STUDIA MATHEMATICA 185 (1) (2008)

On the spectrum of the operator which is a composition of integration and substitution

by

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Abstract. Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0,1)$. Let the operator $V_\phi : f(x) \mapsto \int_0^{\phi(x)} f(t)\,dt$ be defined on $L_2[0,1]$. We prove that V_ϕ has a finite number of nonzero eigenvalues if and only if $\phi(0) > 0$ and $\phi(1-\varepsilon) = 1$ for some $0 < \varepsilon < 1$. Also, we show that the spectral trace of the operator V_ϕ always equals 1.

1. Introduction. It is well known that the Volterra integration operator $V : f(x) \mapsto \int_0^x f(t)\,dt$ defined on $L_2[0,1]$ is quasinilpotent, i.e., $\sigma(V) = \{0\}$. Let $\phi \in C[0,1]$ be such that $\phi(0) = 0$. It was pointed out in [9] and [10] that the operator V_ϕ defined by

$$V_\phi : f(x) \mapsto \int\limits_0^{\phi(x)} f(t)\,dt \tag{1.1}$$

is quasinilpotent on $C[0,1]$ whenever $\phi(x) \le x$ for all $x \in [0,1]$.

Let $\phi : [0,1] \to [0,1]$ be a measurable function and let $V_\phi : L_p[0,1] \to L_p[0,1]$ $(1 \le p < \infty)$ be defined by (1.1). It was proved in [11] and [13] that V_ϕ is quasinilpotent on $L_p[0,1]$ if and only if $\phi(x) \le x$ for almost all $x \in [0,1]$. It was noted in [13] and proved in [15] that the spectral radius of V_{x^α} (defined on $L_p[0,1]$ or $C[0,1]$) is $1-\alpha$ $(0 < \alpha < 1)$. The detailed investigation of the spectrum of the operator V_{x^α} was done in [1], where it was shown that the point spectrum $\sigma_{\mathrm{p}}(V_{x^\alpha})$ of V_{x^α} is simple and $\sigma_{\mathrm{p}}(V_{x^\alpha}) = \{(1-\alpha)\alpha^{n-1}\}_{n=1}^{\infty}$. The oscillation properties of the eigenfunctions of V_{x^α} were also investigated in [1].

The aim of this paper is to prove the following theorem.

2000 *Mathematics Subject Classification*: 34l20, 45C05, 47A10, 47A75.
Key words and phrases: eigenvalue, integral operator, Fredholm determinant.
This research was partially supported by NAS of Ukraine, Grant # 0105U006289.

THEOREM 1.1. *Let* $\phi : [0,1] \to [0,1]$ *be a nondecreasing continuous function such that* $\phi(x) > x$ *for all* $x \in (0,1)$, *and* V_ϕ *be defined on* $L_2[0,1]$ *by* (1.1). *Set also* $\sigma_{\mathrm{p}}(V_\phi) \setminus \{0\} = \{\lambda_n\}_{n=1}^{\omega}$ $(1 \le \omega \le \infty)$. *Then*:

(1) $\omega < \infty$ *if and only if* $\phi(0) > 0$ *and* $\phi(1-\varepsilon) = 1$ *for some* $0 < \varepsilon < 1$;
(2) $\lim_{\varepsilon\to 0} \sum_{|\lambda_n|>\varepsilon} \lambda_n = 1$;
(3) $\sum_{n=1}^{\omega} |\lambda_n|^{1+\varepsilon} < \infty$ *for all* $\varepsilon > 0$.

The paper is organized as follows.

In Section 2 we recall some classical results on trace class operators, Fredholm determinants and entire functions. In Section 3 we calculate the Fredholm determinant $D_{V_\phi}(\lambda)$ of the operator V_ϕ. In Section 4 we estimate the order of growth of $D_{V_\phi}(\lambda)$ and prove Theorem 1.1. It turns out that the matrix trace of V_ϕ is not defined, but the spectral trace of V_ϕ does not depend on ϕ and always equals 1. This contrasts with the fact that $\sigma_{\mathrm{p}}(V_x) = \emptyset$. We also find the spectral (= matrix) traces of the V_ϕ^2 and V_ϕ^3. In Section 5 we assume that $\phi : [0,1] \to [0,1]$ is a strictly increasing continuous function such that $\mathrm{card}\{x : \phi(x) = x\} < \infty$ and describe the spectrum of V_ϕ. Then we consider V_ϕ defined on the space $L_p[0,1]$.

2. Preliminaries. Here we recall some facts about trace class operators, Fredholm determinants and entire functions.

2.1. Let K be a compact operator defined on an infinite-dimensional Hilbert space $\mathfrak{H}$. Let $s_n(K)$ $(n \ge 1)$ be the eigenvalues of KK^*. The operator K is said to be *of class* $\mathbf{S}_p$ if $\sum_{n=1}^{\infty} s_n(K)^p < \infty$. The trace $\operatorname{tr} K$ of an operator $K \in \mathbf{S}_1$ is defined as its *matrix trace*: $\operatorname{tr} K = \sum_{n=1}^{\infty}(Ke_n, e_n)$, where $\{e_n\}_{n=1}^{\infty}$ is some orthonormal basis. It is known that $\operatorname{tr} K$ does not depend on the choice of $\{e_n\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty}(Ke_n, e_n)$ converges absolutely. The celebrated theorem of Lidskiĭ (see [4]) says that the matrix trace of an operator $K \in \mathbf{S}_1$ is equal to its *spectral trace*, which is defined as the sum of the eigenvalues of K (counted with algebraic multiplicity):

$$\operatorname{tr} K = \sum_{n=1}^{\infty}(Ke_n, e_n) = \sum_{n=1}^{\omega} \lambda_n, \qquad \omega \le \infty. \tag{2.1}$$

Let K be an integral operator, $(Kf)(x) = \int_0^1 k(x,t) f(t)\, dt$ on $L_2[0,1]$. It is well known (see [4]) that if $k(x,t)$ is a continuous function on $[0,1]\times[0,1]$, then $K \in \mathbf{S}_1$ and $\operatorname{tr} K$ is given by the integral over the diagonal:

$$\operatorname{tr} K = \int_0^1 k(t,t)\, dt. \tag{2.2}$$

2.2. Now let $k(x,t)$ be a bounded function on $[0,1]\times[0,1]$. Define

$$D_K(\lambda) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n \lambda^n, \tag{2.3}$$

where $A_0 := 1$ and

$$\begin{gathered} A_n := \int_0^1 \dots \int_0^1 K(t_1,\dots,t_n)\,dt_1\dots dt_n, \\ K(t_1,\dots,t_n) := \det \begin{pmatrix} k(t_1,t_1) & \dots & k(t_1,t_n) \\ \vdots & \vdots & \vdots \\ k(t_n,t_1) & \dots & k(t_n,t_n) \end{pmatrix} \end{gathered} \tag{2.4}$$

for $n \ge 1$. The function $D_K(\lambda)$ is called the *Fredholm determinant* of K. Recall (see [6, 8, 12]) that

$$A_n = n! \int_0^1 \int_{t_1}^1 \int_{t_2}^1 \dots \int_{t_{n-1}}^1 K(t_1,\dots,t_n)\,dt_n\dots dt_1, \qquad n \ge 1. \tag{2.5}$$

Moreover, $D_K(\lambda)$ is an entire function of λ of order $\varrho \le 2$, and $D_K(\mu^*) = 0$ if and only if $\lambda^* := 1/\mu^* \in \sigma_{\mathrm{p}}(K)$; moreover, the multiplicity of μ^* as a root of the Fredholm determinant of K is equal to the algebraic multiplicity of the eigenvalue λ^*.

