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Monotone substochastic operators and a new Calderón couple

by

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Abstract. An important result on submajorization, which goes back to Hardy, Littlewood and Pólya, states that $b \preceq a$ if and only if there is a doubly stochastic matrix A such that $b = Aa$. We prove that under monotonicity assumptions on the vectors a and b the matrix A may be chosen monotone. This result is then applied to show that $(\widetilde{L^p}, L^\infty)$ is a Calderón couple for $1 \le p < \infty$, where $\widetilde{L^p}$ is the Köthe dual of the Cesàro space $\mathrm{Ces}_{p'}$ (or equivalently the down space $L_\downarrow^{p'}$). In particular, $(\widetilde{L^1}, L^\infty)$ is a Calderón couple, which gives a positive answer to a question of Sinnamon [Si06] and complements the result of Mastyło and Sinnamon [MS07] that $(L_\downarrow^\infty, L^1)$ is a Calderón couple.

1. Introduction. The classical Hardy, Littlewood and Pólya theorem on submajorization states that for two vectors $a, b \in \mathbb{R}^n$, $a \preceq b$ is equivalent to existence of a doubly stochastic matrix A satisfying $Ab = a$ (see for example [BS88], [MOA11] or [Mi88]). These ideas were developed by Calderón [Ca66] (cf. [Ry65]) who proved that for two functions $f, g \in L^0$ such that $f \prec g$ and $g \in L^1 + L^\infty$ there is a substochastic operator T such that $Tg = f$. This allowed him to complete the "only if" part of his famous theorem which states that a space X is an interpolation space between L^1 and L^∞ if and only if X has the following property: if $f, g \in L^0$ are such that $f \prec g$ and $g \in X$, then also $f \in X$ (see [Ca66]). The "only if" part seems to have a deeper impact and be more spectacular. Moreover, it initiated investigations of the so called Calderón couples and (L^1, L^∞) was the first such couple known. Recall that (X_0, X_1) is a *Calderón couple* if each interpolation space between X_0 and X_1 is K-monotone (with respect to (X_0, X_1)). Thanks to the K-divisibility theorem of Brudnyi–Krugljak each K-monotone space may be represented by the K-method of interpolation, which means that all interpolation spaces for a Calderón couple may be produced by the K-method. Some of the most important papers in interpolation theory deal with Calderón couples: see for example [AC84], [Cw76], [Ka93], [LS71] and [Sp78].

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We have sketched briefly how the Hardy–Littlewood–Pólya theorem evolved into Calderón couples, because we will adopt similar steps to prove that $(\widetilde{L^1}, L^\infty)$ is a Calderón couple. We start by proving a monotone version of the Hardy–Littlewood–Pólya theorem and apply it to provide the corresponding monotone version of Calderón's theorem. Notice that actually the latter (and so also the former) was already proved by Bennett and Sharpley [BS86]. They used it in an alternative proof of the K-divisibility theorem for Gagliardo couples. Our method is however essentially different and we present it in the first part of the paper.

The second part is devoted to proving that $(\widetilde{L^1}, L^\infty)$ is a Calderón couple, where $\widetilde{L^p}$ is defined as

$$\widetilde{L^p} = \{f \in L^0 : \widetilde{f} \in L^p\}$$

for $\widetilde{f}$ being the nonincreasing majorant of f, i.e.

$$\widetilde{f}(t) = \operatorname*{ess\,sup}_{s \geq t} |f(s)|, \qquad t > 0. \tag{1.1}$$

In [MS06] it was shown that $(L^\infty_\downarrow, L^1)$ is a Calderón couple, where $L^\infty_\downarrow$ means the down space of L^∞. On the other hand, the Köthe dual of $L^p_\downarrow$ is $\widetilde{L^{p'}}$, so the couple $(\widetilde{L^1}, L^\infty)$ is just the dual of $(L^\infty_\downarrow, L^1)$. In the remaining part we discuss possible extensions of this result. In particular, we apply Dimitriev's results to show that $(\widetilde{\Lambda_\varphi}, L^\infty)$ is a relative Calderón couple with respect to $(\widetilde{L^1}, L^\infty)$. Moreover, using the Avni–Cwikel theorem [AC12] we conclude that $(\widetilde{L^p}, L^\infty)$ is a Calderón couple also for $1 < p < \infty$.

2. Basic definitions. Let μ be a Lebesgue measure on $\mathbb{R}_+$ and denote by L^0 the space of all (equivalence classes of) real-valued functions on $\mathbb{R}_+$ (we understand that a function in L^0 may attain infinite values but only on a set of measure zero). A Banach space $X \subset L^0$ is called a *Banach function space* if $f \in X, g \in L^0, |g| \leq |f|$ implies $g \in X$ and $\|g\|_X \leq \|f\|_X$. We will also understand that there is $f \in X$ with $f(t) > 0$ for each $t > 0$.

By a *symmetric space* we mean a Banach function space X with the additional property that for any two equimeasurable functions $f, g \in L^0$ (that is, they have the same distribution functions $d_f \equiv d_g$, where $d_f(\lambda) = \mu(\{t > 0 : |f(t)| > \lambda\}), \lambda \geq 0$, if $f \in X$ then $g \in X$ and $\|f\|_X = \|g\|_X$. In particular, $\|f\|_X = \|f^*\|_X$, where $f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}$, $t \geq 0$, is the nonincreasing rearrangement of f. For more information on Banach function spaces and symmetric spaces we refer to [KPS82] or [BS88].

