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Mod 2 normal numbers and skew products

by

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Abstract. Let E be an interval in the unit interval $[0,1)$. For each $x \in [0,1)$ define $d_n(x) \in \{0,1\}$ by $d_n(x) := \sum_{i=1}^{n} 1_E(\{2^{i-1}x\}) \pmod 2$, where $\{t\}$ is the fractional part of t. Then x is called a normal number mod 2 with respect to E if $N^{-1}\sum_{n=1}^{N} d_n(x)$ converges to 1/2. It is shown that for any interval $E \neq (1/6,5/6)$ a.e. x is a normal number mod 2 with respect to E. For $E = (1/6,5/6)$ it is proved that $N^{-1}\sum_{n=1}^{N} d_n(x)$ converges a.e. and the limit equals 1/3 or 2/3 depending on x.

1. Introduction. Let (X,μ) be a probability space and let $T : X \to X$ be an ergodic transformation. Given a measurable subset $E \subset X$, we consider the binary sequence $d_n(x) \in \{0,1\}$ defined by

$$d_n(x) := \sum_{i=1}^{n} 1_E(T^{i-1}x) \pmod 2,$$

where 1_E is the characteristic function of E. The *mod* 2 *normality problem* is to investigate the convergence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n(x)$$

to the limit 1/2 or the convergence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_n(x)$$

to the limit 0, where $e_n(x) = \exp(\pi i d_n(x))$.

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When T is given by an irrational translation on the unit interval $[0,1)$, i.e., $Tx = \{x+\theta\}$, θ irrational, the problem was first investigated by Veech [14] using Furstenberg's idea on coboundaries (see [6]). Let E be an interval. Define

$$\mu_\theta(E) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n(x)$$

if the limit exists. Let t be the length of E. It can be shown that for arbitrary θ if $t \in 2\mathbb{Z}\cdot\theta + \mathbb{Z}$ then $\mu_\theta(E)$ exists but it may not be equal to $1/2$. And for arbitrary θ and t at least one of $\mu_\theta(E)$ or $\mu_\theta([0,1]\setminus E)$ is $1/2$, the limit existing. Veech proved that $\mu_\theta(E)$ exists for every interval E if and only if θ has bounded partial quotients in its continued fraction expansion, and in this case if $t \notin 2\mathbb{Z}\cdot\theta + \mathbb{Z}$ then $\mu_\theta(E) = 1/2$. For closely related results, see [8], [9], [11], [13], [15].

In this article we are interested in the case that $X = [0,1)$ and $Tx = \{2x\}$, where $\{t\}$ is the fractional part of t. Set inclusions and identities are understood modulo measure zero sets.

DEFINITION 1.1. Let E be a measurable subset of $[0,1)$. For $x \in [0,1)$ define $d_n(x) \in \{0,1\}$ as above. Then x is called a *normal number mod* 2 *with respect to* E if $N^{-1}\sum_{n=1}^{N} d_n(x)$ converges to $1/2$.

We state the main result that will be proved in Section 3.

MAIN THEOREM. (i) *For any interval $E \neq (1/6,5/6)$ a.e. $x \in [0,1)$ is a normal number mod* 2 *with respect to E.*

(ii) *For $E = (1/6,5/6)$ a.e. $x \in [0,1)$ is not a normal number mod* 2 *with respect to E. More precisely*, *for $E = (1/6,5/6)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n = \frac{2}{3} \quad \text{for a.e. } x \in \left[\frac{1}{3},\frac{2}{3}\right],$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n = \frac{1}{3} \quad \text{for a.e. } x \in \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right].$$

2. $\mathbb{Z}_2$-extensions. Let T be ergodic on a probability space (X,μ). It is not necessarily invertible. Let G be a compact abelian group. Suppose that $\phi : X \to G$ is measurable. Define a skew product T_ϕ on $X \times G$ by $T_\phi(x,y) = (Tx, \phi(x)+y)$. It is also called a *$G$-extension* of T by ϕ.

DEFINITION 2.1. A function $\phi : X \to G$ is called a *G-coboundary* if there exists a measurable function $q : X \to G$ satisfying $\phi(x) = q(x) - q(Tx)$.

From now on we consider the case $G = \mathbb{Z}_2 = \{0,1\}$. Put $\phi = 1_E$. Define V_ϕ on $L^2(X)$ by $(V_\phi f)(x) = e^{\pi i\phi(x)} f(Tx)$. Since $(V_\phi)^n 1 = e_n$, von

Neumann's Mean Ergodic Theorem implies that $N^{-1}\sum_{n=1}^{N} e_n$ converges to the orthogonal projection of 1 onto the subspace $H = \{h : V_\phi h = h\}$. Now for the pointwise convergence the following is known. For the proof see [3].

