$ such that $[\![\tau(\mathcal{L})]\!] \le N[\![\mathcal{L}]\!]$ for every $\mathcal{L} \subset \mathcal{B}$, and we define $[\![\tau]\!]$ as the infimum of such N. We will see that for $0 < p < 2$ the operator $T_{\tau,p}$ is continuous if and only if τ^{-1} preserves the Carleson constant, while for $2 < p < \infty$ the operator $T_{\tau,p}$ is continuous if and only if τ preserves the Carleson constant. These results appeared in [Mu] for rearrangements in BMO and H_p for $1 \le p < 2$ and $2 < p < \infty$, and were then extended in [GMP] to $0 < p < 1$. Our main result is proved with the use of atomic decomposition of H_p ([CoW], [We]).

2. A sufficient condition. We now give a sufficient condition for $T_{\tau,p}$ to be continuous.

THEOREM 1. *Let $0 < p < 2$. Assume $\mathcal{B}$, $\mathcal{C}$ are families of dyadic intervals from $\mathcal{D}$ such that there exists a bijection $\tau : \mathcal{B} \to \mathcal{C}$. If τ^{-1} preserves* *the Carleson constant, then the operator*

$$T_{\tau,p} : H_p(\mathcal{B}) \to H_p(\mathcal{C})$$

induced by τ *is continuous.*

Proof. We divide the proof into six parts.

1. It suffices to show that for some finite constant $C < \infty$.

$$\|T_{\tau,p}(x)\|_{H_p} \le C\|x\|_{H_p}$$

for all $x \in H_p(\mathcal{B})$ of norm $\|x\|_{H_p} \le 1$ with finite Haar expansion $x = \sum a_{I,p} h_{I,p}$. Indeed, if $x = \sum_{I\in\mathcal{R}} a_{I,p} h_{I,p}$ and $\|T_{\tau,p}(x)\|_{H_p} > N$, then (by simple approximation) there exists a finite family $\mathcal{R}_1 \subset \mathcal{R}$ such that $\|T(\sum_{J\in\mathcal{R}_1} a_{J,p} h_{J,p})\|_{H_p} > N$. By [We, Theorem 2.2] we may also assume that x is a simple $(2, p, \infty)$ atom (see [We] for definition). Moreover, we shall show in the next part that we can assume that the quadratic function $S(x) = (\sum a_{I,p}^2 h_{I,p}^2)^{1/2}$ is bounded,

$$1/8 \le S(x)[t] \le 1 \quad \text{for } t \in [0,1]. \tag{2}$$

2. In order to justify (2) we use an atomic decomposition of x similar to the one used in the proof of [We, Theorem 2.2]. Let $\{h_{i,p}\}_i$ denote the Haar functions $\{h_{I,p}\}$ numbered according to the Haar order. For $s \in \mathbb{N}$ set

$$d_{s,p}(x) = \begin{cases} a_{I,p} h_{I,p} & \text{if } h_{s,p} = h_{I,p} \text{ and } a_{I,p} \neq 0, \\ 0 & \text{if } h_{s,p} = h_{J,p} \text{ and } a_{J,p} = 0. \end{cases}$$

We define stopping times $\nu_{k,p}$ for $k \in \mathbb{Z}$ by

$$\nu_{k,p}(t) = \inf\Big\{n \in \mathbb{N} : \Big(\sum_{s=0}^{n+1} d_{s,p}(x)^2[t]\Big)^{1/2} > 2^k\Big\}.$$

Now x has an atomic decomposition

$$x = \sum_k c_{k,p} A_{k,p} \tag{3}$$

where

$$c_{k,p} = 3 \cdot 2^k |\{t \in [0,1] : S(x)[t] > 2^k\}|^{1/p}$$

and $A_{k,p}$ are simple $(2, p, \infty)$ atoms described by

$$A_{k,p} = \sum_{s\ge0} \chi(\{t : \nu_{k,p}(t) < s \le \nu_{k+1,p}(t)\}) \cdot d_{s,p}(x) \cdot c_{k,p}^{-1}$$

with the property

$$C_p^{-1}\Big(\sum_k |c_{k,p}|^p\Big)^{1/p} \le \|x\|_{H_p} \le \Big(\sum_k |c_{k,p}|^p\Big)^{1/p}. \tag{4}$$