2.3. From Hadamard's theorem ([7, Th. 1, p. 26]) and Lindelöf's theorem ([7, Th. 3, p. 33]), we get the following

THEOREM 2.1. *Let $f(z)$ be an entire function of order $\varrho_f \le 1$ and type $\sigma_f < \infty$. Let also $\{a_n\}_{n=1}^{\omega}$ $(\omega \le \infty)$ be all roots of $f(z)$ and $f(0) = 1$. Then*

(i) *if $\varrho_f = 1$, $\sigma_f = 0$ and $\sum_{n=1}^{\omega} 1/|a_n| < \infty$, then $\omega = \infty$, $f(z) = \prod_{n=1}^{\infty}(1 - z/a_n)$ and $\sum_{n=1}^{\infty} 1/a_n = -f'(0)$;*

(ii) *if $\varrho_f < 1$, then $f(z) = \prod_{n=1}^{\omega}(1 - z/a_n)$ and $\sum_{n=1}^{\omega} 1/a_n = -f'(0)$;*

(iii) *if $\varrho_f = 0$, then $\sum_{n=1}^{\omega} 1/|a_n|^{\varepsilon} < \infty$ for each $\varepsilon > 0$;*

(iv) *if $\varrho_f = 1$, $\sigma_f = 0$ and $\sum_{n=1}^{\infty} 1/|a_n| = \infty$, then*

$$f(z) = e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} \quad \text{and} \quad \limsup_{r\to\infty} \left| a + \sum_{|a_n|<r} \frac{1}{a_n} \right| = 0.$$

In particular,

$$\limsup_{r\to\infty} \left(\sum_{|a_n|<r} \frac{1}{a_n} \right) = -a = -f'(0).$$

(v) *$\sum_{n=1}^{\omega} 1/|a_n|^{1+\varepsilon} < \infty$ for each $\varepsilon > 0$.*

3. The Fredholm determinant of the operator V_ϕ. We begin with an auxiliary lemma.

LEMMA 3.1. *Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix all of whose elements are 0 or 1 and $a_{ij} = 1$ for $1 \le j \le i \le n$. Then*

$$\det A = \prod_{i=2}^{n}(1 - a_{i-1,i}) = \begin{cases} 1, & a_{i-1,i} = 0 \text{ for } 2 \le i \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof is trivial. ∎

THEOREM 3.2. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0,1)$. Let V_ϕ be defined on $L_2[0,1]$ by (1.1). Then*

$$D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_{\phi(t_1)}^1 \ldots \int_{\phi(t_{n-1})}^1 dt_n \ldots dt_1. \tag{3.1}$$

Proof. It is clear that $(V_\phi f)(x) = \int_0^1 k(x,t) f(t)\, dt =: (Kf)(x)$, where

$$k(x,t) = \chi(\phi(x) - t) = \begin{cases} 1, & \phi(x) \ge t, \\ 0, & \phi(x) < t. \end{cases}$$

Assume that $0 \le t_1 \le \cdots \le t_n \le 1$. Then $k(t_i, t_j) = 1$ for $1 \le j \le i \le n$ and the matrix $(k(t_i,t_j))_{i,j=1}^n$ satisfies the assumptions of Lemma 3.1. Hence, $K(t_1, \ldots, t_n) = \prod_{i=2}^n (1 - k(t_{i-1}, t_i))$. Further, using (2.3)–(2.5) we get

$$A_n = n! \int_0^1 \int_{t_1}^1 \int_{t_2}^1 \ldots \int_{t_{n-1}}^1 \prod_{i=2}^n (1 - k(t_{i-1}, t_i))\, dt_n \ldots dt_1 = n! \int_{\Omega_n} 1\, dt_n \ldots dt_1,$$

where

$$\begin{aligned} \Omega_n &:= \{(t_1, \ldots, t_n) : 0 \le t_1 \le \cdots \le t_n \le 1, k(t_1,t_2) = \cdots = k(t_{n-1}, t_n) = 0\} \\ &= \{(t_1, \ldots, t_n) : 0 \le t_1 \le \phi(t_1) \le t_2 \le \phi(t_2) \le \cdots \le \phi(t_{n-1}) \le t_n \le 1\}. \end{aligned}$$

That is,

$$A_n = n! \int_0^1 \int_{\phi(t_1)}^1 \ldots \int_{\phi(t_{n-1})}^1 dt_n \ldots dt_1, \qquad n \ge 1.$$

This completes the proof. ∎

4. The spectrum of the operator V_ϕ. The following proposition immediately follows from Theorem 3.2.

PROPOSITION 4.1. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0,1)$. Then $\sigma_{\mathrm{p}}(V_\phi) \cap \mathbb{R}_- = \emptyset$.*

LEMMA 4.2. *Suppose $\phi : [0,1] \to [0,1]$ is a nondecreasing continuous function and $\phi(x) > x$ for $x \in (0,1)$. Then the following conditions are equivalent*:

(i) $\phi(0) > 0$ *and* $\phi(1-\varepsilon) = 1$ *for some* $0 < \varepsilon < 1$;
(ii) *there exists a unique* $N = N(\phi) \in \{2,3,\ldots\}$ *such that* $\phi^N(x) := \phi(\phi(\ldots\phi(x))) = 1$ *for all* $x \in [0,1]$ *and* $\phi^{N-1}(x_0) \neq 1$ *for some* $x_0 \in [0,1)$.

Proof. The proof is left to the reader. ∎

THEOREM 4.3. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0,1)$. Suppose also that $\phi(0) > 0$, $\phi(1-\varepsilon) = 1$ for some $0 < \varepsilon < 1$, and $N = N(\phi)$ is determined by Lemma* 4.2(ii). *Then*

(1) $\sigma_{\mathrm{p}}(V_\phi) = \{0\} \cup \{\lambda_1, \ldots, \lambda_N\}$, *with all* $\lambda_n \neq 0$;
(2) $\sum_{n=1}^N \lambda_n = 1$.

