STUDIA MATHEMATICA 185 (1) (2008)

On operators from separable reflexive spaces with asymptotic structure

by

BENTUO ZHENG (Austin, TX)

Abstract. Let $1 < q < p < \infty$ and $q \le r \le p$. Let X be a reflexive Banach space satisfying a lower-ℓ_q-tree estimate and let T be a bounded linear operator from X which satisfies an upper-ℓ_p-tree estimate. Then T factors through a subspace of $(\sum F_n)_{\ell_r}$, where (F_n) is a sequence of finite-dimensional spaces. In particular, T factors through a subspace of a reflexive space with an (ℓ_p, ℓ_q) FDD. Similarly, let $1 < q < r < p < \infty$ and let X be a separable reflexive Banach space satisfying an asymptotic lower-ℓ_q-tree estimate. Let T be a bounded linear operator from X which satisfies an asymptotic upper-ℓ_p-tree estimate. Then T factors through a subspace of $(\sum G_n)_{\ell_r}$, where (G_n) is a sequence of finite-dimensional spaces. In particular, T factors through a subspace of a reflexive space with an asymptotic (ℓ_p, ℓ_q) FDD.

1. Introduction. In [10], E. Odell and Th. Schlumprecht prove that a separable reflexive Banach space X which satisfies an (ℓ_p, ℓ_q)-tree estimate for $1 \le q \le p \le \infty$ embeds into a reflexive Banach space with an (ℓ_p, ℓ_q) FDD. In particular, this proves that if every normalized weakly null tree in a separable reflexive Banach space X has a branch equivalent to the unit vector basis of ℓ_p $(1 < p < \infty)$, then X is isomorphic to a subspace of an ℓ_p sum of finite-dimensional spaces. Then, in [11], E. Odell, Th. Schlumprecht and A. Zsák prove that a separable reflexive Banach space X which satisfies an asymptotic (ℓ_p, ℓ_q)-tree estimate embeds into a reflexive Banach space with an asymptotic (ℓ_p, ℓ_q) FDD. So a special case is that every separable reflexive asymptotic ℓ_p space is a subspace of a reflexive Banach space with an asymptotic ℓ_p FDD.

Recall [8] that a Banach space X with an FDD (E_n) is *asymptotic* ℓ_p with respect to (E_n) if there exists a $1 \le C < \infty$ so that for all n, every

2000 *Mathematics Subject Classification*: Primary 46B03; Secondary 46B20.

Key words and phrases: lower-ℓ_p-tree estimate, upper-ℓ_q-tree estimate, asymptotic lower-ℓ_p-tree estimate, asymptotic upper-ℓ_q-tree estimate, factorizations.

Supported in part by NSF/DMS 02-00690 and 05-03688. This paper forms part of the author's doctoral dissertation which is being prepared at Texas A&M University under the direction of W. B. Johnson.

 © Instytut Matematyczny PAN, 2008

normalized block sequence $(x_i)_{i=1}^n$ of $(E_i)_{i=n}^\infty$ is C-equivalent to the unit vector basis of ℓ_p. A coordinate-free version of this notion is in [7]. Let X be an arbitrary Banach space. Then X is said to be *asymptotic* ℓ_p if there exists $1 \le C < \infty$ so that $\forall n \in \mathbb{N}, \exists Y_1 \in \mathrm{cof}(X)\ \forall y_1 \in S_{Y_1}, \ldots, \exists Y_n \in \mathrm{cof}(X)$ $\forall y_n \in S_{Y_n}$, $(y_i)_{i=1}^n$ is C-equivalent to the unit vector basis of ℓ_p^n. An FDD $(E_n)_{i=1}^\infty$ is *asymptotic* (ℓ_p, ℓ_q) if there exists $0 < C < \infty$ such that for all $n \in \mathbb{N}$ and all block sequences $(x_i)_{i=1}^n$ of $(E_n)_{i=n}^\infty$,

$$C^{-1}\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p} \le \Big\|\sum_{i=1}^n x_i\Big\| \le C\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{1/q}.$$

If $p = q$, then we say that (E_n) is *asymptotic* ℓ_p.

The results above can be restated in the way that under some conditions, the identity operator on a separable reflexive Banach space factors through a subspace of an ℓ_p sum of finite-dimensional spaces, and under some other conditions, it factors through a subspace of a space with an asymptotic ℓ_p FDD. From this point of view, it is natural to consider general operators from a separable reflexive Banach space. The goal is to find the right conditions under which the operators factor through a subspace of an ℓ_p sum of finite-dimensional spaces or factor through a subspace of a space with an asymptotic ℓ_p FDD. In [12], the author proves that if X is a Banach space with an FDD satisfying a block lower-p estimate and T is an operator from X which satisfies an upper-ℓ_p-tree estimate, then T factors through $(F_n)_{\ell_p}$, where (F_n) is a sequence of finite-dimensional spaces. An important consequence is that any bounded linear operator from L_p $(2 < p < \infty)$ which satisfies an upper-ℓ_p-tree estimate factors through ℓ_p (actually this is also a necessary condition).