Indeed, if we decompose x using (3), then applying the Abel rearrangement ([We]) we get

$$(5)\qquad \sum_k |c_{k,p}|^p = 3^p \sum_k (2^p)^k |\{t : S^p(x)[t] > (2^p)^k\}|$$
$$= \frac{3^p}{2^p-1} \sum_k [(2^p)^{k+1} - (2^p)^k] |\{t : S^p(x)[t] > (2^p)^k\}|$$
$$\le \frac{6^p}{2^p-1} \sum_k (2^p)^{k-1} |\{t : (2^p)^{k-1} < S^p(x)[t] \le (2^p)^k\}|.$$

3. For $x = \sum x_{I,p} h_{I,p}$ the collection of all dyadic intervals I from $\mathcal{D}$ for which $x_{I,p} \neq 0$ in this Haar decomposition of x will be called the *Haar support* of x. Now we will construct an atomic decomposition of some vector x' such that

$$(6)\qquad \text{Haar support of } x' = \text{Haar support of } x.$$

We assume for the moment that $A_{k,p} = 0$ for $k < 0$. For $t \in [0,1]$ let

$$k_{0,p}(t) = \min\{k : A_{k,p}[t] \neq 0\}$$

and $B_{k_{0,p}(t)} = A_{k_{0,p}(t)}$. Suppose $k_{n-1,p}(t)$ has been defined for some $n \ge 1$. Then we define

$$k_{n,p}(t) = \min\{k > k_{n-1,p}(t) : A_{k,p}[t] \neq 0\}$$

if it exists. Let $I_{k_{n,p}(t)}$ denote the longest interval in the Haar support of $A_{k_{n,p}(t)}$ containing t. We put

$$(7)\qquad B_{k_{n,p}(t)}[t] = c_{k_{n,p}(t)} A_{k_{n,p}(t)}[t] + 2^{k_{n,p}(t)-2} \frac{h_{I_{k_{n,p}(t)}}[t]}{|h_{I_{k_{n,p}(t)}}[t]|}.$$

We can see that

$$2^{k_{n,p}(t)-2} \le S(B_{k_{n,p}(t)}[t]) \le 2^{k_{n,p}(t)+1} \qquad \text{for } t \in \operatorname{supp}(B_{k_{n,p}(t)})$$

so $B_{k_{n,p}(t)}$ multiplied by $2^{-(k_{n,p}(t)+1)}$ satisfies the boundedness condition (2) on its support. Define $B_{k,p}$ (to get rid of t in the index) as follows:

$$B_{k,p}[t] = B_{k_{n,p}(t)}[t] \quad \text{ if } k_{n,p}(t) = k.$$

It is easy to check that $\frac{1}{2}B_{k,p}$ are $(2,p,\infty)$ atoms. To specify x' mentioned in (6), we set

$$x' = \sum_k B_{k,p}.$$

Notice that each $B_{k,p}$ can be easily decomposed into a sum of simple atoms $B_{k_{n,p}(t)}$ (where $k_{n,p}(t) = k$) with pairwise disjoint supports being dyadic intervals. If we can show that

$$(8)\qquad \|T_{\tau,p}(B_{k,p})\|_{H_p} \le C \|B_{k,p}\|_{H_p}$$

for every k and $C = C([\![\tau^{-1}]\!]) < \infty$, then for C_p from (4),

$$\|T_{\tau,p}(x)\|_{H_p}^p \overset{(7)}{\le} \Big\|T_{\tau,p}\Big(\sum_k B_{k,p}\Big)\Big\|_{H_p}^p \le \sum_k \|T_{\tau,p}(B_{k,p})\|_{H_p}^p \overset{(8)}{\le} C^p \sum_k \|B_{k,p}\|_{H_p}^p \overset{(4),(7)}{\le} C^p \cdot C_p^p \cdot 2^p \|x\|_{H_p}^p \tag{9}$$

and we are done.

