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A characterization of Q-algebras of type F

by

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Abstract. We prove that a real or complex unital F-algebra is a Q-algebra if and only if all its maximal one-sided ideals are closed.

A *topological algebra* is a real or complex algebra A which is a topological vector space (t.v.s.) and the multiplication $(x, y) \mapsto xy$ is a jointly continuous map from $A \times A$ to A.

A unital topological algebra A is called a Q-*algebra* if the set (group) $G(A)$ of all its invertible elements is open.

An F-*algebra* (an algebra of type F) is a topological algebra which is an F-*space*, i.e. a complete metrizable t.v.s. The topology of an F-space X can be given by means of an F-*norm*, i.e. a map $x \mapsto \|x\|$ from X to the set of non-negative real numbers such that

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$,
(ii) $\|x + y\| \leq \|x\| + \|y\|$,
(iii) the map $(\lambda, x) \mapsto \|\lambda x\|$ from $\mathbb{K} \times X$ to X is jointly continuous ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

The metric (distance) of an F-space X is given by means of $\|x - y\|$, $x, y \in X$. We shall also write $x_n \to x_0$ if $\lim_n \|x_n - x_0\| = 0$.

For further information on F-spaces the reader is referred to [2] and [5], and for more information on F-algebras, to [3]–[6].

M. Akkar and C. Nacir ([1, Proposition 17]) proved that a commutative unital F-algebra has all maximal ideals closed if and only if it is a Q-algebra. In this paper we extend this result to the non-commutative case. The result seems to be new even in the case of an m-convex B_0-algebra (a locally convex F-algebra whose topology can be given by means of a family of submultiplicative homogeneous seminorms, cf. [3], [4] or [6]). In the case of a commutative m-convex algebra this result is contained in the main result of [7].

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The (unital) topological algebras with all maximal one-sided ideals closed are also called *Mallios algebras*, thus our result says that a unital F-algebra is a Mallios algebra if and only if it is a Q-algebra.

We start our proof with the following simple lemma.

LEMMA 1. *Let A be a real or complex F-algebra with unity e. Then for given $u, v \in A$ and $\varepsilon > 0$ there is a positive $\delta = \delta(\varepsilon, u, v)$ such that*

$$\|x - e\| < \delta \quad \textit{implies} \quad \|uxv - uv\| < \varepsilon. \tag{1}$$

This follows immediately from the fact that the map $x \mapsto uxv$ is continuous at $x = e$.

We shall use the following notation. Let $a = (a_n)$ be a sequence of elements of A. For all integers k and m with $1 \leq k \leq m$, we put

$$u_k^{(m)}(a) = \begin{cases} a_m a_{m-1} \ldots a_k & \text{if } k < m, \\ a_k & \text{if } k = m, \end{cases}$$

$$v_k^{(m)}(a) = \begin{cases} a_k a_{k+1} \ldots a_m & \text{if } k < m, \\ a_k & \text{if } k = m. \end{cases}$$

The following is our crucial lemma. Similarly to [1] (see also [8]) we shall be using infinite products of elements of A.

LEMMA 2. *Let A be a real or complex F-algebra with unity e. Then for every sequence $(x_i) \subset A$ with $x_i \to e$ and sequence $(y_i) \subset A$, there is a subsequence $a_i = x_{k_i}$, $k_i < k_j$ for $i < j$, such that for each natural k the limits*

$$u_k = \lim_i u_k^{(k+i)}(a) \quad (= \lim_i a_{k+i} a_{k+i-1} \ldots a_k) \tag{2}$$

and

$$v_k = \lim_i v_k^{(k+i)}(a) \quad (= \lim_i a_k a_{k+1} \ldots a_{k+i}) \tag{3}$$

exist and satisfy

$$\lim_k u_k = \lim_k v_k = e, \tag{4}$$

and moreover, for each natural k and non-negative integer i, we have

$$\|v_k^{(k+i)}(b) u_k - v_k^{(k+i)}(b) u_k^{(k+i)}(a)\| \leq 2^{-(k+i)}, \tag{5}$$

$$\|v_k u_k^{(k+i)}(b) - v_k^{(k+i)}(a) u_k^{(k+i)}(b)\| \leq 2^{-(k+i)}, \tag{6}$$

where $b_i = y_{k_i}$ and $b = (b_i)$.