For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, a^* is the vector produced by arranging the entries of $|a|$ in nonincreasing order. Writing $b \prec a$ for $a, b \in \mathbb{R}^n$ we understand that

$$\sum_{i=1}^{k} b_i^* \le \sum_{i=1}^{k} a_i^* \quad \text{for each } 0 < k \le n,$$

while $b \preceq a$ means that $b \prec a$ and additionally

$$\sum_{i=1}^{n} b_i^* = \sum_{i=1}^{n} a_i^*.$$

Given a matrix A we shall not distinguish it from the corresponding linear operator and write just Ax for $x \cdot A^T$. A positive matrix $A = (a_{ij})_{i,j=1}^n$ (here positivity means that $a_{ij} \ge 0$ for all i, j, or equivalently $Aa \ge 0$ for each $0 \le a \in \mathbb{R}^n$) is called *doubly stochastic* when

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji} = 1 \quad \text{for each } 0 < i \le n. \tag{2.1}$$

If all the above sums are just less than or equal to one, the matrix is called *substochastic*. Equivalently, a positive matrix A is doubly stochastic [substochastic] if and only if $Aa \preceq a$ [$Aa \prec a$] for each $0 \le a \in \mathbb{R}^n$. A positive matrix A will be called *monotone* if it is positive and Aa is nonincreasing for each nonincreasing $0 \le a \in \mathbb{R}^n$. It is easy to see that a positive square matrix $A = (a_{ij})_{i,j=1}^n$ is monotone if and only if for each $k = 1, \dots, n$,

$$\sum_{j=1}^{k} a_{i,j} \ge \sum_{j=1}^{k} a_{i+1,j} \quad \text{for each } 0 < i \le n-1. \tag{2.2}$$

In fact, the sufficiency of (2.2) is a consequence of the Hardy lemma, and the necessity comes by applying the definition to the vectors $e_1 = [1, 0, \dots, 0]$, $e_1 + e_2 = [1, 1, 0, \dots, 0]$, etc. (see for example [MOA11, Chapter 2.E]).

We shall also need continuous versions of the above objects. A linear positive operator (in the sense that $f \ge 0$ implies $Tf \ge 0$) defined on $L^1 + L^\infty$, mapping continuously L^1 into L^1 and L^∞ into L^∞ with both norms ≤ 1 is called *substochastic*. When T is positive, this is equivalent to $Tf \prec f$ for each $f \in L^1 + L^\infty$ (cf. [KPS82, p. 84]). To talk about monotone functions in the setting of Banach function spaces we need to clarify the notion of monotonicity to make it insensitive to perturbation on a set of measure zero. Hence we will understand that $0 \le f \in L^0$ is nonincreasing if it is nonincreasing (in the classical sense) on some set $S \subset \mathbb{R}_+$ such that $\mu(\mathbb{R}_+ \setminus S) = 0$. It will be useful to use also an equivalent formulation: $0 \le f \in L^0$ is nonincreasing if for each $x > 0$,

$$\operatorname*{ess\,sup}_{s \ge x} |f(s)| \le \operatorname*{ess\,inf}_{s \le x} |f(s)| \tag{2.3}$$

(see [Si94, Theorem 2.4]). Let X, Y be Banach function spaces. We shall say that an operator $T : X \to Y$ is *monotone* if it is positive and for each nonincreasing $0 \le f \in X$, Tf is also nonincreasing.

Let us recall that a *Banach limit* is a linear functional $\eta \in (l^\infty)^*$ with the following properties:

(i) if $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} x_n = \eta((x_n)_{n=1}^\infty)$,
(ii) $x_n \ge 0$ for each n implies $\eta((x_n)_{n=1}^\infty) \ge 0$ (i.e. η is positive),
(iii) $\eta((x_n)_{n=1}^\infty) = \eta((x_{n+1})_{n=1}^\infty)$ (i.e. η is shift invariant).

Such limits arise from applying the Hahn–Banach theorem to subspaces of l^∞. Alternatively, one can define η to be the limit of $([Cx]_n)$ with respect to a free ultrafilter, where $[Cx]_n = \frac{1}{n}\sum_{i=1}^n x_i$.

For a given Banach function space X we define the space $\widetilde{X}$ as

$$\widetilde{X} = \{f \in L^0 : \widetilde{f} \in X\}$$

with the norm given by

$$\|f\|_{\widetilde{X}} = \|\widetilde{f}\|_X.$$

To ensure that such a space is a Banach function space in the sense of our definition, we will assume that for a Banach function space X there is a nonincreasing $f \in X$ with $f(t) > 0$ for each $t > 0$. Spaces $\widetilde{X}$ appear in a natural way in different contexts. It seems that such spaces in general form (for X symmetric) were first defined by Sinnamon, who proved that they are duals of down spaces, i.e. $(X_\downarrow)' = \widetilde{X}$ (see [Si94], [Si01], [Si03] and [Si07]). On the other hand, the space $\widetilde{L^1}$ was found to be the Köthe dual of the Cesàro space Ces_∞ already in the early paper [KKL48]. Recently, it was also proved in [LM15a] that they are duals of Cesàro type spaces even for not necessarily symmetric X, i.e. $(CX)' = \widetilde{X'}$ (for more information on such spaces and their history see [LM15a] and references therein; cf. [AM09], [KMS07]). On the other hand, $\widetilde{X}$ is just the space $X(Q)$ associated with the cone Q of positive nonincreasing functions considered in [CC05] and [CEP99].