FACT 2.2. (i) *Suppose that* $T_\phi : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2$ *is ergodic. Then for a.e.* x,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_n(x) = 0.$$

(ii) *Suppose that* T_ϕ *is not ergodic. Then for a.e.* x,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e_n(x) = h(x) \int_X h\, d\mu,$$

where $e^{\pi i \phi(x)} = h(x)h(Tx)$ *and* $h(x) \in \{\pm 1\}$ *for a.e.* x.

If ϕ is a coboundary then it is possible that H is not trivial and the limit is not zero (see [5]). In this paper we investigate for which E the skew product is ergodic. The following fact is well known.

FACT 2.3. *Let* T *be ergodic on* X. *Suppose* $\phi(x) \in \mathbb{Z}_2$, *or equivalently* $\phi = 1_E$ *for some measurable subset* E. *Then the following are equivalent*:

(i) T_ϕ *is not ergodic on* $X \times \mathbb{Z}_2$.
(ii) *There exists a* $\mathbb{Z}_2$*-valued measurable function* q *such that* $\phi(x) = q(x) + q(Tx) \pmod 2$. *In this case*, $q = 1_F$ *for some measurable* F.
(iii) $E = F \triangle T^{-1}F$ *for some measurable* F, *where* $\triangle$ *denotes the symmetric difference.*

The following fact is a special case of a well known result (see Zimmer [16] and also [10], [7]). We give a proof for the sake of completeness. Throughout the rest of the article addition is done modulo 2.

FACT 2.4. *Let* T *be ergodic on* X *and let* ϕ *be* $\mathbb{Z}_2$*-valued on* X. *If* T_ϕ *is not ergodic*, *then it has exactly two ergodic components*, *each having the same measure. In this case the two ergodic components of* T_ϕ *are* $\{(x, q(x)) : x \in X\}$ *and* $\{(x, 1 + q(x)) : x \in X\}$, *where* q *is* $\mathbb{Z}_2$*-valued and* $\phi(x) = q(x) + q(Tx)$.

Proof. Since T_ϕ is not ergodic, Fact 2.3 implies that ϕ is a coboundary. Define Q and S on $X \times \mathbb{Z}_2$ by $Q(x, y) = (x, q(x) + y)$ and $S(x, y) = (Tx, y)$. Then $Q^{-1} = Q$ and $Q \circ T_\phi = S \circ Q$. Note that T_ϕ is isomorphic to S, which has two ergodic components $X \times \{0\}$ and $X \times \{1\}$. Therefore the ergodic components of T_ϕ are $Q(X \times \{0\}) = \{(x, q(x)) : x \in X\}$ and $Q(X \times \{1\}) = \{(x, 1 + q(x)) : x \in X\}$. ∎

3. Construction of piecewise linear maps. In this section we study $\mathbb{Z}_2$-extensions of $T : x \mapsto \{2x\}$ on $X = [0,1)$ from the viewpoint of interval mappings. The subsets $X_0 = X \times \{0\}$ and $X_1 = X \times \{1\}$ are identified with the intervals $[0,1)$ and $[1,2)$, respectively, and T_ϕ is regarded as being defined on $\widetilde{X} = [0,2)$. The new interval map on $\widetilde{X}$ induced from T_ϕ is denoted by $\widetilde{T}_\phi$. If $\widetilde{T}_\phi$ is not ergodic, then it has two ergodic components in $\widetilde{X}$, each having Lebesgue measure 1.

More precisely, for a measurable subset $E \subset [0,1)$, put $\phi = 1_E$. Recall that $T_\phi(x,i) = (Tx, i + \phi(x))$, $i \in \{0,1\}$. Define $\psi : X \times \{0,1\} \to [0,2)$ by $\psi(x,i) = x + i$. For $u \in [0,2)$ let

$$\widetilde{T}_\phi(u) = \psi \circ T_\phi \circ \psi^{-1}(u).$$

Then $\widetilde{T}_\phi$ preserves Lebesgue measure and satisfies

$$\widetilde{T}_\phi(u) = \begin{cases} \{2u\} + 1_E(u), & 0 \le u < 1, \\ \{2u\} + 1 - 1_E(u-1), & 1 \le u < 2. \end{cases}$$

EXAMPLE 3.1 (An ergodic skew product). Take $E = [1/2, 1]$ and $\phi = 1_E$. Then $\widetilde{T}_\phi$ can be regarded as the piecewise linear map given in Fig. 1. Note that it is ergodic. See the Folklore Theorem given in [1].