In this paper, the author proves the following theorems. Theorem A generalizes the results in [12] we mentioned above and Theorem B gives conditions for operators from spaces with certain asymptotic structure to factor through a subspace of an ℓ_r sum of finite-dimensional spaces.

THEOREM A. *Let $1 < q \le r \le p < \infty$ and let X be a separable reflexive Banach space which satisfies a lower-ℓ_q-tree estimate. Let T be a bounded linear operator from X into Y which satisfies an upper-ℓ_p-tree estimate. Then T factors through a subspace of $(\sum F_n)_{\ell_r}$, where (F_n) is a sequence of finite-dimensional spaces.*

THEOREM B. *Let $1 < q < r < p < \infty$ and let X be a separable reflexive Banach space which satisfies an asymptotic lower-ℓ_q-tree estimate. Let T be a bounded linear operator from X which satisfies an asymptotic upper-ℓ_p-tree estimate. Then T factors through a subspace of $(F_n)_{\ell_r}$, where (F_n) is a sequence of finite-dimensional spaces.*

2. Definitions and notations. Let X be a Banach space and let S_X be the unit sphere of X. Let $[\mathbb{N}]^{<\omega}$ denote all finite subsets of the positive integers. By a *normalized weakly null tree* we mean a family $(x_A)_{A\in[\mathbb{N}]^{<\omega}} \subset S_X$ with the property that every sequence $(x_{A\cup\{n\}})_{n\in\mathbb{N}}$ is weakly null. Let $A = \{n_1, \ldots, n_m\}$ with $n_1 < \cdots < n_m$ and $B = \{j_1, \ldots, j_r\}$ with $j_1 < \cdots < j_r$. Then we say A is an *initial segment* of B if $m \le r$ and $n_i = j_i$ when $1 \le i \le m$. The tree order on $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ is given by $x_A \le x_B$ if A is an initial segment of B. A *branch* of a tree is a maximal linearly ordered subset of the tree under the tree order.

Definition 2.1. Let V be a Banach space with a 1-unconditional and normalized basis (v_i). We say that a Banach space X satisfies a *lower-V-tree estimate* if there exists a $C \ge 1$ such that every normalized weakly null tree in X has a branch (x_i) so that for all $(a_i) \subset \mathbb{R}$,

$$\Big\| \sum a_i x_i \Big\| \ge C^{-1} \Big\| \sum a_i v_i \Big\|.$$

In this paper, we need an analogous definition for operators.

Definition 2.2. Let U be a Banach space with a 1-unconditional and normalized basis (u_i). Let T be a bounded linear operator from X. We say that T satisfies an *upper-U-tree estimate* if there exists a $C > 0$ such that every normalized weakly null tree in X has a branch (x_i) so that for all $(a_i) \subset \mathbb{R}$,

$$\Big\| T\Big(\sum a_i x_i \Big) \Big\| \le C \Big\| \sum a_i u_i \Big\|.$$

Definition 2.3. Let $1 \le p \le \infty$. A reflexive Banach space X satisfies an *asymptotic lower-ℓ_p-tree estimate* if there exists a $1 \le C < \infty$ so that for every $k \in \mathbb{N}$, every normalized weakly null tree of length k in X has a branch $(x_i)_{i=1}^k$ such that for all $(a_i) \subset \mathbb{R}$,

$$\Big\| \sum_{i=1}^k a_i x_i \Big\| \ge C^{-1} \Big(\sum_{i=1}^k \|a_i\|^p \Big)^{1/p}.$$

Definition 2.4. Let $1 \le q \le \infty$ and let T be a bounded linear operator from a Banach space X. Then T satisfies an *asymptotic upper-ℓ_q-tree estimate* if there exists a $1 \le C < \infty$ so that for every $k \in \mathbb{N}$, every normalized weakly null tree of length k in X has a branch $(x_i)_{i=1}^k$ such that

$$\Big\| \sum_{i=1}^k x_i \Big\| \le C \Big(\sum_{i=1}^k \|x_i\|^q \Big)^{1/q}.$$

3. Main results. Let (u_i) be a sequence in a Banach space U and let (v_i) be a sequence in a Banach space V. We say that (u_i) *C-dominates* (v_i)

or (v_i) *is* C^{-1}*-dominated by* (u_i) if for all $(a_i) \subset \mathbb{R}$,

$$\Big\| \sum a_i u_i \Big\|_U \geq C \Big\| \sum a_i v_i \Big\|_V .$$

Let (w_i) be a 1-unconditional normalized basis for W. Let $F = (F_n)$ be an FDD for a Banach space Z. Then we define the space $Z_W(F)$ to be the completion of $c_{00}(\bigoplus F_n)$ under the norm

$$\|(x_i)\| = \Big\| \sum_j \|x_i\| w_j \Big\|.$$

Let (w_i) be a basis for some Banach space W. We say that (w_i) is C-*subsymmetric* if it is 1-unconditional and every subsequence is C-equivalent to (w_i).