For two couples of Banach function spaces $(X_0, X_1), (Y_0, Y_1)$ and a linear operator T acting from X_0+X_1 into Y_0+Y_1 we write $T : (X_0, X_1) \to (Y_0, Y_1)$ when $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ with

$$\|T\|_{(X_0,X_1)\to(Y_0,Y_1)} = \max\{\|T\|_{X_0\to Y_0}, \|T\|_{X_1\to Y_1}\} < \infty.$$

For $f \in X_0 + X_1$ the *K-functional* of f with respect to the couple (X_0, X_1) is defined as

$$K(t, f; X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\} \quad \text{for } t > 0.$$

Having two couples $(X_0, X_1), (Y_0, Y_1)$ of Banach function spaces we say that (X_0, X_1) is a *relative Calderón couple* with respect to (Y_0, Y_1) if for each $f \in X_0 + X_1$, $g \in Y_0 + Y_1$ the inequality $K(t, g; Y_0, Y_1) \le K(t, f; X_0, X_1)$

for all $t > 0$ implies that there exists $T : (X_0, X_1) \to (Y_0, Y_1)$ with $Tf = g$. If $(X_0, X_1) = (Y_0, Y_1)$ we call (X_0, X_1) simply a *Calderón couple*. More information on interpolation spaces and Calderón couples may be found in [AC84], [BK91], [BL76], [BS88], [Cw76], [KPS82] and [Sp78].

3. Monotone version of the Calderón and Hardy–Littlewood–Pólya theorems. The classical theorem of Calderón states that if $g \prec f$, then there is a substochastic operator such that $Tf = g$. It may be regarded as a continuous version of the Hardy, Littlewood and Pólya theorem which ensures existence of a doubly stochastic matrix A such that $Aa = b$ provided $b \preceq a$ where $a, b \in \mathbb{R}^n$ ([BS88, Theorem 2.7, p. 108], cf. [Mi88] and [MOA11]). We will need the following monotone refinements of those theorems.

THEOREM 3.1 (Bennett–Sharpley 1986). *Let $0 \le f, g \in L^1 + L^\infty$ be both nonincreasing and suppose that $g \prec f$. Then there is a substochastic monotone operator T such that $Tf = g$.*

THEOREM 3.2. *Let $0 \le a, b \in \mathbb{R}^n$ be both nonincreasing. If $b \preceq a$, then there exists a doubly stochastic monotone matrix A such that $Aa = b$. If just $b \prec a$, then the matrix A may be chosen to be substochastic and monotone.*

Theorem 3.1 was proved by Bennett and Sharpley in [BS86, Theorem 5] (cf. [BS88, Lemma 7.5]); they used an idea of Lorentz and Shimogaki [LS71], the so called "pushing mass" technique. On the other hand, the original proof of Calderón's theorem [BS88, Theorem 2.10, p. 114] is based on the Hardy–Littlewood–Pólya result. We will prove the above monotone refinement of the Hardy–Littlewood–Pólya theorem and use it in the alternative proof of Theorem 3.1.

Proof of Theorem 3.2. The proof of the first statement is by induction on n. The statement for $n = 1$ is evident. Let $n > 1$, assume the claim is true for all $k = 1, \ldots, n-1$ and let $0 \le a, b \in \mathbb{R}^n$ be both nonincreasing with $b \preceq a$. If $a_1 = b_1$, then $a' = (a_2, \ldots, a_n)$ and $b' = (b_2, \ldots, b_n)$ satisfy $b' \preceq a'$. Thus by induction hypothesis there is an $(n-1) \times (n-1)$ doubly stochastic and monotone matrix B' such that $b' = B'a'$. Moreover, the matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & B' \end{bmatrix}$$

is also doubly stochastic monotone and $Ba = b$. If $a_1 > b_1$, we will find a doubly stochastic and monotone matrix A' such that $b \preceq A'a$ but with $[A'a]_1 = b_1$. Then it is enough to apply the previous step to the vectors $b, A'a$ and the desired matrix will be $A = BA'$. Therefore we only need to find A' as above.

Suppose $a_1 > b_1$. Because $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, we can find $k \leq n$ such that

$$\frac{1}{k}\sum_{i=1}^{k} a_i \leq b_1 < \frac{1}{k-1}\sum_{i=1}^{k-1} a_i.$$

Consider the function

$$f(\eta) = \frac{\eta}{k-1}\sum_{i=1}^{k-1} a_i + (1-\eta)a_k$$

for $0 \leq \eta \leq 1$. We have $f(1) = \frac{1}{k-1}\sum_{i=1}^{k-1} a_i > b_1$ and $f\big(\frac{k-1}{k}\big) = \frac{1}{k}\sum_{i=1}^{k-1} a_i + \frac{1}{k}a_k \leq b_1$. Therefore there is a solution γ of the equation $f(\eta) = b_1$ that belongs to the interval $[(k-1)/k, 1)$. In other words, the equation

$$b_1 = \sum_{i=1}^{n} \lambda_i a_i$$

has a solution $\lambda_i = \gamma/(k-1)$ for $i = 1, \ldots, k-1$, $\lambda_k = 1-\gamma$ and $\lambda_i = 0$ for $i > k$. Moreover, the sequence (λ_i) is nonincreasing because $(k-1)/k \leq \gamma \leq 1$. Then the matrix A' we are looking for is