Proof. It is easily shown that $0 \in \sigma_{\mathrm{p}}(V_\phi)$. Using Theorem 3.2, we get

$$D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} A_n \lambda^n, \quad \text{where} \quad A_n = (-1)^n \int_0^1 \int_{\phi(t_1)}^1 \cdots \int_{\phi(t_{n-1})}^1 dt_n \ldots dt_1.$$

It is easily shown that $\phi^{n-1}(t_1) \le t_n \le 1$. Since $\phi^n(x) = 1$ for $n \ge N$, it follows that $A_n = 0$ for $n \ge N+1$. Therefore $D_{V_\phi}(\lambda)$ is a polynomial of degree N and (1) is proved. Further note that $D_{V_\phi}(\lambda) = \prod_{n=1}^N (1 - \lambda/a_n)$. Thus

$$\sum_{n=1}^N \lambda_n = \sum_{n=1}^N \frac{1}{a_n} = -A_1 = 1. \ \blacksquare$$

Let $\alpha_i, \beta_i \in C[0,1]$ $(1 \le i \le n)$. Define

$$\left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \ldots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\} := \int_{\beta_1(x)}^{\alpha_1(x)} \int_{\beta_2(x_1)}^{\alpha_2(x_1)} \cdots \int_{\beta_n(x_{n-1})}^{\alpha_n(x_{n-1})} dx_n \ldots dx_1.$$

So $\left\{ \begin{smallmatrix} \alpha_1 \\ \beta_1 \end{smallmatrix}, \ldots, \begin{smallmatrix} \alpha_n \\ \beta_n \end{smallmatrix} \right\}$ is a function of x. It is clear that

$$(4.1) \quad \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \ldots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \alpha_i \\ \beta_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \ldots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\} + \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \ldots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \gamma_i \\ \alpha_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \ldots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\}$$
$$= \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \ldots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \alpha_i \\ \beta_i \end{matrix} + \begin{matrix} \gamma_i \\ \alpha_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \ldots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\}$$
$$= \left\{ \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, \ldots, \begin{matrix} \alpha_{i-1} \\ \beta_{i-1} \end{matrix}, \begin{matrix} \gamma_i \\ \beta_i \end{matrix}, \begin{matrix} \alpha_{i+1} \\ \beta_{i+1} \end{matrix}, \ldots, \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\}.$$

The following lemmas are needed.

Lemma 4.4. *Let* $0 < \varepsilon_1 < \varepsilon_2 < 1$ *and*

$$\psi(x) = \begin{cases} \psi_1(x), & x \in [0, \varepsilon_1], \\ \psi_2(x), & x \in [\varepsilon_1, \varepsilon_2], \\ \psi_3(x), & x \in [\varepsilon_2, 1], \end{cases}$$

be a strictly increasing continuous function such that $\psi(\varepsilon_1) = \varepsilon_1$ *and* $\psi(\varepsilon_2) = \varepsilon_2$. *Let also* $a_0 = b_0 = c_0 = 1$ *and* a_k, b_k, c_k, d_k $(k = 1, 2, \dots)$ *be the* k*-fold integrals defined by*

$$a_k := \left\{ \begin{matrix} \varepsilon_1 \\ 0 \end{matrix}, \begin{matrix} \psi_1 \\ 0 \end{matrix}, \dots, \begin{matrix} \psi_1 \\ 0 \end{matrix} \right\}, \quad b_k := \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \dots, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix} \right\},$$
$$c_k := \left\{ \begin{matrix} 1 \\ \varepsilon_2 \end{matrix}, \begin{matrix} \psi_3 \\ \varepsilon_2 \end{matrix}, \dots, \begin{matrix} \psi_3 \\ \varepsilon_2 \end{matrix} \right\}, \quad d_k := \left\{ \begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\}.$$

Then

$$d_n = \sum_{k=0}^{n} c_k \sum_{l=0}^{n-k} b_l a_{n-k-l}, \qquad n = 1, 2, \dots.$$

Proof. Using (4.1), we get

$$\begin{aligned} d_n &= \left\{ \begin{matrix} \varepsilon_1 \\ 0 \end{matrix} + \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix} + \begin{matrix} 1 \\ \varepsilon_2 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\} \\ &= \left\{ \begin{matrix} \varepsilon_1 \\ 0 \end{matrix}, \begin{matrix} \psi_1 \\ 0 \end{matrix}, \dots, \begin{matrix} \psi_1 \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \varepsilon_1 \\ 0 \end{matrix} + \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\} \\ &\quad + \left\{ \begin{matrix} 1 \\ \varepsilon_2 \end{matrix}, \begin{matrix} \varepsilon_1 \\ 0 \end{matrix} + \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix} + \begin{matrix} \psi_3 \\ \varepsilon_2 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\} \\ &=: K_n + L_n + M_n. \end{aligned}$$

By definition $K_n = a_n$. Further, again using (4.1), we get

$$\begin{aligned} L_n &= \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \varepsilon_1 \\ 0 \end{matrix}, \begin{matrix} \psi_1 \\ 0 \end{matrix}, \dots, \begin{matrix} \psi_1 \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \varepsilon_1 \\ 0 \end{matrix} + \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\} \\ &= b_1 a_{n-1} + \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \varepsilon_1 \\ 0 \end{matrix}, \begin{matrix} \psi_1 \\ 0 \end{matrix}, \dots, \begin{matrix} \psi_1 \\ 0 \end{matrix} \right\} \\ &\quad + \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \varepsilon_1 \\ 0 \end{matrix} + \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\} \\ &= b_1 a_{n-1} + b_2 a_{n-2} + \left\{ \begin{matrix} \varepsilon_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi_2 \\ \varepsilon_1 \end{matrix}, \begin{matrix} \psi \\ 0 \end{matrix}, \dots, \begin{matrix} \psi \\ 0 \end{matrix} \right\} = \cdots \\ &= \sum_{k=1}^{n} b_k a_{n-k}, \end{aligned}$$

$$M_n = \left\{\frac{1}{\varepsilon_2}, \frac{\varepsilon_1}{0}, \frac{\psi_1}{0}, \ldots, \frac{\psi_1}{0}\right\} + \left\{\frac{1}{\varepsilon_2}, \frac{\varepsilon_2}{\varepsilon_1}, \frac{\psi_2}{0}, \frac{\psi}{0}, \ldots, \frac{\psi}{0}\right\}$$
$$+ \left\{\frac{1}{\varepsilon_2}, \frac{\psi_3}{\varepsilon_2}, \frac{\psi}{0}, \ldots, \frac{\psi}{0}\right\}$$
$$= c_1 a_{n-1} + c_1 L_{n-1} + \left\{\frac{1}{\varepsilon_2}, \frac{\psi_3}{\varepsilon_2}, \frac{\varepsilon_1}{0} + \frac{\varepsilon_2}{\varepsilon_1} + \frac{\psi_3}{\varepsilon_2}, \frac{\psi}{0}, \ldots, \frac{\psi}{0}\right\}$$
$$= c_1 a_{n-1} + c_1 L_{n-1} + c_2 a_{n-2} + c_2 L_{n-2}$$
$$+ \left\{\frac{1}{\varepsilon_2}, \frac{\psi_3}{\varepsilon_2}, \frac{\psi_3}{\varepsilon_2}, \frac{\varepsilon_1}{0} + \frac{\varepsilon_2}{\varepsilon_1} + \frac{\psi_3}{\varepsilon_2}, \frac{\psi}{0}, \ldots, \frac{\psi}{0}\right\}$$
$$= \cdots = \sum_{k=1}^{n} c_k a_{n-k} + \sum_{k=1}^{n-1} c_k L_{n-k}$$
$$= \sum_{k=1}^{n} c_k a_{n-k} + \sum_{k=1}^{n} c_k \sum_{l=1}^{n-k} b_l a_{n-k-l}.$$