THEOREM 3.1. *Let U and V be two Banach spaces with 1-unconditional normalized bases (u_i) and (v_i). Let (w_i) be a normalized C'-subsymmetric basis for W. Suppose that (u_i) A_1-dominates (w_i) and (w_i) A_2-dominates (v_i). Let X be a separable reflexive Banach space with an FDD (E_n) which satisfies a lower-U-tree estimate. Let T be a bounded linear operator from X into Y which satisfies an upper-V-tree estimate. Then T factors through $X_W(F)$, where $F = (F_n)$ is a blocking of (E_n).*

In the proof of Theorem 3.1, the following lemmas are used.

LEMMA 3.2 (Propositions 2.4 and 2.5 in [10]). *Let X be a separable reflexive Banach space which embeds into a reflexive Banach space with an FDD (E_n). Then for $\mathcal{A} \subset S_X^\omega$, the following are equivalent*:

(a) *For all $\varepsilon > 0$, every weakly null tree in S_X has a branch in $\widetilde{\mathcal{A}}_\varepsilon$.*
(b) *For all $\varepsilon > 0$, there exists a blocking (F_n) of (E_n) and $\delta = (\delta_i)$ with $\delta_i \downarrow 0$ so that if $(x_n) \subset S_X$ is a δ-skipped block sequence with respect to (F_i) then $(x_n) \in \widetilde{\mathcal{A}}_\varepsilon$.*

Here $\mathcal{A}_\varepsilon = \{(x_n) \subset S_X : \text{there exists } (y_n) \in \mathcal{A} \text{ with } \|x_n - y_n\| < \varepsilon/2^n \text{ for all } n\}$ and $\widetilde{\mathcal{A}}_\varepsilon$ denotes the closure of $\mathcal{A}_\varepsilon$ with respect to the product topology of the discrete topology on S_X.

Given an FDD (E_n), (x_n) is said to be a *skipped block sequence with respect to* (E_n) if there exists a sequence of integers $1 = m_1 < m_2 < \cdots$ such that $m_n + 1 < m_{n+1}$ and $x_n \in \bigoplus_{j=m_n+1}^{m_{n+1}-1} E_j$, $\forall n \in \mathbb{N}$. Let $\delta = (\delta_i)$ be a sequence of positive numbers decreasing to 0. We say (y_n) is a δ-*skipped block sequence with respect to* (E_n) if there is a skipped block sequence (x_n) so that $\|y_n - x_n\| < \delta_n \|y_n\|$ for all $n \in \mathbb{N}$. We say (F_n) is a *blocking* of (E_n) if there is a sequence of integers $0 = k_0 < k_1 < \cdots$ so that $F_n = \bigoplus_{j=k_{n-1}+1}^{k_n} E_j$.

Proof of Theorem 3.1. For a blocking $F = (F_i)$ of $E = (E_n)$, let J_F be the natural embedding of X into $X_W(F)$ so that if $x = \sum x_i$ with $x_i \in F_i$,

then $J_F(x) = \sum x_i \in X_W(F)$. We define $\widetilde{T}$ to be the operator from $J_F(X)$ into Y so that $\widetilde{T} \circ J_F = T$. Initially, J_F and $\widetilde{T}$ are only defined on the linear span of the FDD's. Once they are bounded, they have bounded linear extensions to the closures. So our goal is to find an appropriate blocking F of E so that J_F and $\widetilde{T}$ are bounded. Let $C \geq 1$ be a constant associated with the upper-V-tree estimate for the operator T and set

$$\mathcal{A} = \Big\{(x_i) \in S_X^\omega : \forall j \in \mathbb{N},\ \Big\|T\Big(\sum_{i=1}^{j} a_i x_i\Big)\Big\| \leq C\Big\|\sum_{i=1}^{j} a_i v_i\Big\|,\ \forall (a_i) \subset \mathbb{R}\Big\}.$$

Since T satisfies an upper-V-tree estimate, applying Lemma 3.2 to the set $\mathcal{A}$, we get a blocking (G_i) of (E_i) such that given $\varepsilon > 0$, there exists $\delta = (\delta_i)$ so that if $(x_n) \subset S_X$ is a δ-skipped block sequence with respect to (G_n), then whenever $\sum a_i x_i$ converges, we have $\|T(\sum a_i x_i)\| \leq (1+\varepsilon)C\|\sum a_i v_i\|$.