$$A' = \begin{bmatrix} \frac{\gamma}{k-1} & \cdots & \frac{\gamma}{k-1} & 1-\gamma & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \frac{\gamma}{k-1} & \cdots & \frac{\gamma}{k-1} & 1-\gamma & 0 & & 0 \\ 1-\gamma & \cdots & 1-\gamma & \sigma & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where the entry $\sigma = 1-(k-1)(1-\gamma)$ is at position (k,k). In fact, we have

$$\sum_{i=1}^{j} b_i \leq jb_1 = \sum_{i=1}^{j}[A'a]_i \quad \text{for } j \leq k-1,$$
$$\sum_{i=1}^{j} b_i \leq \sum_{i=1}^{j} a_i = \sum_{i=1}^{j}[A'a]_i \quad \text{for } j \geq k,$$

so that $b \preceq A'a$. Moreover, A' is evidently monotone and doubly stochastic.

Note that above we have essentially used the equality $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, so the second case, when we only have $b \prec a$, has to be treated in another way. We divide the interval $(0,n]$ of natural numbers into intervals $(0,i_1], (i_1,i_2], \ldots, (i_{k-1}, i_k = n]$ in such a way that $\delta_j a$ and b restricted to $(i_{j-1}, i_j]$ satisfy the stronger relation $\preceq$ with some constant $\delta_j \leq 1$, i.e. $b\chi_{(i_{j-1},i_j]} \preceq \delta_j a\chi_{(i_{j-1},i_j]}$. Then it will be enough to find doubly stochastic and monotone matrices A_j for each $(i_{j-1}, i_j]$ and to define

$$A = \begin{bmatrix} \delta_1 A_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_k A_k \end{bmatrix}.$$

It remains to find the above partition of $(0, n]$ and the sequence (δ_j). There is nothing to do when $b \preceq a$. Suppose $b \prec a$. Set $i_0 = 0$ and define, for $j = 0, 1, 2, \dots$,

$$\delta_{j+1} = \max_{i_j < k \le n} \frac{\sum_{i=i_j+1}^{k} b_i}{\sum_{i=i_j+1}^{k} a_i}, \quad i_{j+1} = \max\Big\{k \le n : \sum_{i=i_j+1}^{k} b_i = \delta_{j+1} \sum_{i=i_j+1}^{k} a_i\Big\}.$$

We apply these formulas until $i_j = n$ for some j. Denote the last such j by k. Then the sequences $(\delta_j)_{j=1}^k$ and $((i_{j-1}, i_j]_{j=1}^k)$ are as desired. ∎

Remark 3.3. To see that the above method is essentially different from the Lorentz–Shimogaki "pushing mass" technique applied by Bennett and Sharpley [BS86, proof of Lemma 3], consider the vectors $f = (1,1,1,1)$ and $g = (2,1,1,0)$. Of course, $f \preceq g$. Applying the "pushing mass" algorithm from [BS86] we need three steps, namely $g_0 = (3/2, 3/2, 1, 0)$, $g_1 = (4/3, 4/3, 4/3, 0)$, $g_2 = f = (1,1,1,1)$. On the other hand, our method produces the desired matrix immediately (we mean the inductive step from the first part of the above proof). It is also worth mentioning that the maximal number of steps in our method is n, while the "pushing mass" technique may require more than n steps (in general less than $2n$). In fact, consider $f = (12/10, 11/10, 1, 8/10)$ and $g = (2,1,1,0)$. Then the steps of the "pushing mass" method are $g_0 = (3/2, 3/2, 1, 0)$, $g_1 = (4/3, 4/3, 4/3, 0)$, $g_2 = (12/10, 12/10, 12/10, 4/10)$, $g_3 = (12/10, 11/10, 11/10, 6/10)$, $g_4 = f$, and so four steps are not enough to get f.

Remark 3.4. There is also another important connection of the proof of Theorem 3.2 with the classical results. Namely, a careful reading reveals that it may be regarded as a discrete version of the "cutting corners" method of Arazy and Cwikel [AC84, Figure 1]. It is however disappointing that their method cannot be applied to get the (L^p, L^∞) version of Theorem 3.1 in case $p > 1$ because then the operators S from [AC84, pp. 258–260] are not monotone.

Theorem 3.1 was stated without proof as a corollary from its "simple–function" version in [BS86] and [BS88]. It seems to be intuitively evident, but we explain a little more carefully that the standard "limit" argument preserves monotonicity of the resulting operator. In order to do so, we start by sketching the main steps of the proof of [BS88, Proposition 2.9, p. 110] and then we complete the required explanation.

Proof of Theorem 3.1. Let $0 \leq f, g \in L^1 + L^\infty$ be both nonincreasing with $g \prec f$.

Suppose first that

$$g = \sum_{k=1}^{n} b_k \chi_{A_k}, \tag{3.1}$$

where $A_k = [(k-1)d, kd)$ for some $d > 0$. Define $G : L^1 + L^\infty \to \mathbb{R}^n$ and $H : \mathbb{R}^n \to L^1 + L^\infty$ by

$$G : h \mapsto \left(\frac{1}{\mu(A_k)} \int_{A_k} h \, d\mu \right)_{k=1}^{n} \tag{3.2}$$

and

$$H : (a_k)_{k=1}^n \mapsto \sum_{k=1}^{n} a_k \chi_{A_k}. \tag{3.3}$$

Then the composition HG is just an averaging operator and $g = HGg$. Moreover, $g \prec f$ implies $Gg \prec Gf$ as well as $HGg \prec HGf$. We apply Theorem 3.2 to find a substochastic monotone matrix B such that $Gg = BGf$. Then set $T = HBG$ so that

$$g = HGg = HBGf = Tf,$$

and T is monotone because each of its components evidently is. Thus we have proved the conclusion in the case when g is of the form (3.1).