Finally, we obtain

$$d_n = K_n + L_n + M_n$$
$$= c_0 a_n + \sum_{k=1}^{n} b_k a_{n-k} + \sum_{k=1}^{n} c_k a_{n-k} + \sum_{k=1}^{n} c_k \sum_{l=1}^{n-k} b_l a_{n-k-l}$$
$$= \sum_{k=0}^{n} c_k \sum_{l=0}^{n-k} b_l a_{n-k-l}. \blacksquare$$

LEMMA 4.5. *Let* $0 < \varepsilon \le 1/4$, $\beta > 1$, *and*

$$\psi_{\varepsilon,\beta}(x) = \begin{cases} x, & x \in [0, \varepsilon], \\ \varepsilon + (1-2\varepsilon)^{1-\beta}(x-\varepsilon)^{\beta}, & x \in [\varepsilon, 1-\varepsilon], \\ x, & x \in [1-\varepsilon, 1]. \end{cases}$$

Then

$$d_n := \left\{\frac{1}{0}, \frac{\psi_{\varepsilon,\beta}}{0}, \ldots, \frac{\psi_{\varepsilon,\beta}}{0}\right\} \tag{4.2}$$
$$= \frac{(2\varepsilon)^n}{n!} + \frac{(1-2\varepsilon)(2\varepsilon)^{n-1}}{(n-1)!}$$
$$+ \sum_{l=2}^{n} \frac{(1-2\varepsilon)^l (2\varepsilon)^{n-l}}{(n-l)!(1+\beta)\ldots(1+\beta+\ldots\beta^{l-1})}, \qquad n = 1, 2, \ldots.$$

Moreover,

$$d_n < \operatorname{const}(\varepsilon, \beta)\, \frac{(4\varepsilon)^n}{n!}, \qquad n = 1, 2, \ldots,$$

where $\operatorname{const}(\varepsilon, \beta)$ *does not depend on* n.

Proof. Substituting $\psi_{\varepsilon,\beta}$ for $\psi(x)$ in Lemma 4.4, we get (4.2). Indeed, it is easily proved that $a_l = c_l = \varepsilon^l/l!$ $(l = 0, 1, \dots n)$. Define $\widetilde{b}_1(x) := (1-2\varepsilon)^{1-\beta}(x-\varepsilon)^\beta$, $\psi_2(x) := \varepsilon + \widetilde{b}_1(x)$, and

$$\widetilde{b}_l(x) := \underbrace{\begin{Bmatrix} \psi_2 & & \psi_2 \\ \varepsilon & ,\dots, & \varepsilon \end{Bmatrix}}_{l}, \qquad l = 2, 3, \dots .$$

Then $\widetilde{b}_{l+1}(x) = \int_\varepsilon^{\psi_2(x)} \widetilde{b}_l(t)\,dt$. It can easily be checked (by induction on l) that

$$\widetilde{b}_l(x) = \frac{(1-2\varepsilon)^{l-\beta-\dots-\beta^l}(x-\varepsilon)^{\beta+\beta^2+\dots+\beta^l}}{(1+\beta)\dots(1+\beta+\dots+\beta^{l-1})}, \qquad l = 2, 3, \dots .$$

Since $b_l = \widetilde{b}_l(1-\varepsilon)$, we see that

$$\begin{aligned} &b_0 = 1, \qquad b_1 = 1 - 2\varepsilon, \\ &b_l = \frac{(1-2\varepsilon)^l}{(1+\beta)\dots(1+\beta+\dots+\beta^{l-1})}, \qquad l = 2, 3, \dots . \end{aligned} \tag{4.3}$$

Using Lemma 4.4, we get

$$\begin{aligned} d_n &= \sum_{k=0}^{n} c_k \sum_{l=0}^{n-k} b_l a_{n-k-l} = \sum_{l=0}^{n} b_l \sum_{k=0}^{n-l} c_k a_{n-k-l} \\ &= \sum_{l=0}^{n} b_l \sum_{l=0}^{n-l} \frac{\varepsilon^k}{k!} \frac{\varepsilon^{n-k-l}}{(n-k-l)!} \\ &= \sum_{l=0}^{n} b_l \frac{\varepsilon^{n-l}}{(n-l)!} \sum_{l=0}^{n-l} \frac{(n-l)!}{k!(n-l-k)!} \\ &= \sum_{l=0}^{n} b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!}, \qquad n = 1, 2, \dots . \end{aligned} \tag{4.4}$$

Substituting (4.3) for b_l in (4.4) we get (4.2).

To estimate d_n, taking into account the inequality of arithmetic and geometric means, we obtain

$$(1+\beta)\cdots(1+\beta+\dots+\beta^{l-1}) \geq 2\beta^{1/2}3\beta^{2/2}\dots l\beta^{(l-1)/2} = l!\beta^{(l-1)l/4}. \tag{4.5}$$

Hence,

$$b_l \leq \frac{(1-2\varepsilon)^l}{l!}\left(\frac{1}{\beta^{1/4}}\right)^{l^2-l} < \frac{(1-2\varepsilon)^l}{l!}.$$

Let N be a number such that

$$\left(\frac{1}{\beta^{1/4}}\right)^{l^2-l} < \left(\frac{2\varepsilon}{1-2\varepsilon}\right)^l \quad \text{for } l > N$$

(for example, $N = [4\log_\beta(1/(2\varepsilon) - 1)] + 2$). Then $b_l < (2\varepsilon)^l/l!$ for $l > N$. Using (4.4), we get, for $n > N$,

$$\begin{aligned} d_n &= \sum_{l=0}^{N} b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} + \sum_{l=N+1}^{n} b_l \frac{(2\varepsilon)^{n-l}}{(n-l)!} \\ &\le \frac{(2\varepsilon)^n}{n!} \sum_{l=0}^{N} \frac{n!}{l!(n-l)!} \left(\frac{1-2\varepsilon}{2\varepsilon}\right)^l + \frac{(2\varepsilon)^n}{n!} \sum_{l=N+1}^{n} \frac{n!}{l!(n-l)!} \\ &\le \frac{(2\varepsilon)^n}{n!} \left(\frac{1-2\varepsilon}{2\varepsilon}\right)^N \sum_{l=0}^{N} \frac{n!}{l!(n-l)!} + \frac{(2\varepsilon)^n}{n!} \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \\ &\le \frac{(4\varepsilon)^n}{n!} \left(\left(\frac{1-2\varepsilon}{2\varepsilon}\right)^N + 1 \right). \end{aligned}$$

This completes the proof. ∎

LEMMA 4.6. *Let $\beta > 1$ and*

$$\psi_\beta(x) := \begin{cases} 2^{\beta-1} x^\beta =: \psi_1(x), & x \in [0, 1/2], \\ 2^{\beta-1}(x - 1/2)^\beta + 1/2 =: \psi_2(x), & x \in [1/2, 1]. \end{cases}$$