Let $\widetilde{C}$ be a constant associated with the lower-U-tree estimate for X and set

$$\mathcal{B} = \Big\{(x_i) \in S_X^\omega : \forall j \in \mathbb{N},\ \Big\|\sum_{i=1}^{j} a_i x_i\Big\| \geq \widetilde{C}^{-1}\Big\|\sum_{i=1}^{j} a_i u_i\Big\|,\ \forall (a_i) \subset \mathbb{R}\Big\}.$$

Since X satisfies a lower-U-tree estimate, applying Lemma 3.2 again to the set $\mathcal{B}$ and properly shrinking δ, we get a blocking (H_i) of (G_i) such that if $(x_n) \subset S_X$ is a δ-skipped block sequence with respect to (H_n), then whenever $\sum a_i x_i$ converges, we have $\|\sum a_i x_i\| \geq (\widetilde{C}^{-1}/(1+\varepsilon))\|\sum a_i u_i\|$. From the above arguments, we get a blocking (H_i) of (E_i) so that (u_i) is $\widetilde{C}^{-1}/(1+\varepsilon)$-dominated by any δ-skipped block sequence with respect to (H_i) in X while the image of any δ-skipped block sequence with respect to (H_i) in X under T is $(1+\varepsilon)C$-dominated by (v_i). Let $K = \sup_{m<n} \|P_n - P_m\|$ be the projection constant for (E_i), where P_n is the canonical projection from X onto $\bigoplus_{i=1}^n E_i$. Using the "killing the overlap technique" [3], we can find a further blocking $F = (F_n)$ of (H_n) with $F_n = \bigoplus_{j=l(n)+1}^{l(n+1)} H_j$, so that for any $x = \sum x_j \in S_X$, $x_j \in H_j$, there are t_n's with $l(n) < t_n < l(n+1)$ such that $\|x_{t_j}\| < \delta_i$, where $0 = l(1) < l(2) < \cdots$.

First, we prove $\widetilde{T}$ is bounded. Let $\varepsilon > 0$ and let $\sum \delta_i < \varepsilon$. Without loss of generality, we assume $\|T\| = 1$. Let $x = \sum x_i = \sum \widetilde{x}_j \in S_X$ with $x_i \in F_i$ and $\widetilde{x}_j \in H_j$. Then

$$\begin{aligned}
\|\widetilde{T}(x)\| &= \Big\|T\Big(\sum x_i\Big)\Big\| \leq \Big\|T\Big(\sum x_{2i}\Big)\Big\| + \Big\|T\Big(\sum x_{2i-1}\Big)\Big\| \\
&\leq (1+\varepsilon)C\Big(\Big\|\sum \|x_{2i}\| v_i\Big\| + \Big\|\sum_{i>1} \|x_{2i-1}\| v_i\Big\| + \|x_1\|\Big) \\
&\leq (1+\varepsilon)CA_2^{-1}\Big(\Big\|\sum \|x_{2i}\| w_i\Big\| + \Big\|\sum_{i>1} \|x_{2i-1}\| w_i\Big\| + \|x_1\|\Big)
\end{aligned}$$

$$\le (1+\varepsilon)CC'A_2^{-1}\Big(\Big\|\sum\|x_{2i}\|w_{2i}\Big\|+\Big\|\sum_{i>1}\|x_{2i-1}\|w_{2i-1}\Big\|+\|x_1\|\Big)$$
$$\le (1+\varepsilon)CC'A_2^{-1}\Big(\Big\|\sum\|x_i\|w_i\Big\|+\Big\|\sum\|x_i\|w_i\Big\|+\Big\|\sum\|x_i\|w_i\Big\|\Big)$$
$$= 3(1+\varepsilon)CC'A_2^{-1}\|x\|_{X_W(F)}.$$

Hence $\widetilde{T}$ is bounded. It remains to prove that J_F is bounded. Let $t_0=1$ and let $y_i=\sum_{j=t_{i-1}}^{t_i-1}\widetilde{x}_j$. Define $\widetilde{y}_1=y_1$ and $\widetilde{y}_i=y_i-\widetilde{x}_{t_{i-1}}$ for $i\ge 2$. Then

$$\begin{aligned}\|x\|_{X_W(F)} &= \Big\|\sum\|x_i\|w_i\Big\| \le A_1^{-1}\Big\|\sum\|x_i\|u_i\Big\|\\
&\le A_1^{-1}\Big\|\sum K(\|y_i\|+\|y_{i+1}\|)u_i\Big\|\\
&\le KA_1^{-1}\Big(\Big\|\sum_{i>1}\|y_i\|u_i\Big\|+\Big\|\sum\|y_{i+1}\|u_i\Big\|+\|y_1\|\Big)\\
&\le KA_1^{-1}\Big(\Big\|\sum_{i>1}\|\widetilde{y}_i\|u_i\Big\|+\Big\|\sum\|\widetilde{y}_{i+1}\|u_i\Big\|+\|y_1\|+2\varepsilon\Big)\\
&\le (1+\varepsilon)KA_1^{-1}\widetilde{C}\Big(\Big\|\sum_{i>1}\widetilde{y}_i\Big\|+\Big\|\sum\widetilde{y}_{i+1}\Big\|+\widetilde{C}^{-1}\|y_1\|+2\widetilde{C}^{-1}\varepsilon\Big)\\
&\le (1+\varepsilon)KA_1^{-1}\widetilde{C}(K(\|x\|+\varepsilon)+K(\|x\|+\varepsilon)+\widetilde{C}^{-1}K\|x\|+2\widetilde{C}^{-1}\varepsilon)\\
&\le (1+\varepsilon)K(2K+2K\varepsilon+\widetilde{C}^{-1}K+2\widetilde{C}^{-1}\varepsilon)A_1^{-1}\widetilde{C}\|x\|.\end{aligned}$$

For the last step, we use the condition that $x\in S_X$. So J_F is bounded. ∎

Theorem 3.1 concerns operators from spaces with an FDD. For operators from spaces without an FDD, we have the following corollaries.