Proof. In choosing the subsequences $a_i = a_{k_i}$ and $b_i = y_{k_i}$, and an auxiliary sequence of positive numbers δ_i, which is necessary to obtain (5) and (6), we proceed by induction. First we choose a_1 and b_1 arbitrarily, say $a_1 = x_1, b_1 = y_1$, and using Lemma 1, we choose δ_1 so that $\|x - e\| < \delta_1$ implies $\|b_1 x a_1 - b_1 a_1\| < 2^{-1}$ and $\|a_1 x b_1 - a_1 b_1\| < 2^{-1}$. Suppose now that we have chosen $a_1, \dots, a_n, b_1, \dots, b_n$ ($a_i = x_{k_i}, b_i = y_{k_i}$) and $\delta_1, \dots, \delta_n$ so that for all $n \geq m > k \geq 1$ the following relations hold (they are well defined in spite of the fact that we do not know yet the whole sequences $a = (a_i)$ and $b = (b_i)$):

$$\|u_k^{(m)}(a) - u_k^{(m-1)}(a)\| < 2^{-m}, \tag{7}$$

$$\|v_k^{(m)}(a) - v_k^{(m-1)}(a)\| < 2^{-m}, \tag{8}$$

$$\|e - u_{k+1}^{(m)}(a)\| < \delta_k, \tag{9}$$

$$\|e - v_{k+1}^{(m)}(a)\| < \delta_k, \tag{10}$$

$$0 < \delta_m < \min\{\delta(2^{-m}, v_k^{(m)}(b), u_k^{(m)}(a)), \delta(2^{-m}, v_k^{(m)}(a), u_k^{(m)}(b)) : 1 \leq k \leq m\}, \tag{11}$$

where $\delta(\varepsilon, v, u)$ is given by Lemma 1.

By (9) and (10), there are $\delta'_1, \dots, \delta'_n > 0$ such that

$$\|e - u_{k+1}^{(m)}(a)\| + \delta'_k < \delta_k, \qquad \|e - v_{k+1}^{(m)}(a)\| + \delta'_k < \delta_k \tag{12}$$

for $1 \leq k < m \leq n$.

We want to find an index $j > k_n$ such that setting $a_{n+1} = x_j$ and $b_{n+1} = y_j$, and choosing δ_{n+1} in a suitable way, the conditions (7)–(11) will be satisfied with n replaced by $n+1$. For $m \leq n$ these conditions are satisfied by the inductive assumption, so that we have to consider the case $m = n+1$ only. Since $x_i \to e$, we can find $i' > k_n$ so that for each $i \geq i'$,

$$\|u_{k+1}^{(n)}(a) - x_i u_{k+1}^{(n)}(a)\| < \delta'_k, \qquad \|v_{k+1}^{(n)}(a) - v_{k+1}^{(n)}(a) x_i\| < \delta'_k \tag{13}$$

for $1 \leq k < n$, and

$$\|e - x_i\| < \delta_n. \tag{14}$$

There is also an index $i'' \geq i'$ such that

$$\|u_k^{(n)}(a) - x_i u_k^{(n)}(a)\| < 2^{-(n+1)}, \quad \|v_k^{(n)}(a) - v_k^{(n)}(a) x_i\| < 2^{-(n+1)}, \tag{15}$$

$$\|a_n - x_i a_n\| < 2^{-(n+1)} \tag{16}$$

for all $i \geq i''$ and $k = 1, \dots, n-1$. We now put $j = i''$ and $a_{n+1} = x_j$, $b_{n+1} = y_j$. Inequalities (9), (10), (12) and (13) imply

$$\begin{aligned}\|e - u_{k+1}^{(n+1)}(a)\| &= \|e - a_{n+1} u_{k+1}^{(n)}(a)\| \\ &\leq \|e - u_{k+1}^{(n)}(a)\| + \|u_{k+1}^{(n)}(a) - a_{n+1} u_{k+1}^{(n)}(a)\| \\ &\leq \|e - u_{k+1}^{(n)}(a)\| + \delta'_k < \delta_k\end{aligned}$$