Let $a_0 = b_0 = 1$ and a_k, b_k, and d_k $(k = 1, 2 \ldots)$ be the k-fold integrals defined by

$$a_k := \left\{ \begin{matrix} 1/2 \\ 0 \end{matrix}, \begin{matrix} \psi_1 \\ 0 \end{matrix}, \ldots, \begin{matrix} \psi_1 \\ 0 \end{matrix} \right\}, \qquad b_k := \left\{ \begin{matrix} 1 \\ 1/2 \end{matrix}, \begin{matrix} \psi_2 \\ 1/2 \end{matrix}, \ldots, \begin{matrix} \psi_2 \\ 1/2 \end{matrix} \right\},$$

$$d_k := \left\{ \begin{matrix} 1 \\ 0 \end{matrix}, \begin{matrix} \psi_\beta \\ 0 \end{matrix}, \ldots, \begin{matrix} \psi_\beta \\ 0 \end{matrix} \right\}.$$

Then

$$d_n = \sum_{l=0}^{n} b_l a_{n-l}, \qquad n = 1, 2, \ldots, \tag{4.6}$$

$$d_n < \frac{\beta^{(-n^2/2+n)/4}}{n!}, \qquad n = 1, 2, \ldots .$$

Proof. Substituting $1/2$ for ε_1 and 1 for ε_2 in Lemma 4.4, we get (4.6). Further, it is not hard to prove that $a_1 = b_1 = 1/2$ and

$$a_l = b_l = 2^{-l}((\beta + 1) \cdots (\beta^{l-1} + \cdots + 1))^{-1} \quad \text{for } l \ge 2.$$

Now, by (4.5), $a_l \le 2^{-l}/\beta^{(l-1)l/4} l!$ and

$$\begin{aligned} d_n &\le \sum_{l=0}^{n} \frac{2^{-l}}{\beta^{(l-1)l/4} l!} \, \frac{2^{-n+l}}{\beta^{(n-l-1)(n-l)/4}(n-l)!} \\ &= \frac{2^{-n}}{n!} \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} \, \beta^{(-2(l-n/2)^2 - n^2/2 + n)/4} < \frac{\beta^{(-n^2/2+n)/4}}{n!}. \ ∎ \end{aligned}$$

Proposition 4.7. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function.*

(1) *If $\phi(x) > x$ for $x \in (0,1)$ then the order of $D_{V_\phi}(\lambda)$ does not exceed 1, and if it equals 1, $D_{V_\phi}(\lambda)$ is of minimal type;*
(2) *if for some $0 < a < b < 1$,*

$$\phi(x) \ge f_{a,b}(x) := \begin{cases} \dfrac{b}{a}x, & x \in [0,a], \\ \dfrac{1-b}{1-a}x + \dfrac{b-a}{1-a}, & x \in [a,1], \end{cases}$$

for $x \in [0,1]$, then the order of $D_{V_\phi}(\lambda)$ equals 0.

Proof. (1) Taking into account Theorem 3.2, we obtain

$$D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n A_n \lambda^n, \quad \text{where} \quad A_n = \left\{ \frac{1}{0}, \frac{1}{\phi}, \dots, \frac{1}{\phi} \right\}.$$

Since $\phi(x) > x$ for each $0 < \varepsilon < 1/4$, it follows that there exists $\beta > 1$ such that $\phi(x) \ge \psi_{\varepsilon,\beta}^{-1}(x)$. Using Lemma 4.5, we get

$$\begin{aligned} A_n = d_n = \left\{ \frac{1}{0}, \frac{1}{\phi}, \dots, \frac{1}{\phi} \right\} &< \left\{ \frac{1}{0}, \frac{1}{\psi_{\varepsilon,\beta}^{-1}}, \dots, \frac{1}{\psi_{\varepsilon,\beta}^{-1}} \right\} \\ &= \left\{ \frac{1}{0}, \frac{\psi_{\varepsilon,\beta}}{0}, \dots, \frac{\psi_{\varepsilon,\beta}}{0} \right\} < \mathrm{const}(\varepsilon,\beta) \frac{(4\varepsilon)^n}{n!}. \end{aligned}$$

Therefore the order of growth of $D_{V_\phi}(\lambda)$ does not exceed 1. Assume that this order is 1. Then the type of $D_{V_\phi}(\lambda)$ does not exceed 4ε for each $\varepsilon < 1/4$. Thus $D_{V_\phi}(\lambda)$ is of minimal type.

(2) Since $\phi(x) \ge f_{a,b}(x)$ for some $0 < a < b < 1$, it follows that there exists $\beta > 1$ such that $\phi(x) \ge \psi_\beta^{-1}(x)$. Using Lemma 4.6, we get

$$\begin{aligned} A_n = d_n = \left\{ \frac{1}{0}, \frac{1}{\phi}, \dots, \frac{1}{\phi} \right\} &< \left\{ \frac{1}{0}, \frac{1}{\psi_\beta^{-1}}, \dots, \frac{1}{\psi_\beta^{-1}} \right\} \\ &= \left\{ \frac{1}{0}, \frac{\psi_\beta}{0}, \dots, \frac{\psi_\beta}{0} \right\} < \frac{\beta^{(-n^2/2+n)/4}}{n!}. \end{aligned}$$

Therefore the order of growth of $D_{V_\phi}(\lambda)$ equals 0. ∎

Theorem 4.8. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0,1)$. Suppose that either $\phi(0) = 0$ or $\phi(1-\varepsilon) \ne 1$ for all $0 < \varepsilon < 1$. Then*

(1) *$\sigma_{\mathrm{p}}(V_\phi) \setminus \{0\} =: \{\lambda_1, \lambda_2, \dots\}$ is an infinite set;*
(2) *$\lim_{\varepsilon \to 0} \sum_{|\lambda_n| > \varepsilon} \lambda_n = 1$;*
(3) *$\sum_{n=1}^{\omega} |\lambda_n|^{1+\varepsilon} < \infty$ for all $\varepsilon > 0$.*

Proof. Using Theorem 3.2, we get

$$D_{V_\phi}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n A_n \lambda^n, \quad \text{where} \quad A_n = \left\{ \frac{1}{0}, \frac{1}{\phi}, \dots, \frac{1}{\phi} \right\}.$$

It is easy to see that if either $\phi(0) = 0$ or $\phi(1-\varepsilon) \neq 1$ for all $0 < \varepsilon < 1$, then $A_n > 0$ for $n \geq 0$. Therefore $D_{V_\phi}(\lambda)$ is not a polynomial in λ. Now we apply Proposition 4.7(1). Suppose that the order of $D_{V_\phi}(\lambda)$ is less than 1; then using Theorem 2.1(ii), we get $D_{V_\phi}(\lambda) = \prod_{n=1}^{\omega}(1-\lambda/a_n)$. Since $D_{V_\phi}(\lambda)$ is not a polynomial, it follows that $\omega = \infty$ and $\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} 1/a_n = -A_1/A_0 = 1$. Now suppose that the order of $D_{V_\phi}(\lambda)$ is 1; then $D_{V_\phi}(\lambda)$ is of minimal type. Thus the spectrum of V_ϕ is an infinite set. Now, the application of Theorem 2.1(i), (iv) yields (2).