COROLLARY 3.3. *Let (u_i) be a normalized 1-subsymmetric basis for a reflexive space U. Let (v_i) be a normalized 1-subsymmetric basis for a reflexive space V and let (w_i) be a normalized subsymmetric basis for a reflexive space W. Suppose that (u_i) dominates (w_i) and (w_i) dominates (v_i). Also assume that (u_i) is dominated by every normalized block basis of (u_i) and (v_i^*) is dominated by every normalized block basis of (v_i^*). Let X be a separable reflexive Banach space which satisfies a lower-U-tree estimate and let Y be a separable reflexive Banach space which satisfies an upper-V-tree estimate. Then any bounded linear operator T from X into Y factors through a subspace of $Z_W(F)$, where $F=(F_n)$ is a sequence of finite-dimensional spaces.*

Proof. By Theorem 4.4 in [11], Y embeds into a reflexive space $\widetilde{Y}$ with an FDD (G_n) which satisfies an upper-V-tree estimate. We use $\widetilde{U}$, $\widetilde{V}$ and $\widetilde{W}$ to denote the closed linear spans of (u_i^*), (v_i^*) and (w_i^*) respectively, where $(u_i^*),(v_i^*)$ and (w_i^*) are the biorthogonal functionals of (u_i), (v_i) and (w_i)

respectively. Let T^* be the adjoint operator of T. Since the image of T^* is inside X^* which satisfies an upper-$\widetilde{U}$-tree estimate (Corollary 4.3 in [11]), T^* satisfies an upper-$\widetilde{U}$-tree estimate. As $\widetilde{Y}$ satisfies an upper-V-tree estimate, $\widetilde{Y}^*$ satisfies a lower-$\widetilde{V}$-tree estimate. By Theorem 3.1, T^* factors through $Z_{\widetilde{W}}(\widetilde{F}_n)$. By considering T^{**}, which is T, we conclude that T factors through a subspace of $Z_{\widetilde{W}}(\widetilde{F}_n)^*$, which is $Z_W(F)$. ∎

Proof of Theorem A. By Theorem 2.1 in [10], X is a quotient of a reflexive space Z with an FDD which satisfies a lower-ℓ_q-tree estimate. Let Q be a quotient map from Z onto X. Then it is easy to see that $T \circ Q$ satisfies an upper-ℓ_p-tree estimate. By Theorem 3.1, $\widetilde{T} = T \circ Q$ factors through $(E_n)_{\ell_r}$, where (E_n) is a sequence of finite-dimensional spaces. Let $\widetilde{T}^*$ be the adjoint operator of $\widetilde{T}$. Then $\widetilde{T}^* = J \circ T^*$ factors through $(E_n^*)_{\ell_{r'}}$, where J is an embedding of X^* into Z^* and $1/r' + 1/r = 1$. This implies that T^* factors through a subspace H of $(E_n^*)_{\ell_{r'}}$. By [5], H is also a quotient of some $(F_n)_{\ell_{r'}}$. By considering T^{**}, which is T, we deduce that T factors through a subspace of $(F_n^*)_{\ell_r}$. ∎

Let $1 < p < \infty$ and let $\ell_{p,1}$ be the completion of c_{00} under

$$\|(a_i)\|_{\ell_{p,1}} = \sum i^{1/p-1} a_i^*,$$

where (a_i^*) is the decreasing rearrangement of $(|a_i|)$.

Remark 3.4. Let $1 \le r < p \le \infty$. Then the canonical basis of ℓ_r dominates the canonical basis of $\ell_{p,1}$.

The Tsirelson space corresponding to p and $0 < \gamma < 1$ is denoted by $T(\ell_p, \gamma)$ (see X.E in [1]). The standard Tsirelson space is $T = T(\ell_1, 1/2)$ and the original p-convexification of T is $T(\ell_p, 2^{-1/p})$.

To prove Theorem B, we need the following lemmas. The first one is proved by E. Odell, Th. Schlumprecht and A. Zsák.