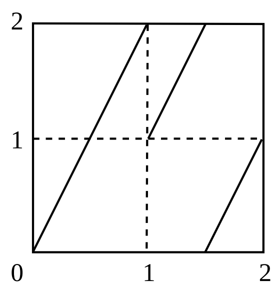

Fig. 1. An ergodic skew product T_ϕ generated by the interval $E = [1/2, 1) \subset X = [0,1)$ is regarded as an interval map $\widetilde{T}_\phi$ on $\widetilde{X} = [0,2)$.

EXAMPLE 3.2 (A nonergodic skew product). Take $E = [1/4, 3/4]$ and $\phi = 1_E$. Put $F = [1/2, 1]$ and $q = 1_F$. Then $E = F \triangle T^{-1}F$ and ϕ is a coboundary. Then $\widetilde{T}_\phi$ can be regarded as the piecewise linear map given in Fig. 2. It has two ergodic components: $[0, 1/2) \cup [3/2, 2)$ and $[1/2, 3/2)$. This can be seen from Fact 2.4.

Let J be an interval and τ be a piecewise C^2 mapping on J. Assume that $\inf_{x \in J_1} |\tau'(x)| > 1$, where $J_1 = \{x \in J : \tau'(x) \text{ exists}\}$. The points of $J - J_1$ are called the *points of discontinuity*. For $x \in J$, let $\Lambda(x)$ be the set of limit points of $\tau^n(x)$, that is, $\Lambda(x) = \bigcap_{N=1}^{\infty} \overline{\{\tau^n(x) : n \ge N\}}$. Note that $\tau(\Lambda(x)) = \Lambda(x)$. Li and Yorke [12] proved the following.

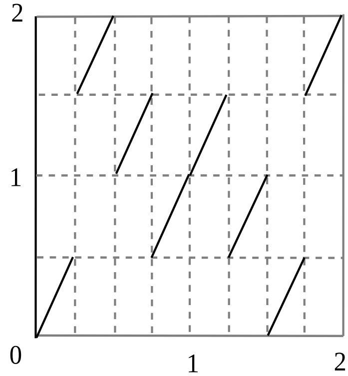

Fig. 2. A nonergodic skew product T_ϕ generated by the interval $E = [1/4, 3/4) \subset X = [0, 1)$ is regarded as an interval map $\widetilde{T}_\phi$ on $\widetilde{X} = [0, 2)$. It has two ergodic components: $K_1 = [1/2, 3/2]$ and $K_2 = \widetilde{X} - K_1$.

FACT 3.3. *For a.e. x, $\Lambda(x)$ is the union of (one or more) intervals of positive length. Furthermore, there is a finite collection of sets $L_1, \ldots, L_n$, where each L_i is a union of finitely many disjoint intervals, such that for a.e. x, $\Lambda(x)$ is one of the sets L_i. (The sets L_i are ergodic components of τ.) If $i \neq j$ then $L_i \cap L_j$ contains at most a finite number of points and each L_i contains in its interior a point of discontinuity of τ and/or τ'. For each L_i there exists a unique absolutely continuous τ-invariant measure μ_i such that $\mu_i(L_i) = 1$ and $\mu_i(L_j) = 0$ for $j \neq i$.*

REMARK 3.4. Suppose that E is an interval. Then we have the following: (i) $\widetilde{T}_\phi$ is piecewise linear and satisfies $\widetilde{T}'_\phi(x) = 2$ except at finitely many points. (ii) If $\widetilde{T}_\phi$ is not ergodic, then an ergodic component is a union of finitely many disjoint intervals. More precisely, if $E = F \triangle T^{-1}F$, then Fact 2.4 implies that the ergodic components are given by $([0, 1) \setminus F) \cup (F + 1)$ and $F \cup ([1, 2) \setminus (F + 1))$.

PROPOSITION 3.5. *Suppose E is a finite union of nonempty open intervals such that $E \subset [0, 1/2)$ or $E \subset [1/2, 1)$. Then T_ϕ is ergodic.*

Proof. First, consider the case $E \subset [1/2, 1)$. We regard T_ϕ as a map $\widetilde{T}_\phi$ on $[0, 2)$. Note that an ergodic component has Lebesgue measure 1 or 2. Since E is a union of finitely many intervals, $\widetilde{T}_\phi$ is an interval map on $[0, 2)$ that satisfies the condition from Fact 3.3. This implies that an ergodic component of $\widetilde{T}_\phi$ is a union of finitely many intervals. Thus $[0, 1/2^n)$ is in an ergodic component of $\widetilde{T}_\phi$ for sufficiently large n. This implies that this ergodic component contains $(0, 1)$. Note that the intersection of $\widetilde{T}_\phi([0, 1))$ and $[1, 2)$ is of positive measure. Consequently, the Lebesgue measure of the ergodic component is greater than 1, which implies that the measure is equal to 2. Thus there is only one ergodic component.