Let now g be arbitrary with $g \prec f$. We find a sequence $(g_m)_{m=1}^\infty$ such that $g_m \to g$ μ-a.e. and $g_m \prec f$ with g_m nonincreasing for each m. Moreover, we may assume that each g_m is as in (3.1) (for some n and d depending on m). Then one can apply the previous part to find a sequence of substochastic monotone operators T_m satisfying $T_m(f) = g_m$. Following the proof of [BS88, Proposition 2.9, p. 110], it remains to define the set function $\nu_h : \Sigma|_E \to \mathbb{R}$, where E is a measurable set with finite measure, by

$$\nu_h(F) = \eta\Big(\Big(\int_F T_m h \, d\mu \Big)_{m=1}^\infty \Big)$$

for η being a fixed Banach limit. Then, exactly as in [BS88], we conclude that ν_h is absolutely continuous with respect to μ (restricted to E), and consequently Th on E is defined to be the Radon–Nikodym derivative of ν_h with respect to μ. Since E was arbitrary and by uniqueness of the Radon–Nikodym derivative (up to a set of measure 0), one can "glue together" all parts to define Th on the whole semiaxis. Also $Tf = g$ and the only thing we need to explain more carefully is the monotonicity of T. Let $0 \leq h \in L^1 + L^\infty$ be nonincreasing. Notice first that by the Lebesgue theorem, $\frac{1}{\mu(F_n(t))} \int_{F_n(t)} Th \, d\mu \to Th(t)$ for almost all $t > 0$, where $F_n(t) =$

$(t-1/n, t+1/n) \cap \mathbb{R}_+$. Denote by Z the set of such t. Choose $0 < t_0 < t_1$ from Z. Then for each m, and each n such that $F_n(t_0) \cap F_n(t_1) = \emptyset$,

$$\frac{1}{\mu(F_n(t_0))} \int\limits_{F_n(t_0)} T_m h \, d\mu \geq \frac{1}{\mu(F_n(t_1))} \int\limits_{F_n(t_1)} T_m h \, d\mu$$

by monotonicity of $T_m h$. Finally, by positivity of the functional η we conclude that also

$$\frac{1}{\mu(F_n(t_0))} \int\limits_{F_n(t_0)} Th \, d\mu \geq \frac{1}{\mu(F_n(t_1))} \int\limits_{F_n(t_1)} Th \, d\mu$$

for each such n, and thus $Th(t_0) \geq Th(t_1)$, which means that Th is nonincreasing in the sense of (2.3). ∎

REMARK 3.5. Dmitriev [Dm81] considered the so-called *positively K-monotone interpolation*, where having positive $f \in X_0 + X_1, g \in Y_0 + Y_1$ with $K(t, g; Y_0, Y_1) \leq K(t, f; X_0, X_1)$ for all $t > 0$ one asks if there is a positive operator $T : (X_0, X_1) \to (Y_0, Y_1)$ satisfying $Tf = g$. Therefore Theorem 3.1 may be read as: the couple (L^1, L^∞) satisfies the "monotone" version of the above property, i.e. in addition to positivity we also require monotonicity of T, assuming that f, g are nondecreasing.

4. Calderón couples. We start with the following theorem which is the second of the main two steps toward the proof that $(\widetilde{L^1, L^\infty})$ is a Calderón couple.

THEOREM 4.1. *Let $f \in \widetilde{L^0} = \{f \in L^0 : \tilde{f} \in L^0\}$. Then for each $q > 1$ there is a linear operator S defined on $\widetilde{L^0}$ such that $Sf = \tilde{f}$ and $\|S\|_{\widetilde{X} \to \widetilde{X}} \leq q$ for each Banach function space X.*

Proof. Let $f \in \widetilde{L^0}$, $f \neq 0$ and $q > 1$. Since $h \mapsto \operatorname{sign}(f) h$ acts boundedly with norm one in each Banach function space, we may assume that $0 \leq f$. For each $n \in \mathbb{Z}$ define

$$A'_n = (q^{-n}, q^{-n+1}] \quad \text{and} \quad A_n = (\tilde{f})^{-1}(A'_n).$$