Lemma 3.5 (Proposition 5.5 in [11]). *Let $1 < p < \infty$. For a separable reflexive Banach space X, the following are equivalent*:

(a) *X satisfies an asymptotic lower-ℓ_p-tree estimate.*
(b) *There exists $\gamma \in (0,1)$ such that X satisfies a lower-$T(\ell_p, \gamma)$-tree estimate.*

Let (v_i) be a sequence in a Banach space. We say that (v_i) satisfies an *asymptotic upper-ℓ_p estimate* if there is a $C > 0$ so that for every finite sequence of natural numbers $n_1 < \cdots < n_m$ with $m < n_1$ and $(a_i)_{i=1}^m \subset \mathbb{R}$,

$$\Big\| \sum_{i=1}^m a_i v_{n_i} \Big\| \le C \Big(\sum_{i=1}^m |a_i|^p \Big)^{1/p}.$$

LEMMA 3.6. *Let $1 < p < \infty$. Let (u_i) be the canonical basis for $\ell_{p,1}$ and let (v_i) be a sequence in a Banach space which has an asymptotic upper-ℓ_p estimate. Then (v_i) is dominated by (u_i).*

Proof. By definition, we need to prove that there is a $C_0 > 0$ so that for any $(a_i)_{i=1}^{\infty} \subset \mathbb{R}$,

$$\Big\| \sum_{i=1}^{\infty} a_i v_i \Big\| \le C_0 \Big\| \sum_{i=1}^{\infty} a_i u_i \Big\|.$$

By scaling and taking small perturbations, we may assume $0 < |a_i| \le 1$, $\forall i \in \mathbb{N}$. Let (A_k) be the partition of $\mathbb{N}$ defined as

$$A_k = \{i \in \mathbb{N} : 1/2^k < |a_i| \le 1/2^{k-1}\}.$$

Let $I = \{k \in \mathbb{N} : |A_k| > 2^{-1} \sum_{j<k} |A_j|\}$. There are two cases for I, finite or infinite. Since the proof when I is finite is essentially the same as when I is infinite, here we just give the proof for the case when I is infinite.

Let $I = \{m_1, m_2, \ldots\}$, where $m_1 < m_2 < \cdots$. Set $m_0 = 1$ and let $B_n = \bigcup_{k=m_n}^{m_{n+1}-1} A_k$ for $n \ge 0$. Let C be a constant associated with the weak asymptotic upper-ℓ_p estimate for (v_i). Suppose $2^{t_n} \le |A_{m_n}| < 2^{t_n+1}$. Then

$$\begin{aligned}\Big\| \sum_{i \in A_{m_n}} a_i v_i \Big\| &\le \frac{C}{2^{m_n-1}} (2^{t_n/p} + 2^{(t_n-1)/p} + \cdots + 1) \le \frac{C}{2^{m_n-1}} \frac{2^{t_n/p}}{1 - 2^{-1/p}} \\ &\le \frac{C}{1 - 2^{-1/p}} \frac{1}{2^{m_n-1}} |A_{m_n}|^{1/p}.\end{aligned}$$

Noticing that for all $m_n < k < m_{n+1}$,

$$|A_k| \le (3/2)^{k-m_n} |A_{m_n}|,$$

we get

$$\begin{aligned}\sum_{k=m_n+1}^{m_{n+1}-1} \frac{1}{2^{k-1}} |A_k|^{1/p} &\le \sum_{k=m_n+1}^{m_{n+1}-1} \frac{1}{2^{k-1}} (3/2)^{(k-m_n)/p} |A_{m_n}|^{1/p} \\ &= \sum_{k=m_n+1}^{m_{n+1}-1} \frac{1}{2^{m_n-1}} \frac{(3/2)^{(k-m_n)/p}}{2^{k-m_n}} |A_{m_n}|^{1/p}.\end{aligned}$$

Now let

$$\widetilde{C}_p = \frac{C}{1 - 2^{-1/p}} \Big(1 + \sum_{k=m_n+1}^{\infty} \frac{(3/2)^{(k-m_n)/p}}{2^{k-m_n}} \Big) = \frac{C}{1 - 2^{-1/p}} \Big(1 + \sum_{k=1}^{\infty} \frac{(3/2)^{k/p}}{2^k} \Big).$$

We have

$$\Big\| \sum_{i \in B_n} a_i v_i \Big\| \le \sum_{k=m_n}^{m_{n+1}-1} \Big\| \sum_{i \in A_k} a_i v_i \Big\| \le \frac{1}{2^{m_n-1}} \widetilde{C}_p |A_{m_n}|^{1/p}.$$

Next, we are going to estimate $\sum_{k=\sum_{j=1}^{m_n-1}|A_j|+1}^{\sum_{j=1}^{m_n}|A_j|} k^{1/p-1}$. From the definition of A_{m_n}, $|A_{m_n}|$ is always greater than or equal to 1. If $|A_{m_n}|=1$, then $\sum_{j=1}^{m_n-1}|A_j|$ can only be 0 or 1. In either case,

$$\sum_{k=\sum_{j=1}^{m_n-1}|A_j|+1}^{\sum_{j=1}^{m_n}|A_j|} k^{1/p-1} \geq 2^{1/p-1} \geq (3^{1/p}-(5/2)^{1/p})|A_{m_n}|^{1/p}.$$