and similarly

$$\|e - v_{k+1}^{(n+1)}(a)\| < \delta_k$$

for $1 \le k < n$. Thus (9) and (10) hold if we replace n by $n+1$, except for the case $k = n$. If $k = n$ both inequalities read $\|e - a_{n+1}\| < \delta_n$, and this follows from (14) since $j = i''$. Similarly, (7) and (8) for $m = n+1$ and $k < n$ follow from (15), and for $k = n$ from (16). In order to obtain δ_{n+1}, one can simply define it to be any number satisfying (11) with $m = n+1$, since the right-hand expression is now well defined. This completes the induction.

Having defined the sequences a and b, observe now that for a fixed natural k, inequality (7) implies

$$\begin{aligned} (17)\quad & \|u_k^{(m)}(a) - u_k^{(n)}(a)\| \\ &\le \|u_k^{(m)}(a) - u_k^{(m+1)}(a)\| + \dots + \|u_k^{(n-1)}(a) - u_k^{(n)}(a)\| \\ &\le 2^{-(m+1)} + 2^{-(m+2)} + \dots + 2^{-n} < 2^{-m} \quad \text{for } n > m \ge k \ge 1. \end{aligned}$$

Thus $(u_k^{(i)}(a))_{i=k}^{\infty}$ is a Cauchy sequence converging to some element u_k in A, and (2) follows. Similarly, (8) implies (3). Since $u_k^{(k)}(a) = a_k$, the estimate (17) implies

$$\|a_k - u_k^{(n)}(a)\| < 2^{-k} \quad \text{for } n \ge k,$$

and by letting $n \to \infty$ we obtain

$$(18)\qquad \|a_k - u_k\| \le 2^{-k},$$

and similarly

$$(19)\qquad \|a_k - v_k\| \le 2^{-k}.$$

Since (a_k) is a subsequence of (x_k) and $x_k \to e$, inequalities (18) and (19) imply (4).

In order to obtain (5) write

$$\begin{aligned} (20)\quad & \|v_k^{(k+i)}(b)u_k^{(n)}(a) - v_k^{(k+i)}(b)u_k^{(k+i)}(a)\| \\ &= \|v_k^{(k+i)}(b)u_{k+i+1}^{(n)}(a)u_k^{(k+i)} - v_k^{(k+i)}(b)u_k^{(k+i)}(a)\|. \end{aligned}$$

Inequalities (10), (11), and Lemma 1 imply that the right-hand side of (20) is estimated from above by $2^{-(k+i)}$ for all $n \ge k+i+1$. Letting $n \to \infty$, we obtain (5). The proof of (6) is analogous.

We can now prove our main result.

Theorem. *Let A be a real or complex F-algebra with unity e. Then A is a Q-algebra if and only if all its one-sided maximal ideals are closed.*

Proof. If A is a Q-algebra and I a proper (i.e. $\ne A$) left or right maximal ideal, then I is disjoint from the set $G(A)$ which is a neighbourhood of the unity. Thus the closure $\bar{I}$ is also disjoint from $G(A)$, and so is a proper ideal. Hence $I = \bar{I}$ by the maximality of I.

It remains to show that if A is not a Q-algebra, then it contains either a dense left ideal or a dense right ideal I, for then any maximal ideal containing I is dense and so non-closed. Since A is not a Q-algebra, there is a sequence (x_i) of non-invertible elements tending to e. By passing to a subsequence if necessary, we can assume that no x_i is left invertible (otherwise we could either consider A with the reversed multiplication $x \circ y = yx$, or treat the case when no x_i is right invertible in an analogous way).

By Lemma 2, we find a subsequence (a_i) of (x_i) such that the elements v_k given by (3) are convergent for all k, and $\lim_k v_k = e$. If no v_k is left invertible, then the left ideals $I_k = Av_k$ are proper and satisfy $I_k = Aa_k v_{k+1} \subset Av_{k+1} = I_{k+1}$. Thus $J = \bigcup_{k=1}^{\infty} I_k$ is also a proper left ideal and $v_k \in J$ for all natural k. For every x in A, we have $xv_k \in J$ and $xv_k \to x$ by (4). Consequently, J is dense and we are done.