(3) follows from Theorem 2.1. ∎

Now we are ready to prove the main result of the paper.

Proof of Theorem 1.1. (1) follows from Theorem 4.3(1) and Theorem 4.8(1).

(2)–(3) follow from Theorem 4.3(2) and Theorem 4.8(2)–(3). ∎

THEOREM 4.9. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function and for some $0 < a < b < 1$,*

$$\phi(x) \geq \begin{cases} \dfrac{b}{a} x, & x \in [0,a], \\ \dfrac{1-b}{1-a} x + \dfrac{b-a}{1-a}, & x \in [a,1], \end{cases}$$

for all $x \in [0,1]$. Suppose that either $\phi(0) = 0$ or $\phi(1-\varepsilon) \neq 1$ for all $0 < \varepsilon < 1$. Then

(1) $\sigma_{\mathrm{p}}(V_\phi) \setminus \{0\} =: (\lambda_1, \lambda_2, \dots)$ *is an infinite set*;

(2) $\sum_{n=1}^{\infty} \lambda_n = 1$;

(3) $\sum_{n=1}^{\infty} |\lambda_n|^{\varepsilon} < \infty$ *for all $\varepsilon > 0$.*

Proof. (1) follows from Theorem 4.8(1). By Proposition 4.7(2), the order of $D_{V_\phi}(\lambda)$ equals 0. Thus (2) and (3) follow from (ii) and (iii) of Theorem 2.1. ∎

REMARK 4.10. (i) Suppose ϕ is strictly increasing and $\phi(x) > x$ for all $x \in (0,1)$. Assume that also $\phi \in C^1[0,1]$ and $(\phi')^{-1/2} \in L_\infty[0,1]$. We claim that $V_\phi \notin \mathbf{S}_1$. Indeed, let $c := (\int_0^1 (\phi'(s))^{1/2}\, ds)^{-1}$ and let W_ϕ and T_ϕ be the linear operators on $L_2[0,1]$ defined by

$$(W_\phi f)(x) = \int_0^x (\phi'(t))^{1/2} f(t)\, dt, \qquad (T_\phi f)(x) = f\Big(c \int_0^x (\phi'(s))^{1/2}\, ds\Big).$$

It can easily be checked (see [2]–[3]) that T_ϕ and T_ϕ^{-1} are bounded operators and $cV_x = T_\phi^{-1} W_\phi T_\phi$. Hence (see [5]),

$$s_n(W_\phi) \ge \|T_\phi\|^{-1}\|T_\phi^{-1}\|^{-1} s_n(cV_x) = \|T_\phi\|^{-1}\|T_\phi^{-1}\|^{-1} c\,\frac{2}{(2n-1)\pi}.$$

Further,

$$(V_\phi V_\phi^* f)(x) = \int\limits_0^{\phi(x)} \int\limits_{\phi^{-1}(t)}^{1} f(s)\,ds\,dt = \int\limits_0^{x} \phi'(t) \int\limits_t^1 f(s)\,ds\,dt = (W_\phi W_\phi^* f)(x).$$

Thus $s_n(V_\phi) = s_n(W_\phi) \ge \|T_\phi\|^{-1}\|T_\phi^{-1}\|^{-1} c\,\frac{2}{(2n-1)\pi}$. Hence, $V_\phi \notin \mathbf{S}_1$.

(ii) Since $V_\phi \notin \mathbf{S}_1$, it follows that the matrix trace of the operator V_ϕ is not defined. Hence we cannot use (2.1)–(2.2) to prove Theorem 4.9(2). Nevertheless, (2.1)–(2.2) hold for $K = V_\phi$ and the orthonormal basis $\{e_n\}_{n=1}^\infty$ defined by: $e_1 \equiv 1$, $e_{2n} := e^{2\pi i n x}$ and $e_{2n+1} := e^{-2\pi i n x}$ ($n = 1, 2, \dots$). Indeed, since $\sum_{n=1}^\infty (\sin nx)/n = (\pi - x)/2$ for $x \in (0, 2\pi)$, it follows that

$$\begin{aligned}
\sum_{n=0}^\infty (V_\phi e_n, e_n) &= \int\limits_0^1 \phi(x)\,dx \\
&\quad + \sum_{n=1}^\infty \Bigg(\int\limits_0^1 \frac{(e^{2\pi i n \phi(x)} - 1)e^{-2\pi i n x}}{2\pi i n}\,dx + \int\limits_0^1 \frac{(e^{-2\pi i n \phi(x)} - 1)e^{2\pi i n x}}{-2\pi i n}\,dx \Bigg) \\
&= \int\limits_0^1 \phi(x)\,dx + \sum_{n=1}^\infty \int\limits_0^1 \frac{\sin(2\pi n(\phi(x) - x))}{\pi n}\,dx \\
&= \int\limits_0^1 \phi(x)\,dx + \int\limits_0^1 \frac{1}{\pi}\,\frac{\pi - 2\pi(\phi(x) - x)}{2}\,dx = 1.
\end{aligned}$$

Further, $\int_0^1 \chi(\phi(x) - x)\,dx = 1$. Thus formulas (2.1)–(2.2) hold. This contrasts with the fact that $\sum_{n=0}^\infty (V_x e_n, e_n) = \infty$.

(iii) Theorem 1.1 states that the spectral trace of an operator V_ϕ always equals 1. This also contrasts with the fact that the operator V_x is quasinilpotent.

To estimate the spectral radius $r(V_\phi)$ we recall (see [14]) some results on integral operators with nonnegative kernels. Let $(Kf)(x) = \int_0^1 k(x,t) f(t)\,dt$ and $k(x,t) \ge 0$ for $(x,t) \in [0,1]\times[0,1]$. If there exist $\alpha > 0$ and a nonnegative function f such that $(Kf)(x) \ge \alpha f(x)$ for $x \in [0,1]$, then $r(K) \in \sigma_{\mathrm{p}}(K)$ and $r(K) > \alpha$.

Proposition 4.11. *Let $\phi : [0,1] \to [0,1]$ be a strictly increasing continuous function such that $\phi(x) \ge x$ for all $x \in [0,1]$. Write $\sigma_{\mathrm{p}}(V_\phi) = \{\lambda_n\}_{n=1}^\omega$ ($\omega \le \infty$). Then*

(1) $r(V_\phi) \geq \max_{x\in[0,1]} (\phi(x) - x)$, $r(V_\phi) \in \sigma_{\mathrm{p}}(V_\phi)$.

Suppose moreover that $\phi(0) = 0$. *Then* $\omega = \infty$ *and*

(2) $\sum_{n=1}^{\infty} \lambda_n^2 = 2\int_0^1 \phi(t)\,dt - 1$;

(3) $\sum_{n=1}^{\infty} \lambda_n^3 = 1 - 3\int_0^1 \phi(t)\phi^{-1}(t)\,dt$.