Next, consider $E \subset [0,1/2)$. In this case we start with $(1-1/2^n,1)$, and proceed as before. ■

The following example shows that the above theorem cannot be extended to general measurable sets.

EXAMPLE 3.6. Put $I=[3/4,1)$ and $F=\bigcup_{k=0}^{\infty} 2^{-k}I$. Let $q=1_F$. Then $q(x)+q(Tx)=1_E(x) \pmod 2$ for $E=F \triangle T^{-1}F \subset [1/2,7/8]$. In this case T_ϕ is not ergodic.

Using $\widetilde{T}_\phi$ we obtain the following result. A similar result was also obtained by Ahn [2]. His proof uses the argument previously employed in [3], [4], where the problem was investigated for intervals with dyadic rational endpoints.

PROPOSITION 3.7. *Let E be an interval in $[0,1)$ of length less than* 1, *and let $\phi=1_E$. Then T_ϕ is not ergodic if and only if $E=(1/4,3/4)$ or $E=(1/6,5/6)$.*

Proof. If $E=(1/4,3/4)$ or $E=(1/6,5/6)$, then $E=F\triangle T^{-1}F$ for $F=(1/3,2/3)$ or $F=(1/2,1)$, respectively. So T_ϕ is not ergodic by Fact 2.3. Conversely, assume that T_ϕ is not ergodic. Choose F such that $E=F\triangle T^{-1}F$. Note that F is a disjoint union of n intervals for some $n\ge 1$ by Fact 3.3. Here a set of the form $(\alpha,\beta)\cup(\beta,\gamma)$ is regarded as a single interval (α,γ). We may assume that $F\subset(\delta,1)$ for some $\delta>0$ by taking its complement if necessary. Note that

$$T^{-1}F \subset (\delta/2,1/2)\cup(1/2+\delta/2,1).$$

Since F has $2n$ boundary points, $T^{-1}F$ has $4n$ boundary points, hence the interval $F\triangle T^{-1}F$ has at least $2n$ boundary points. This is possible only if $n=1$, so F is an interval of the form $F=(a,b)$, $a>0$. Then $(a,b)\triangle((a/2,b/2)\cup(1/2+a/2,1/2+b/2))$ is an interval only if (i) $a=b/2$ and $b=1/2+a/2$, or (ii) $a=b/2$ and $b=1/2+b/2$. Thus we have $a=1/3$, $b=2/3$ or $a=1/2$, $b=1$, which gives $E=(1/6,5/6)$, $F=(1/3,2/3)$ or $E=(1/4,3/4)$, $F=(1/2,1)$, respectively. ■

COROLLARY 3.8. *Let E be of the form $E=[0,a)\cup[b,1)$, $0<a<b<1$. Assume that T_ϕ is not ergodic, where $\phi=1_E$. Then T_ψ is ergodic, where $\psi=1_B$, $B=(a,b)$. Hence $(a,b)\ne(1/6,5/6)$ and $(a,b)\ne(1/4,3/4)$.*

Proof of Main Theorem. First, Proposition 3.7 implies that for any interval E such that $E\ne(1/6,5/6)$ and $E\ne(1/4,3/4)$ almost every x is a normal number mod 2 with respect to E.

Next, consider $E = [1/4, 3/4]$. Put $F = [1/2, 1)$. Then $E = F \triangle T^{-1}F$, and $\phi(x) = 1_E(x) = q(x) + q(Tx) \pmod 2$, where $q = 1_F$. Then $\int_0^1 e^{\pi i q}\,dx = 0$, and Fact 2.2(ii) implies $\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} d_n = 1/2$ for a.e. x. In this case we have mod 2 normality with respect to E even though 1_E is a coboundary.

There remains the only exceptional case $E = [1/6, 5/6]$ for which the normality mod 2 with respect to E does not hold. Put $F = [1/3, 2/3]$. Then $E = F \triangle T^{-1}F$. Now use $\int_0^1 e^{\pi i q}\,dx = 1/3$. ■

REMARK 3.9. Fig. 3 illustrates the fact that $E = [1/6, 5/6]$ gives a coboundary.

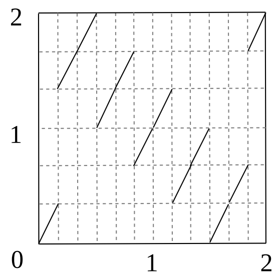

Fig. 3. A nonergodic skew product T_ϕ generated by the interval $E = [1/6, 5/6] \subset X = [0, 1)$ is regarded as an interval map $\widetilde{T}_\phi$ on $\widetilde{X} = [0, 2)$. It has two ergodic components: $K_1 = [0, 1/3] \cup [2/3, 1] \cup [4/3, 5/3]$ and $K_2 = \widetilde{X} - K_1$.

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