Because $\tilde{f}$ is nonincreasing and right-continuous, the A_n are either empty or are left-closed intervals. Moreover, we can choose a nondecreasing sequence $(a_n)_{n=m_0}^{m_1}$ of real numbers in such a way that each nonempty A_n is of the form

$$A_n = [a_n, a_{n+1}),$$

or $A_n = [a_n, \infty)$ for some $n = m_1$. Let us say a few words about this sequence. We have three possibilities for the "right side" of the sequence $(a_n)_{n=m_0}^{m_1}$. If $0 < \lim_{t\to\infty} \tilde{f}(t) \in A'_k$ for some k, then we set $A_k = [a_k, \infty)$, $m_1 = k$ and we additionally define $a_{m_1+1} = \infty$. The second case with finite

m_1 occurs when there is $0 < c < \infty$ such that $\tilde{f}(t) = 0$ for each $t \in [c, \infty)$ but $0 < \lim_{t\to c^-} \tilde{f}(t) \in A'_k$ for some k. Then we understand that $m_1 = k$, $A_k = [a_k, c)$ and $a_{m_1+1} = c$. In the remaining case, $m_1 = \infty$ and for each n_0 there is $n_1 > n_0$ such that $a_{n_1} > a_{n_0}$. The situation on the "left" of (a_n) is easier because either $0 < a_n \downarrow 0$ with $n \to -\infty$ and we then set $m_0 = -\infty$, or $a_k = 0$ for some k and we set $m_0 = k$ for the largest such k (notice that if $f \neq 0$ then $a_k \neq 0$ for some k). Note also that A_n may be nonempty only for $m_0 \leq n \leq m_1$, where we understand $n < \infty$ if $m_1 = \infty$ and $n > -\infty$ if $m_0 = -\infty$. Of course, it may happen that A_n is empty for some n between m_0 and m_1; then we understand that $a_n = a_{n+1}$.

We can now proceed with the construction of the desired operator. For each $m_0 \leq n \leq m_1$ the limits

$$\tilde{f}(a_{n+1}^-) = \lim_{t\to a_{n+1}^-} \tilde{f}(t)$$

are well defined and finite by monotonicity of $\tilde{f}$. Therefore, for each $m_0 \leq n \leq m_1$ such that $A_n \neq \emptyset$ there exists a sequence (B_k^n) of sets of finite, positive measure such that

$$B_k^n \subset ([a_{n+1} - 1/k] \vee a_{n+1}/2, \infty)$$

and

$$\tilde{f}([a_{n+1} - 1/k] \vee a_{n+1}/2) - 1/k \leq f(t) \leq \tilde{f}([a_{n+1} - 1/k] \vee a_{n+1}/2) \tag{4.1}$$

for each $t \in B_k^n$ and $k \in \mathbb{N}$, with the only exception in case $a_{m_1+1} = \infty$ (i.e. when $\tilde{f}(\infty) > 0$), in which we just take

$$B_k^{m_1} \subset (ka_{m_1}, \infty)$$

satisfying

$$\tilde{f}(\infty) - 1/k \leq f(t) \leq \tilde{f}(\infty) \quad \text{ for } t \in B_k^{m_1}.$$

Let η be a Banach limit. Then for each n as above,

$$\tilde{f}(a_{n+1}^-) = \lim_{k\to\infty} \frac{1}{\mu(B_k^n)} \int_{B_k^n} f\,d\mu = \eta\bigg(\bigg(\frac{1}{\mu(B_k^n)} \int_{B_k^n} f\,d\mu\bigg)_{k=1}^{\infty}\bigg). \tag{4.2}$$

We define an operator F on $\widetilde{L^0}$ by the formula

$$Fh = \sum_{n=m_0}^{m_1} \lambda_n(h)\chi_{A_n},$$

where

$$\lambda_n(h) = \eta\bigg(\bigg(\frac{1}{\mu(B_k^n)} \int_{B_k^n} h\,d\mu\bigg)_{k=1}^{\infty}\bigg)$$

for those n with nonempty A_n, and $\lambda_n(h) = 0$ for the remaining n's. The operator F is well defined on $\widetilde{L^0}$ and linear. We will show that

$$|Fh| \le \tilde{h}. \tag{4.3}$$

In fact, choose nonempty A_n and let $t \in A_n$. We can find l such that $B_k^n \subset [t,\infty)$ for all $k \ge l$. Then by property (iii) of the Banach limit we get

$$\begin{aligned}|Fh(t)| = |\lambda_n(h)| &= \left|\eta\left(\left(\frac{1}{\mu(B_k^n)}\int_{B_k^n} h\,d\mu\right)_{k=1}^{\infty}\right)\right| \\ &= \left|\eta\left(\left(\frac{1}{\mu(B_{k+l}^n)}\int_{B_{k+l}^n} h\,d\mu\right)_{k=1}^{\infty}\right)\right| \\ &\le \eta\left(\left(\frac{1}{\mu(B_{k+l}^n)}\int_{B_{k+l}^n} |h|\,d\mu\right)_{k=1}^{\infty}\right) \le \operatorname*{ess\,sup}_{s\ge t}|h(s)| = \tilde{h}(t).\end{aligned}$$

This means that $\|F\|_{\widetilde{X}\to\widetilde{X}} \le 1$ for each Banach function space X. Thus we have found the main part of the desired operator S. The second part will be simpler, just the multiplication operator $M_v : h \mapsto vh$ whose symbol v is given by

$$v = \sum_{n=m_0}^{m_1} \frac{1}{\tilde{f}(a_{n+1}^-)}\tilde{f}\chi_{A_n}.$$

It is well defined because $\tilde{f}(a_{n+1}^-) > 0$ for each $m_0 \le n \le m_1$. Moreover, if $t \in A_n$ then

$$\frac{\tilde{f}(t)}{\tilde{f}(a_{n+1}^-)} \le \frac{q^{-n+1}}{q^{-n}} = q.$$

Then $\|v\|_{L^\infty} \le q$, which implies that $\|M_v\|_{\widetilde{X}\to\widetilde{X}} \le q$ for each Banach function space X (see for example [MP89]). The proof is finished now, since for $S = M_vF$,

$$Sf = v\Big(\sum_{n=m_0}^{m_1}\lambda_n(f)\chi_{A_n}\Big) = \Big(\sum_{n=m_0}^{m_1}\frac{1}{\tilde{f}(a_{n+1}^-)}\tilde{f}\chi_{A_n}\Big)\Big(\sum_{n=m_0}^{m_1}\tilde{f}(a_{n+1}^-)\chi_{A_n}\Big) = \tilde{f}$$

and $\|S\|_{\widetilde{X}\to\widetilde{X}} \le \|M_v\|_{\widetilde{X}\to\widetilde{X}}\|F\|_{\widetilde{X}\to\widetilde{X}} \le q$. ∎