Proof. (1) Let $f_a(x) = 1 - \chi(a - x)$, $a \in (0,1)$. Then

$$(V_\phi f_a)(x) = \left.\begin{cases} 0, & [0,\phi^{-1}(a)] \\ \phi(x) - a, & [\phi^{-1}(a), 1] \end{cases}\right\} \geq (\phi(a) - a) f_a(x),$$

and (1) is proved.

(2), (3) It is easy to check that $\phi^{-1}(x)$ is well defined and

$$(V_\phi^2 f)(x) = \int_0^1 \chi(\phi^2(x) - t)(\phi(x) - \phi^{-1}(t)) f(t)\,dt =: \int_0^1 k_2(x,t) f(t)\,dt,$$

$$(V_\phi^3 f)(x) = \int_0^1 \chi(\phi^3(x) - t) \int_{\phi^{-2}(t)}^{\phi(x)} (\phi(s) - \phi^{-1}(t))\,ds\, f(t)\,dt =: \int_0^1 k_3(x,t) f(t)\,dt.$$

Further, k_2 and k_3 are continuous functions on $[0,1]\times[0,1]$. Hence, $V_\phi^2 \in \mathbf{S}_1$ and $V_\phi^3 \in \mathbf{S}_1$. Now if we recall (2.2), we get

$$\sum_{n=1}^{\infty} \lambda_n^2 = \int_0^1 k_2(t,t)\,dt = \int_0^1 (\phi(t) - \phi^{-1}(t))\,dt = 2\int_0^1 \phi(t)\,dt - 1,$$

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n^3 = \int_0^1 k_3(t,t)\,dt &= \int_0^1 \int_{\phi^{-2}(t)}^{\phi(t)} (\phi(s) - \phi^{-1}(t))\,ds \\ &= \int_0^1 (\phi(t)\phi^2(t) - 2\phi^{-1}(t)\phi(t) + \phi^{-1}(t)\phi^{-2}(t))\,dt \\ &= 1 - 3\int_0^1 \phi(t)\phi^{-1}(t)\,dt. \ \blacksquare\end{aligned}$$

EXAMPLE 4.12. Let $\phi(x) = x^\alpha$ $(0 < \alpha < 1)$. It can be proved by direct calculations that

$$D_{V_{x^\alpha}}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_{t_1^\alpha}^1 \ldots \int_{t_{n-1}^\alpha}^1 dt_n \ldots dt_1$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \frac{\alpha^{n(n-1)/2}(1-\alpha)^n}{(1-\alpha)\dots(1-\alpha^n)} = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{(1-\alpha)\alpha^{n-1}}\right).$$

Hence, $\sigma_{\mathrm{p}}(V_{x^\alpha}) = \{(1-\alpha)\alpha^{n-1}\}_{n=1}^{\infty}$ and each eigenvalue of V_{x^α} is of algebraic multiplicity one. Further,

$$\sum_{n=1}^{\infty} (1-\alpha)\alpha^{n-1} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} ((1-\alpha)\alpha^{n-1})^{\varepsilon} = \frac{(1-\alpha)^{\varepsilon}}{1-\alpha^{\varepsilon}} < \infty$$

for each $\varepsilon > 0$.

5. Some generalizations

5.1. The following lemma can be proved by direct calculations.

LEMMA 5.1. *Let A be a compact operator defined on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{H} = \bigoplus_{i=1}^{k} \mathfrak{H}_i$ and $A_i := P_i A : \mathfrak{H}_i \to \mathfrak{H}_i$, where P_i be the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{H}_i$. Suppose that $\{\bigoplus_{j=1}^{i} \mathfrak{H}_j\}_{i=1}^{k}$ is invariant for A. Then $1/\lambda$ is an eigenvalue of A of algebraic multiplicity $m \geq 1$ if and only if $1/\lambda$ is an eigenvalue of A_i of algebraic multiplicity $m_i \geq 0$ and $\sum_{i=1}^{k} m_i = m$.*

Proof. The proof is omitted. ∎

THEOREM 5.2. *Let $\phi : [0,1] \to [0,1]$ be a strictly increasing continuous function. Let $\{x : \phi(x) = x,\ x \in (0,1)\} = \{a_i\}_{i=1}^{k-1}$, where $0 < a_1 < \dots < a_{k-1} < 1$ $(k \geq 2)$. Define $a_0 := 0$, $a_k := 1$, and*

$$\phi_i(x) := (\phi(x(a_i - a_{i-1}) + a_{i-1}) - a_{i-1})/(a_i - a_{i-1}), \qquad 1 \leq i \leq k.$$

$$D_{V_{\phi_i}}(\lambda) := \begin{cases} 1 + \displaystyle\sum_{n=0}^{\infty} (-\lambda)^n \left\{ \frac{1}{0}, \frac{1}{\phi_i}, \dots, \frac{1}{\phi_i} \right\} & \text{if } \phi_i(x) > x \text{ for } x \in (0,1), \\ 1 & \text{if } \phi_i(x) < x \text{ for } x \in (0,1). \end{cases}$$

Then

(1) *$1/\lambda \in \sigma_{\mathrm{p}}(V_\phi)$ if and only if $\prod_{i=1}^{k} D_{V_{\phi_i}}((a_i - a_{i-1})\lambda) = 0$;*
(2) *the algebraic multiplicity of the eigenvalue $1/\lambda$ is equal to the multiplicity of λ as a root of the entire function $\prod_{i=1}^{k} D_{V_{\phi_i}}((a_i - a_{i-1})\lambda)$.*

Proof. Set $\mathfrak{H} := L_2[0,1]$, $\mathfrak{H}_i := L_2[a_{i-1}, a_i]$ and

$$P_i : f(x) \to \begin{cases} f(x), & x \in [a_{i-1}, a_i], \\ 0, & x \notin [a_{i-1}, a_i], \end{cases} \qquad P_i : \mathfrak{H} \to \mathfrak{H}_i,$$

$$A := V_\phi, \qquad A_i := P_i A \restriction_{\mathfrak{H}_i},$$

$$T_i : \left\{ \begin{matrix} f(x), & x \in [a_{i-1}, a_i] \\ 0, & x \notin [a_{i-1}, a_i] \end{matrix} \right\} \mapsto f((a_i - a_{i-1})x + a_{i-1}), \qquad T_i : \mathfrak{H}_i \to \mathfrak{H}.$$