4. Suppose $x \in H_p(\mathcal{B})$, $1/8 \le S(x) \le 1$ on $[0,1]$, $\mathcal{A}$ is the Haar support of x, and x has finite Haar expansion (i.e. $\mathcal{A}$ is finite)

$$x = \sum_{I\in\mathcal{A}} a_{I,p} h_{I,p}.$$

For $k \in \mathbb{N}^+$ define

$$\mathfrak{K}_{k,\tau,p}(x) = \{t \in [0,1] : (2^p)^{k-1} < S^p(T_{\tau,p}(x))[t] \le (2^p)^k\}.$$

We shall always assume that $|L_s| = \frac{1}{2}\min\{|\tau(I)| : I \in \mathcal{A}\}$ and $L_s \in \mathcal{D}$ for each s. If $L_s \subset \mathfrak{K}_{k,\tau,p}(x)$ and $J_1,\dots,J_{\mathfrak{g}_s}$ are all intervals from $\tau(\mathcal{A})$ containing L_s, we define

$$\mathcal{B}_s = \{J_1,\dots,J_{\mathfrak{g}_s}\}$$

and

$$\begin{aligned}\alpha_J &= |\tau^{-1}(J)|/|J|,\\ \widehat{\alpha}_s &= \max\{\alpha_J : J \in \mathcal{B}_s\},\\ \beta_J &= \|a_{\tau^{-1}(J),p} h_{\tau^{-1}(J),p}\|_\infty,\end{aligned}$$

and choose an interval $\widehat{J}_s \in \mathcal{B}_s$ such that

$$|\widehat{J}_s| = \max\{|J| : J \in \mathcal{B}_s,\ \alpha_J = \widehat{\alpha}_s\}. \tag{10}$$

Then, if $L_s \subset \mathfrak{K}_{k,\tau,p}(x)$, we have

$$2^{(k-1)p} \le \Big(\sum_{J\in\mathcal{B}_s} \alpha_J^{2/p}\beta_J^2\Big)^{p/2}.$$

5. We consider three possible cases:

(i) $\sum_{J\in\mathcal{B}_s} \beta_J^2 \le 1$;

(ii) $\sum_{J\in\mathcal{B}_s} \beta_J^2 > 1$ but $\sum_{J\in\mathcal{B}_s} \alpha_J\beta_J^2 - \alpha_K < 0$ for some $K \in \mathcal{B}_s$;

(iii) $\sum_{J\in\mathcal{B}_s} \beta_J^2 > 1$ and $\sum_{J\in\mathcal{B}_s} \alpha_J\beta_J^2 - \alpha_K \ge 0$ for each $K \in \mathcal{B}_s$.

We write $s \in A_{\rm i}$, $s \in A_{\rm ii}$, or $s \in A_{\rm iii}$, according to the case. Then

$$\sum_s \Big(\sum_{J\in\mathcal{B}_s} \alpha_J^{2/p}\beta_J^2\Big)^{p/2} |L_s| = \sum_{\rm i} + \sum_{\rm ii} + \sum_{\rm iii}. \tag{11}$$

We will estimate each sum separately.

Case (i). We have

$$\sum_{\mathrm{i}} = \sum_{s\in A_{\mathrm{i}}} \Big(\sum_{J\in\mathcal{B}_s} \alpha_J^{2/p}\beta^2\Big)^{p/2} |L_s| \le \sum_{s\in A_{\mathrm{i}}} \widehat{\alpha}_s |L_s|. \tag{12}$$

Now we define

$$\mathfrak{s}(J) = \Big| \bigcup_{E\in\{\widehat{J}_s\,:\,\widehat{J}_s\subset J,\,\widehat{J}_s\ne J,\,s\in A_{\mathrm{i}}\}} E\Big| \cdot |J|^{-1}.$$

From the sequence $(\widehat{J}_s)_{s\in A_{\mathrm{i}}}$ we choose a subsequence $(\widehat{J}_s)_{s\in A'_{\mathrm{i}}}$ such that:

1. $\widehat{J}_k \ne \widehat{J}_j$ for all $k, j \in A'_{\mathrm{i}}$, $k \ne j$,
2. for each $j \notin A'_{\mathrm{i}}$ there exists $k \in A'_{\mathrm{i}}$ such that $\widehat{J}_j = \widehat{J}_k$.