The K-functional for the couple $(\widetilde{L^1}, L^\infty)$ was already calculated by Sinnamon [Si91] and it is given by the formula

$$K(t,f;\widetilde{L^1},L^\infty) = K(t,\tilde{f};L^1,L^\infty) = \int_0^t \tilde{f}(s)\,ds. \tag{4.4}$$

It is also not difficult to see that the first equality may be extended to all couples (X, L^∞) with X being an arbitrary Banach function space.

PROPOSITION 4.2. *Let X be a Banach function space and $f \in \widetilde{X} + L^\infty$. Then*

$$K(t, f; \widetilde{X}, L^\infty) = K(t, \tilde{f}; X, L^\infty).$$

Proof. Without loss of generality we may assume that $0 \le f \in \widetilde{X} + L^\infty$. It is enough to notice that

$$\widetilde{[(f-a)_+]} = (\tilde{f} - a)_+,$$

where, as usual, $g_+ = g\chi_{\{s:\, g(s) \ge 0\}}$. Then

$$\begin{aligned} K(t, f; \widetilde{X}, L^\infty) &= \inf_{a>0} \{ \|(f-a)_+\|_{\widetilde{X}} + at \} \\ &= \inf_{a>0} \{ \|(\tilde{f}-a)_+\|_X + at \} \\ &= K(t, \tilde{f}; X, L^\infty). \ \blacksquare \end{aligned}$$

REMARK 4.3. If we replace the equality $K(t, f; \widetilde{X}, L^\infty) = K(t, \tilde{f}; X, L^\infty)$ by equivalence, then it holds in a much more general setting. Namely, the equivalence $K(\cdot\,, f; \widetilde{X}, \widetilde{Y}) \approx K(\cdot\,, \tilde{f}; X, Y)$ for symmetric spaces X, Y is a straightforward consequence of the general property that the operation $X \mapsto \widetilde{X}$ commutes with the Calderón–Lozanovskii construction, i.e. $\varphi(\widetilde{X}, \widetilde{Y}) = \widetilde{\varphi(X, Y)}$ with norms satisfying

$$\|x\|_{\widetilde{\varphi(X,Y)}} \le \|x\|_{\varphi(\widetilde{X}, \widetilde{Y})} \le C \|x\|_{\widetilde{\varphi(X,Y)}},$$

where $1 \le C \le 2$ (see [LM15b] and note that this equivalence also holds for some nonsymmetric spaces). In particular, taking $\varphi(u, v) = u + v$ and understanding that the space tY contains the same elements as Y with $\|f\|_{tY} = t\|f\|_Y$, we obtain

$$\widetilde{X + tY} = \widetilde{X} + t\widetilde{Y}$$

for each $t > 0$. Consequently,

$$\begin{aligned} K(t, \tilde{g}; X, Y) = \|\tilde{g}\|_{X+tY} &= \|g\|_{\widetilde{X+tY}} \le \|g\|_{\widetilde{X}+t\widetilde{Y}} \\ &\le 2\|g\|_{\widetilde{X+tY}} = 2\|\tilde{g}\|_{X+tY} = 2K(t, \tilde{g}; X, Y), \end{aligned}$$

and since $K(t, g; \widetilde{X}, \widetilde{Y}) = \|g\|_{\widetilde{X}+t\widetilde{Y}}$, the claim follows. Notice that the above equivalences may also be deduced from [CEP99] (cf. [CC05]).

LEMMA 4.4. *If an operator $T : X \to Y$ is monotone, then $T : \widetilde{X} \to \widetilde{Y}$ with $\|T\|_{\widetilde{X} \to \widetilde{Y}} \le \|T\|_{X \to Y}$.*

Proof. Let $f \in \widetilde{X}$. Then, by monotonicity of T, we have

$$\widetilde{T(f)} = \widetilde{|T(f)|} \le \widetilde{T(|f|)} \le \widetilde{T(\tilde{f})} = T(\tilde{f}),$$

which means that

$$\|Tf\|_{\widetilde{Y}} = \|\widetilde{Tf}\|_Y \le \|T(\tilde{f})\|_Y \le \|T\|_{X\to Y}\|\tilde{f}\|_X = \|T\|_{X\to Y}\|f\|_{\widetilde{X}}. \ \blacksquare$$

We are now ready to state the main theorem of this section.