If $|A_{m_n}|\geq 2$, then

$$\begin{aligned}\sum_{k=\sum_{j=1}^{m_n-1}|A_j|+1}^{\sum_{j=1}^{m_n}|A_j|} k^{1/p-1} &\geq p\Big(\Big(\sum_{j=1}^{m_n}|A_j|\Big)^{1/p}-\Big(\sum_{j=1}^{m_n-1}|A_j|+1\Big)^{1/p}\Big)\\ &\geq p((3|A_{m_n}|)^{1/p}-(2|A_{m_n}|+1)^{1/p})\\ &\geq p(3^{1/p}-(2+|A_{m_n}|^{-1})^{1/p})|A_{m_n}|^{1/p}\\ &\geq p(3^{1/p}-(5/2)^{1/p})|A_{m_n}|^{1/p}\\ &\geq (3^{1/p}-(5/2)^{1/p})|A_{m_n}|^{1/p}.\end{aligned}$$

We get the first step by using the definite integral of the function $x^{1/p-1}$. The second step follows from the fact that $(x+a)^{1/p}-x^{1/p}$ is a decreasing function for $p>1$, where a is a positive constant. So for any $n\in\mathbb{N}$, we have

$$\sum_{k=\sum_{j=1}^{m_n-1}|A_j|+1}^{\sum_{j=1}^{m_n}|A_j|} k^{1/p-1} \geq (3^{1/p}-(5/2)^{1/p})|A_{m_n}|^{1/p}.$$

Hence,

$$\begin{aligned}\Big\|\sum_{i=1}^{\infty}a_iu_i\Big\| &\geq \sum_{n=1}^{\infty}\frac{1}{2^{m_n}}\Big(\sum_{k=\sum_{j=1}^{m_n-1}|A_j|+1}^{\sum_{j=1}^{m_n}|A_j|} k^{1/p-1}\Big)\\ &\geq \sum_{n=1}^{\infty}\frac{1}{2^{m_n}}(3^{1/p}-(5/2)^{1/p})|A_{m_n}|^{1/p}\\ &\geq \frac{3^{1/p}-(5/2)^{1/p}}{2\widetilde{C}_p}\sum_{n=1}^{\infty}\Big\|\sum_{i\in B_n}a_iv_i\Big\| \geq \frac{3^{1/p}-(5/2)^{1/p}}{2\widetilde{C}_p}\Big\|\sum_{i=1}^{\infty}a_iv_i\Big\|.\end{aligned}$$

The first step above holds because we only compute the contribution of those a_iu_i's with $a_i\in A_{m_n}$. This finishes the proof. ∎

LEMMA 3.7. *Let $1<p<\infty$ and let X be a separable reflexive Banach space. Let T be a bounded linear operator from X into a Banach space Y. Then* (a)⇒(b), *where*

(a) *T satisfies an asymptotic upper-ℓ_p-tree estimate,*
(b) *T satisfies an upper-$\ell_{p,1}$-tree estimate.*

Proof. Since X is a separable reflexive Banach space, by Zippin's theorem [13], we can assume that X is a subspace of a reflexive space with an FDD (E_n). Let $C > 0$ be the constant associated with the asymptotic upper-ℓ_p-tree estimate. For $k \in \mathbb{N}$, let

$$\mathcal{A}^k = \Big\{(x_i) \in S_X^\omega : \Big\|T\Big(\sum_{i=1}^k a_i x_i\Big)\Big\| \le C\Big(\sum_{i=1}^k |a_i|^p\Big)^{1/p}, \forall (a_i)_{i=1}^k \subset \mathbb{R}\Big\}.$$

Let $\varepsilon > 0$ be so small that for all $k \in \mathbb{N}$,

$$\overline{\mathcal{A}_\varepsilon^k} \subset \Big\{(x_i) \in S_X^\omega : \Big\|T\Big(\sum_{i=1}^k a_i x_i\Big)\Big\| \le 2C\Big(\sum_{i=1}^k |a_i|^p\Big)^{1/p}, \forall (a_i)_{i=1}^k \subset \mathbb{R}\Big\}.$$

Let $(E_i^0) = (E_i)$. Applying Lemma 3.2, we get a decreasing null sequence $\delta^1 = (\delta_i^1)$ and a blocking (E_i^1) of (E_i^0) so that if $(x_i) \subset S_X$ is a δ^1-skipped block sequence of (E_i^1), then (x_i) lies in $\overline{\mathcal{A}_\varepsilon^1}$. By repeating this procedure, we obtain decreasing null sequences $\delta^k = (\delta_i^k)$ and blockings (E_i^k) of (E_i^{k-1}) so that if $(x_i) \subset S_X$ is a (δ_i^k)-skipped block sequence of (E_i^k), then (x_i) lies in $\overline{\mathcal{A}_\varepsilon^k}$. Let $E_k^k = \bigoplus_{i=n_k}^{m_k} E_i$, and let (F_i) be a blocking of (E_i) so that $F_k = \bigoplus_{i=n_k}^{n_{k+1}-1} E_i$. We can then choose a decreasing null sequence $\delta = (\delta_i)$ so that if $(x_i) \subset S_X$ is a δ-skipped block sequence of (F_i), then (x_i) is a basic sequence and any normalized block sequence $(z_i)_{i=1}^k$ of $(x_i)_{i=k}^\infty$ is a δ^k-skipped block sequence of (E_i^k). Hence, by Lemma 3.6, (Tx_i) is dominated by the canonical basis of $\ell_{p,1}$. This shows T satisfies an upper-$\ell_{p,1}$-tree estimate. ∎