Consider now the case when $v_{k_0} \in G^{(l)}(A)$, the set of left invertible elements in A, for some k_0, so that $dv_{k_0} = e$ for some d in A. In this case we have $e = dv_{k_0} = da_{k_0}a_{k_0+1}\cdots a_{k-1}v_k$, and so v_k is left invertible for all $k \geq k_0$. It is not right invertible, except for at most one index k_1. For, if v_{k_1} and v_{k_2} are right invertible, and so invertible, and $k_0 \leq k_1 < k_2$, then $v_{k_1} = a_{k_1}\cdots a_{k_2-1}v_{k_2}$, and $a_{k_1}\cdots a_{k_2-1} = v_{k_1}v_{k_2-1}^{-1}$ is an invertible element in A. Thus there is a c in A with $ca_{k_1}\cdots a_{k_2-1} = e$, which is impossible, since a_{k_2-1} is not left invertible.

In this situation we can start our proof again with a new sequence $x_i = v_{k_1+i}$ of left invertible, but not right invertible elements, tending to e. As above, we consider a subsequence (a_i) of (x_i) such that the conclusion of Lemma 2 is satisfied. If no u_k given by (2) is right invertible, we put, as above, $J = \bigcup_{k=1}^{\infty} u_k A$. This is a proper dense right ideal and we are done.

It remains to show that the elements u_i cannot be right invertible. For if some u_{k_0} is right invertible, then, as above, so are all u_k for $k \geq k_0$. Without loss of generality we can assume that all u_i are right invertible. Denote by b_i the left inverse of a_i, and by c_k the right inverse of u_k. By (5), since $b_k\cdots b_{k+i}a_{k+i}\cdots a_k = e$, we obtain

$$\|v_k^{(k+i)}(b)u_k - e\| \leq 2^{-(k+i)}.$$

Thus for a fixed k, we have $\lim_m v_k^{(m)}(b)u_k = e$, which implies that $\lim_m v_k^{(m)}(b)u_k c_k = c_k$. But $v_k^{(m)}(b)u_k c_k = v_k^{(m)}(b)$, so that $\lim_m v_k^{(m)}(b) = c_k$, and thus $c_k u_k = \lim_m v_k^{(m)}(b)u_k = e$. Consequently, u_k is left invertible, and hence invertible for all k. Writing $u_k = a_k u_{k+1}$, we obtain $a_k = u_k u_{k+1}^{-1}$, so that a_k is also invertible. This is a contradiction, since no a_k is right invertible.

From the proofs of the Theorem and of Lemma 2 we can obtain the following corollary which will be useful in the forthcoming paper [9], where we show that a unital F-algebra has all one-sided ideals closed if and only if it is noetherian.

COROLLARY. *Let A be a real or complex F-algebra with unity e.*

(i) *Let (x_n) be a sequence of non-invertible elements of A tending to e. Then there is a subsequence (a_i) of (x_n) such that for each natural k the infinite products u_k and v_k given by* (2) *and* (3) *exist*, *and* $\lim u_k = \lim v_k = e$.

(ii) *If A is not a Q-algebra*, *then there is a sequence $(a_i) \subset A$ with $a_i \to e$ such that either $\bigcup_{i=1}^{\infty} Av_i$ or $\bigcup_{i=1}^{\infty} u_i A$ is a proper one-sided dense ideal.*

We do not know whether an F-algebra A which is not a Q-algebra must contain both left and right dense proper ideals. The answer would be negative if there existed proper Q_l-algebras (and Q_r-algebras), i.e. non-Q-algebras for which the set $G_l(A)$ (resp. $G_r(A)$) is open. On the other hand, the answer would be affirmative, and the proof of our theorem would be much simpler, if in every non-Q-algebra of type F there were a sequence (x_i) of non-invertible, pairwise commuting elements tending to the unity, i.e. if no non-Q-algebra of type F contained a maximal commutative subalgebra which were also a non-Q-algebra. In that case, no proper Q_l- and Q_r-algebras would exist.

Our result does not extend to the non-metrizable case. In [8, Example 7] we give an example of a complete commutative unital topological algebra which is not a Q-algebra but it has all ideals closed.

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