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Erratum to "Stability of the index of a linear relation under compact perturbations"

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by

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Abstract. We give a corrected version of the main result from the paper cited in the title. We obtain necessary and sufficient conditions for the stability of the topological index of an open linear relation under compact perturbations.

Proposition 1 and Theorem 1 in [4] fail to hold for general Fredholm relations between normed spaces having closed range. The argument given in [4] relies on the unproven result $R(T')^{\perp} = N(T)$ from Lemma 2(i). That result does not hold unless additional conditions are imposed on the bounded operator T (see [3, Proposition III.1.4(a), Proposition III.4.6(b)] in the more general context of linear relations).

The main result of this note is a replacement of Theorem 1 in [4] but its proof does not depend on results from [4]. It also completes recent results of Alvarez [1]. We would like to mention that our method from [4] remains valid.

Our notation is the same as in [4]. A linear relation T is said to be *open* if whenever U is a neighbourhood in $D(T)$, the image $T(U) := \bigcup_{x\in U} T(x)$ is a neighbourhood in $R(T)$. Note that T is open iff there exists $\varrho > 0$ such that $\varrho B_Y \cap R(T) \subset T(B_X \cap D(T))$, where B_X and B_Y denote the closed unit balls of X and Y respectively (see Propositions II.2.4 and II.3.2 (b) in [3]). Assume that the linear relation T satisfies $\dim(N(T)) < \infty$ or $\operatorname{codim}(\overline{R(T)}) < \infty$. Then the quantity $\overline{\operatorname{ind}}(T) = \dim(N(T)) - \operatorname{codim}(\overline{R(T)})$ is called the *topological index* of T. We shall denote by $G(T)$ the graph of the linear relation T.

THEOREM 1. *Let X, Y be normed spaces such that Y is complete and let $T \in LR(X,Y)$ be an open linear relation satisfying*

$$\dim(N(T)) < \infty \quad \textit{and} \quad \operatorname{codim}(\overline{R(T)}) < \infty. \tag{1}$$

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If $K \in \mathcal{K}(X,Y)$ *then*

$$\dim(N(T+K)) < \infty, \quad \operatorname{codim}(\overline{R(T+K)}) < \infty, \tag{2}$$

$$\overline{\operatorname{ind}}(T+K) \le \overline{\operatorname{ind}}(T). \tag{3}$$

Moreover, $\overline{\operatorname{ind}}(T+K) = \overline{\operatorname{ind}}(T)$ *iff* $T+K$ *is open.*

Proof. I. Consider the sequence

$$0 \to T(0) \xrightarrow{i_T} G(T) \times T(0) \xrightarrow{j_{T,K}} Y \to 0 \tag{4}$$

with

$$i_T(y_0) = ((0,y_0),y_0), \qquad \forall y_0 \in T(0),$$
$$j_{T,K}((x,y),y_0) = y + Kx - y_0, \qquad \forall((x,y),y_0) \in G(T) \times T(0).$$

The operators i_T and $j_{T,K}$ are bounded. A simple calculation shows that

$$\dim(N(j_{T,K})/R(i_T)) = \dim(N(T+K)). \tag{5}$$

On the other hand,

$$N(i_T) = \{0\}, \quad R(j_{T,K}) = R(T+K). \tag{6}$$

One can prove that $R(i_T)$ is closed in $G(T) \times T(0)$ and

$$T+K \text{ open} \Leftrightarrow j_{T,K} \text{ open}. \tag{7}$$

Indeed, assume that $T+K$ is open. Consider $y \in R(j_{T,K}) = R(T+K)$ such that $y \neq 0$. It follows that $\varrho\|y\|^{-1}y \in \varrho B_Y \cap R(T+K)$, so there exists $x \in B_X \cap D(T)$ such that $\varrho\|y\|^{-1}y \in (T+K)(x)$, hence $y - K(\|y\|\varrho^{-1}x) \in T(\|y\|\varrho^{-1}x)$. Therefore $((\|y\|\varrho^{-1}x, y-K(\|y\|\varrho^{-1}x)),0) \in G(T) \times M(T)$ and $\|((\|y\|\varrho^{-1}x, y-K(\|y\|\varrho^{-1}x)),0)\| \le k\|y\|$, where $k = \varrho^{-1}(\|K\|+1)+1$. Note that $j_{T,K}((\|y\|\varrho^{-1}x, y-K(\|y\|\varrho^{-1}x)),0) = y$. Hence, if $\|y\| \le k^{-1}$ then $\|((\|y\|\varrho^{-1}x, y-K(\|y\|\varrho^{-1}x)),0)\| \le 1$. Consequently, $j_{T,K}$ is open.

Conversely, assume that $j_{T,K}$ is open and let $y \in \varrho B_Y \cap R(T+K)$. It follows that there exists $((x,y_1),y_0) \in B_{G(T)\times T(0)}$ (in particular $x \in B_X$) such that $j_{T,K}((x,y_1),y_0) = y_1 + Kx - y_0 = y$. We have $(x,y_1) \in G(T)$ and $(0,y_0) \in G(T)$, hence $(x, y_1 - y_0) \in G(T)$ and $y \in (T+K)(x)$. Consequently, the relation $T+K$ is open.