Proof of Theorem B. By Lemma 3.5, X satisfies a lower-$T(\ell_q, \gamma)$-tree estimate for some $0 < \gamma < 1$. By Lemma 3.7, T satisfies an upper-ℓ_{p1}-tree estimate. Since the canonical basis of $T(\ell_q, \gamma)$ dominates the canonical basis of ℓ_r and the canonical basis of ℓ_r dominates the canonical basis of ℓ_{p1} when $1 < q < r < p < \infty$, by Theorem 3.1, we deduce that T factors through a subspace of $(F_n)_{\ell_r}$, where $F = (F_n)$ is a sequence of finite-dimensional spaces. ∎

By Theorem B, we have

COROLLARY 3.8. *Let $1 < q < p < \infty$ and let X be a reflexive asymptotic ℓ_q space. Let T be a bounded linear operator from X which satisfies an asymptotic upper-ℓ_p-tree estimate. Then T factors through a subspace of a space with an (ℓ_p, ℓ_q) FDD.*

REMARK 3.9. Theorem B and Corollary 3.8 start with asymptotic conditions while end up with factorizations through subspaces of spaces with properties much stronger than asymptotic properties. This gives us some information on the relations between asymptotic ℓ_p spaces and $(F_n)_{\ell_r}$ spaces. However, they do not tell us what happens when $p = q$.

The following theorem provides a result for the special case when $p = q$.

THEOREM 3.10. *Let $2 < p < \infty$. Let X be a separable reflexive asymptotic ℓ_p space. Let T be a bounded linear operator from X into L_p which satisfies an asymptotic upper-ℓ_p-tree estimate. Then T factors through ℓ_p.*

Proof. W. B. Johnson proved in [2] that for $p > 2$, a bounded linear operator T into L_p factors through ℓ_p if and only if T is compact when considered as an operator into L_2. So it is enough to show that T is compact as an operator into L_2. By Corollary 4.8 in [11], X embeds into a reflexive Banach space with an asymptotic ℓ_p FDD (E_n). Let (h_n) be the canonical Haar basis of L_2. If T is not compact as an operator into L_2, then there are a $\delta > 0$ and a perturbation of a normalized block sequence (x_i) with respect to (E_n) so that $(i_{p,2} \circ Tx_i)$ is essentially a block sequence with respect to (h_n) and $\|i_{p,2} \circ Tx_i\| > \delta$, $\forall i \in \mathbb{N}$, where $i_{p,2}$ is the formal identity map from L_p into L_2. This gives a contradiction since, on the normalized weakly null tree $(x_A)_{A \in [\mathbb{N}]^{<\omega}}, x_A = x_{\max\{A\}}$, T does not satisfy an asymptotic upper-ℓ_p-tree estimate. ∎

REMARK 3.11. Theorem 3.10 holds even if we only assume that for every normalized weakly null sequence in X there is a subsequence whose image under T satisfies an asymptotic upper-ℓ_r estimate for some $2 < r < \infty$.

Acknowledgments. The author thanks E. Odell for useful conversations and the referee for careful reading.

References

[1] P. G. Casazza and T. J. Shura, *Tsirelson's Space*, Lecture Notes in Math. 1363, Springer, Berlin, 1989.

[2] W. B. Johnson, *Operators into L_p which factor through l_p*, J. London Math. Soc. (2) 14 (1976), 333–339.

[3] —, *On quotients of L_p which are quotients of ℓ_p*, Compos. Math. 34 (1977), 69–89.

[4] W. B. Johnson and M. Zippin, *On subspaces of quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$*, Israel J. Math. 13 (1972), 311–316.

[5] —, —, *Subspaces and quotient spaces of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$*, ibid. 17 (1974), 50–55.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, Berlin, 1977.

[7] B. Maurey, V. D. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, in: Oper. Theory Adv. Appl. 77, Birkhäuser, Basel, 1995, 149–175.

[8] V. D. Milman and N. Tomczak-Jaegermann, *Asymptotic ℓ_p spaces and bounded distortions*, in: Contemp. Math. 144, Amer. Math. Soc., Providence, RI, 1993, 173–195.

[9] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. 354 (2002), 4085–4108.

[10] E. Odell and Th. Schlumprecht, *A universal reflexive space for the class of uniformly convex Banach spaces*, Math. Ann. 335 (2006), 901–916.

[11] E. Odell, Th. Schlumprecht and A. Zsák, *On the structure of asymptotic ℓ_p spaces*, Quart. J. Math., to appear.

[12] B. Zheng, *On operators which factor through ℓ_p or c_0*, Studia Math. 176 (2006), 177–190.

[13] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. 310 (1988), 371–379.

Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-0257, U.S.A.
E-mail: btzheng@math.utexas.edu

Received August 13, 2007
Revised version December 24, 2007 (6198)