It follows easily that $\bigoplus_{j=1}^{i}\mathfrak{H}_j$ $(= L_2[0,a_i])$ is invariant for A and

$$A_i : \begin{Bmatrix} f(x), & x\in[a_{i-1},a_i] \\ 0, & x\notin[a_{i-1},a_i] \end{Bmatrix} \mapsto \begin{cases} \int\limits_{a_{n-1}}^{\phi(x)} f(t)\,dt, & x\in[a_{i-1},a_i], \\ 0, & x\notin[a_{i-1},a_i], \end{cases}$$

$$T_i^{-1} : f(x) \mapsto \begin{cases} f\left(\dfrac{x-a_{i-1}}{a_i-a_{i-1}}\right), & x\in[a_{i-1},a_i], \\ 0, & x\notin[a_{i-1},a_i], \end{cases} \qquad T_i : \mathfrak{H}\to\mathfrak{H}_i,$$

$$T_iA_iT_i^{-1} = (a_i-a_{i-1})V_{\phi_i}.$$

The application of Theorem 3.2 yields

$$1/\lambda\in\sigma_{\mathrm{p}}(A_i) \Leftrightarrow 1/\lambda\in\sigma_{\mathrm{p}}((a_i-a_{i-1})V_{\phi_i}) \Leftrightarrow D_{V_{\phi_i}}((a_i-a_{i-1})\lambda)=0.$$

Now applying Lemma 5.1 completes the proof. ∎

COROLLARY 5.3. *Suppose* ϕ *satisfies the conditions of Theorem* 5.2 *and* $\mathrm{mes}\{x:\phi(x)\ge x,\ x\in[0,1]\}>0$. *Set also* $\sigma_{\mathrm{p}}(V_\phi)\setminus\{0\}=\{\lambda_n\}_{n=1}^{\omega}$ $(1\le\omega\le\infty)$. *Then*

(1) $\omega<\infty$ *if and only if* $\phi(0)>0$, $\phi(1-\varepsilon)=1$ *for some* $0<\varepsilon<1$ *and* $\phi(x)>x$ *for all* $x\in(0,1)$;
(2) $\lim_{\varepsilon\to0}\sum_{|\lambda_n|>\varepsilon}\lambda_n=\mathrm{mes}\{x:\phi(x)\ge x,\ x\in[0,1]\}$.

Proof. (1) follows from Theorems 4.3, 4.8, 5.2.

(2) By definition, put

$$\Omega := \{i:\phi(x)\ge x \text{ for } x\in[a_{i-1},a_i]\} = \{i:\phi_i(x)\ge x \text{ for } x\in[0,1]\},$$
$$\sigma_{\mathrm{p}}(V_{\phi_i}) := \{\lambda_{in}\}_{n=1}^{\omega_i}, \qquad 1\le\omega\le\infty, \qquad i\in\Omega.$$

By Theorem 5.2,

$$\{\lambda_n\}_{n=1}^{\omega}=\sigma_{\mathrm{p}}(V_\phi)=\bigcup_{i\in\Omega}\sigma_{\mathrm{p}}((a_i-a_{i-1})V_{\phi_i})=\bigcup_{i\in\Omega}(a_i-a_{i-1})\{\lambda_{in}\}_{n=1}^{\omega_i}.$$

By Theorem 4.8,

$$\lim_{\varepsilon\to0}\sum_{|\lambda_{in}|>\varepsilon}\lambda_{in}=1.$$

Thus

$$\lim_{\varepsilon\to0}\sum_{|\lambda_n|>\varepsilon}\lambda_n=\sum_{i\in\Omega}(a_i-a_{i-1})\lim_{\varepsilon\to0}\sum_{|\lambda_{in}|>\varepsilon}\lambda_{in}$$
$$=\sum_{i\in\Omega}(a_i-a_{i-1})=\mathrm{mes}\{x:\phi(x)\ge x,\ x\in[0,1]\}. \ ∎$$

REMARK 5.4. It is interesting to note that the case of nonincreasing function ϕ can be more multifarious. In particular, if ϕ is a strictly decreasing continuous function such that $\phi(0) = 1$, $\phi(1) = 0$ and $\phi(\phi(x)) = x$ then V_ϕ is a selfadjoint operator in $L_2[0,1]$. For example,

$$\sigma_{\mathrm{p}}(V_{1-x}) = \left\{ \frac{2(-1)^n}{(2n+1)\pi} \right\}_{n=1}^{\infty}$$

and

$$\sum_{n=1}^{\infty} \frac{2(-1)^n}{(2n+1)\pi} = \frac{2}{\pi}\frac{\pi}{4} = \frac{1}{2} = \operatorname{mes}\{x : 1 - x \geq x\}.$$

5.2. In this subsection we consider an operator V_ϕ defined on $L_p[0,1]$ $(1 \leq p < \infty)$.

Let A_i be a bounded operator defined on Banach space X_i $(i = 1,2)$. Recall that A_1 is said to be *quasisimilar* to A_2 if there exist deformations $K : X_1 \to X_2$ and $L : X_2 \to X_1$ (i.e. $\overline{\mathfrak{R}(K)} = X_2$, $\ker K = \{0\}$, $\overline{\mathfrak{R}(L)} = X_1$, $\ker L = \{0\}$) such that $A_1 L = L A_2$ and $K A_1 = A_2 K$. It is clear that $\sigma_{\mathrm{p}}(A_1) = \sigma_{\mathrm{p}}(A_2)$.

Proposition 5.5. *Let $\phi : [0,1] \to [0,1]$ be a strictly increasing continuous function such that $\phi(0) = 0$ and $\phi(1) = 1$. Let A_1 denote an operator V_ϕ defined on $L_p[0,1]$ $(1 \leq p < \infty)$ and let A_2 denote V_ϕ defined on $L_2[0,1]$. Then A_1 is quasisimilar to A_2, and hence $\sigma_{\mathrm{p}}(A_1) = \sigma_{\mathrm{p}}(A_2)$.*

Proof. Set $K := V_\phi : L_p[0,1] \to L_2[0,1]$, $L := V_\phi : L_2[0,1] \to L_p[0,1]$. It is clear that K and L are deformations and $A_1 L = L A_2$, $K A_1 = A_2 K$. ∎

5.3. Now we consider the operator

$$(V_{\phi,q,w} f)(x) := q(x) \int_0^{\phi(x)} f(t) w(t)\, dt$$

defined on $L_2[0,1]$. The proof of the following theorem is similar to the proof of Theorem 3.2.

Theorem 5.6. *Let $\phi : [0,1] \to [0,1]$ be a nondecreasing continuous function such that $\phi(x) > x$ for all $x \in (0,1)$. Let $q, w \in L_2[0,1]$. Then*

$$D_{V_{\phi,q,w}}(\lambda) = 1 + \sum_{n=1}^{\infty} (-1)^n \lambda^n \int_0^1 \int_{\phi(t_1)}^1 \cdots \int_{\phi(t_{n-1})}^1 q(t_1) w(t_1) \ldots q(t_n) w(t_n)\, dt_n \ldots dt_1.$$

Corollary 5.7. *Under the assumptions of Theorem 5.6 suppose that $q(x)w(x) > 0$ for a.a. $x \in [0,1]$. Then $\sigma_{\mathrm{p}}(V_{\phi,q,w}) \setminus \{0\}$ is a finite set if and only if $\phi(0) > 0$ and $\phi(1-\varepsilon) = 1$ for some $0 < \varepsilon < 1$.*

Acknowledgments. The author wishes to thank Professor J. Zemánek for setting the problem.

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Received January 12, 2006
Revised version December 17, 2007 (5843)