II. Consider the sequence

$$0 \to Y' \xrightarrow{j'_{T,K}} G(T)' \times T(0)' \xrightarrow{i'_T} T(0)' \to 0. \tag{8}$$

We know that $R(i_T) \subset N(j_{T,K})$, which implies that $N(j_{T,K})^\perp \subset R(i_T)^\perp$, which together with

$$R(i_T)^\perp = N(i'_T), \quad \overline{R(j'_{T,K})} \subset N(j_{T,K})^\perp \tag{9}$$

implies that $R(j'_{T,K}) \subset N(i'_T)$, that is, (8) is a Banach space complex. Note that i'_T is surjective. On the other hand, from $R(j_{T,K})^\perp = N(j'_{T,K})$ it follows that

$$\operatorname{codim}(\overline{R(j_{T,K})}) = \dim(N(j'_{T,K})). \tag{10}$$

Using (9) and the fact that $R(i_T)$ is closed in $G(T) \times T(0)$ we deduce that

$$\begin{aligned} \dim(N(j_{T,K})/R(i_T)) &= \dim(R(i_T)^\perp/N(j_{T,K})^\perp) \\ &= \dim(N(i'_T)/N(j_{T,K})^\perp). \end{aligned} \tag{11}$$

Note that $N(i'_T)/N(j_{T,K})^\perp \simeq (N(i'_T)/R(j'_{T,K}))/(N(j_{T,K})^\perp/R(j'_{T,K}))$, so

$$\begin{aligned} \dim(N(i'_T)/N(j_{T,K})^\perp) + \dim(N(j_{T,K})^\perp/R(j'_{T,K})) \\ = \dim(N(i'_T)/R(j'_{T,K})). \end{aligned} \tag{12}$$

From (11) and (12) we deduce that

$$\dim(N(j_{T,K})/R(i_T)) \leq \dim(N(i'_T)/R(j'_{T,K})). \tag{13}$$

III. If $T + K$ is open, then using (7) it follows that $j_{T,K}$ is open, hence $R(j'_{T,K}) = N(j_{T,K})^\perp$, which together with (11) implies

$$\dim(N(j_{T,K})/R(i_T)) = \dim(N(i'_T)/R(j'_{T,K})). \tag{14}$$

Assume that $\dim(N(i'_T)/R(j'_{T,K}))$ is finite and (14) holds. This together with (11) and (12) implies that $\dim(N(j_{T,K})^\perp/R(j'_{T,K})) = 0$, hence $R(j'_{T,K}) = N(j_{T,K})^\perp$, which implies that $j_{T,K}$ is open, so by (7) one has $T + K$ open.

IV. Consider $\widetilde{K} : G(T) \times T(0) \to Y$ defined by $\widetilde{K}((x, y), y_0) = Kx$. It is clear that $\widetilde{K}$ is compact and $j_{T,K} = j_{T,0} + \widetilde{K}$, which implies that $j'_{T,K} = j'_{T,0} + \widetilde{K}'$. Note that the completeness of Y and the compactness of $\widetilde{K}$ imply that $\widetilde{K}'$ is compact (see [3, Theorem V.14.5]). Now, consider the sequences

$$0 \to Y' \xrightarrow{j'_{T,0}} G(T)' \times T(0)' \xrightarrow{i'_T} T(0)' \to 0, \tag{15}$$

$$0 \to Y' \xrightarrow{j'_{T,0} + \widetilde{K}'} G(T)' \times T(0)' \xrightarrow{i'_T} T(0)' \to 0. \tag{16}$$

From II we know that (15) and (16) are Banach space complexes. Using the fact that T is open and (1), we deduce from (14), (5) and (10) (in the case $K = 0$) that

$$\begin{aligned} \dim(N(i'_T)/R(j'_{T,0})) &= \dim(N(T)) < \infty, \\ \dim(N(j'_{T,0})) &= \operatorname{codim}(\overline{R(T)}) < \infty. \end{aligned} \tag{17}$$

It follows that the complex (15) is Fredholm and

$$-\overline{\operatorname{ind}}(T) = \operatorname{ind}(15). \tag{18}$$

Hence, using the main result in [2] it follows that

$$\dim(N(j'_{T,K})) < \infty, \qquad \dim(N(i'_T)/R(j'_{T,K})) < \infty, \tag{19}$$

$$\operatorname{ind}(15) = \operatorname{ind}(16). \tag{20}$$

From (5), (10), (13), (18), (19) and (20) we deduce that (2) and (3) hold.

Now, assume that $\overline{\operatorname{ind}}(T+K) = \overline{\operatorname{ind}}(T)$. This together with (5), (10), (13), (18) and (20) implies that

$$\begin{aligned}\overline{\operatorname{ind}}(T) &= \dim(N(T+K)) - \operatorname{codim}(\overline{R(T+K)})\\ &= \dim(N(j_{T,K})/R(i_T)) - \dim(N(j'_{T,K}))\\ &\le \dim(N(i'_T)/R(j'_{T,K})) - \dim(N(j'_{T,K}))\\ &= -\operatorname{ind}(16) = -\operatorname{ind}(15) = \overline{\operatorname{ind}}(T).\end{aligned}$$

It follows that $\dim(N(j_{T,K})/R(i_T)) = \dim(N(i'_T)/R(j'_{T,K})) < \infty$ and III implies that $T+K$ is open. To prove that the openness of $T+K$ implies $\overline{\operatorname{ind}}(T+K) = \overline{\operatorname{ind}}(T)$ we proceed in the same way.

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