Then by definition of $\widehat{J}_s$, $\widehat{\alpha}_s$ and $\mathfrak{s}(\widehat{J}_s)$ we get

$$\begin{aligned}
\sum_{s\in A_{\mathrm{i}}} \widehat{\alpha}_s|L_s| &= \sum_{s\in A_{\mathrm{i}}} \frac{|\tau^{-1}(\widehat{J}_s)|}{|\widehat{J}_s|}|L_s| \\
&= \sum_{s\in A'_{\mathrm{i}}} \Big[\frac{|\tau^{-1}(\widehat{J}_s)|}{|\widehat{J}_s|} \sum_{\substack{L_k\subset \widehat{J}_s\\ \widehat{J}_k=\widehat{J}_s}} |L_k|\Big] \\
&\overset{(10)}{=} \sum_{s\in A'_{\mathrm{i}}} \frac{|\tau^{-1}(\widehat{J}_s)|}{|\widehat{J}_s|} \Big(|\widehat{J}_s| - \Big| \bigcup_{E\in\{\widehat{J}_k\,:\,\widehat{J}_k\subset\widehat{J}_s,\,\widehat{J}_k\ne\widehat{J}_s,\,s\in A_{\mathrm{i}}\}} E\Big|\Big) \\
&= \sum_{s\in A'_{\mathrm{i}}} |\tau^{-1}(\widehat{J}_s)|(1-\mathfrak{s}(\widehat{J}_s)) \\
&= \sum_{s\in A'_{\mathrm{i}}} \sum_n \sum_{2^{-n}\le 1-\mathfrak{s}(\widehat{J}_s)<2^{-n+1}} |\tau^{-1}(\widehat{J}_s)|(1-\mathfrak{s}(\widehat{J}_s)) \\
&\le 2\sum_{s\in A'_{\mathrm{i}}} \sum_n \sum_{2^{-n}\le 1-\mathfrak{s}(\widehat{J}_s)<2^{-n+1}} |\tau^{-1}(\widehat{J}_s)|2^{-n}
\end{aligned} \tag{13}$$

We shall show that

$$\sum_{s\in A'_{\mathrm{i}}} \sum_n \sum_{2^{-n}\le 1-\mathfrak{s}(\widehat{J}_s)<2^{-n+1}} |\tau^{-1}(\widehat{J}_s)|2^{-n} \le 6[\![\tau^{-1}]\!]. \tag{14}$$

To do this we use some ideas of Jones ([Jo, p. 201]).

Fix $I \in \mathfrak{max}[\tau^{-1}(\{\widehat{J}_s : s \in A'_{\mathrm{i}}\})]$. Suppose that for some natural numbers L and l_s ($s \in A'_{\mathrm{i}}$), whenever $1-\mathfrak{s}(\widehat{J}_s) \in [2^{-n}, 2^{-n+1})$ we have

$$2^{-n} = l_s/L. \tag{15}$$

Now if $A'_{\mathrm{i}} = \{s_1, \ldots, s_r\}$, we define

$$x = (\underbrace{\widehat{J}_{s_1}, \ldots, \widehat{J}_{s_1}}_{l_{s_1}\,\text{times}}, \underbrace{\widehat{J}_{s_2}, \ldots, \widehat{J}_{s_2}}_{l_{s_2}\,\text{times}}, \ldots, \underbrace{\widehat{J}_{s_r}, \ldots, \widehat{J}_{s_r}}_{l_{s_r}\,\text{times}})$$

and write x as $(x_n)_{n=1}^{\bar{r}}$ where $\bar{r} = \sum_{m=1}^{r} l_{s_m}$. Then we split $(x_n)_{n=1}^{\bar{r}}$ into L subsequences $S_1, \ldots, S_L$ by evenly distributing the entries x_n: put x_n in S_q if $n \equiv q \pmod{L}$, so that if $x_n = \widehat{J}_s$ for some $s \in A'_{\mathrm{i}}$, we put a copy of $\widehat{J}_s$ in S_q. Notice that each sequence S_j consists of pairwise different elements. Thus from now on, S_j's will be families of intervals. Then

$$(16)\qquad \sum_n \sum_{\substack{s\in A'_{\mathrm{i}} \\ \tau^{-1}(\widehat{J}_s)\subset I \\ 2^{-n}\le 1-\mathfrak{s}(\widehat{J}_s)<2^{-n+1}}} 2^{-n}|\tau^{-1}(\widehat{J}_s)| \overset{(15)}{=} \frac{1}{L} \sum_{\substack{s\in A'_{\mathrm{i}} \\ \tau^{-1}(\widehat{J}_s)\subset I}} l_s|\tau^{-1}(\widehat{J}_s)| = \frac{1}{L}\sum_{j=1}^{L} \sum_{\substack{\widehat{J}_s\in S_j \\ \tau^{-1}(\widehat{J}_s)\subset I}} |\tau^{-1}(\widehat{J}_s)|.$$

Moreover, for each $J_0 \in \mathcal{D}$ and $j \le L$ the number of intervals $\widehat{J}_s$ from the family S_j with $|\widehat{J}_s| = 2^{-m}$ and $\widehat{J}_s \subset J_0$ satisfies

$$(17)\qquad |\{\widehat{J}_s \subset J_0 : s \in A'_{\mathrm{i}},\ \widehat{J}_s \in S_j \cap \mathcal{D}_m\}| \le 1 + \frac{1}{L} \sum_{\substack{s\in A'_{\mathrm{i}} \\ \widehat{J}_s\in\mathcal{D}_m\cap J_0}} l_s.$$

By definition of $\mathfrak{s}$,

$$(18)\qquad \sum_{\substack{s\in A'_{\mathrm{i}} \\ \widehat{J}_s\subset J_0}} (1-\mathfrak{s}(\widehat{J}_s))|\widehat{J}_s| \le |J_0|,$$

so (for $j \le L$)

$$(19)\qquad \sum_{\widehat{J}_s\in S_j\cap J_0} |\widehat{J}_s| = \sum_{m\ge -\log_2|J_0|} 2^{-m}|(S_j\cap\mathcal{D}_m)\cap J_0| \overset{(17)}{\le} 2|J_0| + \frac{1}{L}\sum_{\substack{s\in A'_{\mathrm{i}} \\ \widehat{J}_s\subset J_0}} l_s|\widehat{J}_s| \overset{(18)}{\le} 3|J_0|.$$

This gives $[\![S_j]\!] \le 3$ for $1 \le j \le L$. Now for all $j \le L$ and all $K \in \mathfrak{max}[\tau^{-1}(S_j)]$, by definition of the Carleson constant, we have

$$(20)\qquad \frac{1}{|K|} \sum_{\substack{\widehat{J}_s\in S_j \\ \tau^{-1}(\widehat{J}_s)\subset K}} |\tau^{-1}(\widehat{J}_s)| \le [\![\tau^{-1}(S_j)]\!].$$

So for $j \le L$ we get

$$\sum_{\substack{\widehat{J}_s \in S_j \\ \tau^{-1}(\widehat{J}_s) \subset I}} |\tau^{-1}(\widehat{J}_s)| \le [\![\tau^{-1}(S_j)]\!] |I|. \tag{21}$$

Similarly, by (20) we obtain

$$\begin{aligned} \sum_{\widehat{J}_s \in S_j} |\tau^{-1}(\widehat{J}_s)| &\le [\![\tau^{-1}(S_j)]\!] \sum_{K \in \mathfrak{max}[\tau^{-1}(S_j)]} |K| \\ &= [\![\tau^{-1}(S_j)]\!] \Big| \bigcup_{\widehat{J}_s \in S_j} \tau^{-1}(\widehat{J}_s) \Big|. \end{aligned} \tag{22}$$

Because $[\![\tau^{-1}]\!]$ preserves the Carleson constant, for $[\![\tau^{-1}]\!] = M$ we have

$$[\![\tau^{-1}(S_j)]\!] \le [\![\tau^{-1}]\!][\![S_j]\!] \overset{(19)}{\le} M \cdot 3, \tag{23}$$

thus

$$\begin{aligned} \sum_{\substack{s \in A'_{\mathrm{i}} \\ \tau^{-1}(\widehat{J}_s) \subset I}} (1 - \mathfrak{s}(\widehat{J}_s)) |\tau^{-1}(\widehat{J}_s)| &\overset{(13),(16)}{\le} 2\, \frac{1}{L} \sum_{j=1}^{L} \sum_{\substack{\widehat{J}_s \in S_j \\ \tau^{-1}(\widehat{J}_s) \subset I}} |\tau^{-1}(\widehat{J}_s)| \\ &\overset{(21),(23)}{\le} 6M|I|. \end{aligned}$$

But I was chosen from $\mathfrak{max}[\tau^{-1}\{\widehat{J}_s : s \in A'_{\mathrm{i}}\}]$, so by (22) we get

$$\sum_{\mathrm{i}} \le 6M. \tag{24}$$

Case (ii). Since for $s \in A_{\mathrm{ii}}$,

$$\sum_{J \in \mathcal{B}_s} \alpha_J \beta_J^2 - \alpha_K < 0 \tag{25}$$

for some $K \in \mathcal{B}_s$, by definition of $\widehat{\alpha}_s$ we can of course assume that $K = \widehat{J}_s$, i.e.

$$\sum_{J \in \mathcal{B}_s} \alpha_J \beta_J^2 < \widehat{\alpha}_s. \tag{26}$$

Thus for $s \in A_{\mathrm{ii}}$ we have

$$\Big(\sum_{J \in \mathcal{B}_s} \alpha_J^{2/p} \beta_J^2 \Big)^{p/2} \le \Big(\widehat{\alpha}_s^{2/p-1} \sum_{J \in \mathcal{B}_s} \alpha_J \beta_J^2 \Big)^{p/2} \overset{(26)}{\le} \widehat{\alpha}_s^{1-p/2} \widehat{\alpha}_s^{p/2} = \widehat{\alpha}_s.$$

Now we can repeat the argument used in Case (i) to show that

$$\sum_{\mathrm{ii}} \le 6M. \tag{27}$$

CASE (iii). Since $\sum_{J\in\mathcal{B}_s}\alpha_J\beta_J^2-\alpha_K\ge 0$ for each $K\in\mathcal{B}_s$, by direct computation for $s\in A_{\mathrm{iii}}$ we get

$$\Big(\sum_{J\in\mathcal{B}_s}\alpha_J^{2/p}\beta_J^2\Big)^{p/2}\le\Big(\widehat{\alpha}_s^{2/p-1}\sum_{J\in\mathcal{B}_s}\alpha_J\beta_J^2\Big)^{p/2}$$
$$\le\Big[\Big(\sum_{J\in\mathcal{B}_s}\alpha_J\beta_J^2\Big)^{2/p-1}\Big(\sum_{J\in\mathcal{B}_s}\alpha_J\beta_J^2\Big)\Big]^{p/2}=\sum_{J\in\mathcal{B}_s}\alpha_J\beta_J^2.$$

Let $\{K_t\}_{t\in T}$ be the family of all dyadic intervals such that:

1. $|K_t|=\frac12\min\{|N|:N\in\tau^{-1}(\{J:J\in\mathcal{B}_s,\ s\in A_{\mathrm{iii}}\})$ for all $t\in T$,
2. $\bigcup_{t\in T}K_t=\bigcup\{J:J\in\mathcal{B}_s,\ s\in A_{\mathrm{iii}}\}$,
3. $K_{t_1}\ne K_{t_2}$ whenever $t_1,t_2\in T$, $t_1\ne t_2$.

Surprisingly easily, we get

$$\sum_{s\in A_{\mathrm{iii}}}\Big(\sum_{J\in\mathcal{B}_s}\alpha_J\beta_J^2\Big)|L_s|\le\sum_{s\in A_{\mathrm{iii}}}\sum_{J\in\mathcal{B}_s}|\tau^{-1}(J)|\beta_J^2\le\sum_t\sum_{\substack{s\in A_{\mathrm{iii}}\\ J\in\mathcal{B}_s\\ \tau^{-1}(J)\supset K_t}}|K_t|\beta_J^2$$
$$=\sum_t|K_t|\Big(\sum_{\substack{s\in A_{\mathrm{iii}}\\ J\in\mathcal{B}_s\\ \tau^{-1}(J)\supset K_t}}\beta_J^2\Big)\le\sum_t|K_t|\le 1.$$

So we have proved that

$$\sum_{\mathrm{iii}}\le 1. \tag{28}$$

6. Now we only need to summarize the above observations. The operator quasinorm of $T_{\tau,p}:H_p\to H_p$ satisfies (the first 2^p on the right hand side below comes from (9)), by (3) and (5),

$$\|T_{\tau,p}\|^p\le 2^p\cdot 2^p\cdot\frac{6^p}{2^p-1}\cdot(6M+6M+1)\cdot\|x\|_{H_p}^{-p}$$
$$\le 4^p\cdot\frac{6^p}{2^p-1}\cdot(12M+1)\cdot 8^p,$$

and we are done. ∎

REMARK 1. In case (iii) of the above proof we have found an analytic condition on τ guaranteeing the continuity of $T_\tau:H_p(\mathcal{B})\to H_p$ for arbitrary $\mathcal{B}\subset\mathcal{D}$. This condition does not make use of $[\![\tau^{-1}]\!]$ at all.

QUESTION 1. *Does the condition from Case* (iii) *characterize contractive rearrangements in* H_p*?*

We can now apply Theorem 1 and duality to prove our main result. There already exists a proof of our next theorem in the literature: see Geiss et al. [GMP] who used general concepts such as complex interpolation of quasi-Banach lattices.

THEOREM 2. *Let* $\mathcal{B} \cup \mathcal{C} \subset \mathcal{D}$ *and let* $\tau : \mathcal{B} \to \mathcal{C}$ *be a bijection. If*

(a) $0 < p < 2$ *and* τ^{-1} *preserves the Carleson constant*, *or*

(b) $2 < p < \infty$ *and* τ *preserves the Carleson constant*,

then the operator $T_{\tau,p} : H_p(\mathcal{B}) \to H_p(\mathcal{C})$ *induced by* τ *is continuous.*

3. A necessary condition. Now we formalize a necessary condition for the continuity of $T_{\tau,p}$. We simply prove the converse to Theorem 2.

THEOREM 3. *Let* $\mathcal{B} \cup \mathcal{C} \subset \mathcal{D}$ *and let* $\tau : \mathcal{B} \to \mathcal{C}$ *be a bijection. Suppose* $T_{\tau,p} : H_p(\mathcal{B}) \to H_p(\mathcal{C})$ *induced by* τ *is a continuous operator. Then*

(a) τ^{-1} *preserves the Carleson constant if* $0 < p < 2$;

(b) τ *preserves the Carleson constant if* $2 < p < \infty$.

Proof. (a) Suppose that $T_{\tau,p}$ is continuous but τ^{-1} does not preserve the Carleson constant. By [Jo, Lemma 2.1] and [Mu, Proposition 2] this implies that

$$\forall M \geq 1 \ \exists \mathcal{L}^M \subset \mathcal{C} : \ [\![\mathcal{L}^M]\!] \leq 4 \quad \text{and} \quad [\![\tau^{-1}(\mathcal{L}^M)]\!] > 4M.$$

By [Ga, Lemma 3.2 in Chapter 10], there exists an interval $I \in \mathcal{D}$ and $2M$ pairwise disjoint families $\mathcal{E}^i \subset \mathcal{Q}(I) \cap \tau^{-1}(\mathcal{L}^M)$, $i = 1, \ldots, 2M$, such that $\mathcal{E}^i$ covers at least half of I. By [W2, Lemma 3.3], $\overline{\operatorname{span}}\{h_{I,p} : I \in \bigcup_{i=1}^{2M} \mathcal{E}^i\}$, i.e. $H_p(\bigcup_{i=1}^{2M} \mathcal{E}^i)$, contains a space X spanned by vectors with pairwise disjoint Haar supports and isomorphic to ℓ_2^{2M} with constant C_p. But $\{h_{J,p} : J \in \mathcal{L}^M\}$ spans ℓ_p with constant $C_{p,4}$, in particular, $T_{\tau,p}(X) \overset{C_{p,4}}{\sim} \ell_p$ because we can divide $\mathcal{L}^M$ into eight disjoint parts $\mathcal{L}_1^M, \ldots, \mathcal{L}_8^M$ such that for $1 \leq i \leq 8$ and $I \in \mathcal{L}_i^M$ we have

$$\Big| \bigcup_{I \neq J \in I \cap \mathcal{L}_i^M} J \Big| < \tfrac{1}{2}|I|,$$

so $\{h_{J,p} : J \in \mathcal{L}_i^M\}$ spans ℓ_p with constant $2^{1/p}$ for $1 \leq i \leq 8$ (cf. [Sm, Lemma 2]). Since M can be arbitrarily large, and since $T_{\tau,p}$ is continuous and a rearrangement, and H_p is p-convex, this leads to a contradiction.

(b) follows by duality from the case $1 < p < 2$. ∎

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