THEOREM 4.5. *The couple $(\widetilde{L^1}, L^\infty)$ is a Calderón couple.*

Proof. Let $f, g \in \widetilde{L^1} + L^\infty$ with

$$K(t, g; \widetilde{L^1}, L^\infty) \le K(t, f; \widetilde{L^1}, L^\infty) \quad \text{for all } t > 0.$$

We will find H satisfying $Hf = g$ according to the following scheme:

$$f \overset{S}{\mapsto} \tilde{f} \overset{T}{\mapsto} \tilde{g} \overset{W}{\mapsto} g,$$

where all of S, T, W act boundedly from $(\widetilde{L^1}, L^\infty)$ into itself. Firstly, we find the last operator W which is just multiplication by the function

$$g/\tilde{g} \le 1,$$

where we understand $g(t)/\tilde{g}(t) = 0$ when $\tilde{g}(t) = 0$. As a consequence, $\|W\|_{(\widetilde{L^1},L^\infty)\to(\widetilde{L^1},L^\infty)} = 1$. Also the operator S is already known, because it is exactly the one from Theorem 4.1, say with $\|S\|_{(\widetilde{L^1},L^\infty)\to(\widetilde{L^1},L^\infty)} \le \gamma$, where $\gamma > 1$. It remains to find T. By Proposition 4.2, the assumption

$$K(t, g; \widetilde{L^1}, L^\infty) \le K(t, f; \widetilde{L^1}, L^\infty)$$

means that

$$\int_0^t \tilde{g}(s)\, ds \le \int_0^t \tilde{f}(s)\, ds$$

for all $t > 0$. Therefore, applying Theorem 3.1 to $\tilde{f}, \tilde{g}$, we find a monotone operator T such that $T\tilde{f} = \tilde{g}$ and $\|T\|_{(L^1,L^\infty)\to(L^1,L^\infty)} \le 1$. Monotonicity of T and Lemma 4.4 imply that also $\|T\|_{(\widetilde{L^1},L^\infty)\to(\widetilde{L^1},L^\infty)} \le 1$. Finally, $H = WTS$ and the proof is finished with $\|H\|_{(\widetilde{L^1},L^\infty)\to(\widetilde{L^1},L^\infty)} \le \gamma$. ■

Suppose we want to show that $(\widetilde{X_0}, \widetilde{X_1})$ and $(\widetilde{Y_0}, \widetilde{Y_1})$ are relative Calderón couples. The proof of the above theorem suggests the following approach. Fix $g \in \widetilde{Y_0} + \widetilde{Y_1}$ and $f \in \widetilde{X_0} + \widetilde{X_1}$ with $K(t, g; \widetilde{Y_0}, \widetilde{Y_1}) \le K(t, f; \widetilde{X_0}, \widetilde{X_1})$ for all $t > 0$ and consider the scheme

$$(\widetilde{X_0}, \widetilde{X_1}) \xrightarrow{\ A\ } (\widetilde{X_0}, \widetilde{X_1}) \xrightarrow{\ S\ } (\widetilde{Y_0}, \widetilde{Y_1}) \xrightarrow{\ B\ } (\widetilde{Y_0}, \widetilde{Y_1}),$$

$$f \xmapsto{\quad A \quad} \tilde{f} \xmapsto{\quad S \quad} \tilde{g} \xmapsto{\quad B \quad} g.$$

Notice that once again existence of A is a consequence of Theorem 4.1, and B is just multiplication by $g/\tilde{g}$. The only one missing is S. However, the assumption on f, g is, by Remark 4.3, equivalent (up to some constant) with

$K(t,\tilde{g};Y_0,Y_1) \le K(t,\tilde{f};X_0,X_1)$ for all $t>0$. Therefore, if we prove that for given positive, nonincreasing functions $h \in X_0+X_1, w \in Y_0+Y_1$ with $K(t,w;Y_0,Y_1) \le K(t,h;X_0,X_1)$ for all $t>0$, there is a positive monotone operator $S:(X_0,X_1)\to(Y_0,Y_1)$ with $Sh=w$ then, thanks to Lemma 4.4, we will have the desired operator $S:(\widetilde{X_0},\widetilde{X_1})\to(\widetilde{Y_0},\widetilde{Y_1})$. According to this observation and using Dmitriev's results [Dm74] we can generalize the main theorem to the Lorentz space setting.

Recall that the Lorentz space Λ_φ is defined by

$$\Lambda_\varphi = \Big\{ f \in L^0 : \|f\|_{\Lambda_\varphi} = \int f^*(t)\,d\varphi(t) < \infty \Big\},$$

where φ is a concave, positive and increasing function on $[0,\infty)$ with $\varphi(0^+)=0$ and $\varphi(\infty)=\infty$ (cf. [BS86], [KPS82]). We get the following monotone version of Dmitriev's theorem [Dm74].

Theorem 4.6. *Let $0 \le f \in \Lambda_\varphi + L^\infty$ and $0 \le g \in L^1+L^\infty$ be both nonincreasing and such that*

$$K(t,g;L^1,L^\infty) \le K(t,f;\Lambda_\varphi,L^\infty) \quad \text{for all } t>0.$$

Then there exists a monotone $S:(\Lambda_\varphi,L^\infty)\to(L^1,L^\infty)$ such that $Sf=g$.

Proof. As in the proof of Theorem 4.5, we sketch the important steps of Dmitriev's proof to demonstrate where the monotone modification is necessary. Let $0\le f\in\Lambda_\varphi+L^\infty$ and $0\le g\in L^1+L^\infty$ satisfy our assumptions.

Suppose first that

$$g = \sum_{k=1}^{n} b_k \chi_{A_k}, \tag{4.5}$$

where $A_k = [(k-1)d, kd)$ for some $d>0$. For $h \in \Lambda_\varphi + L^\infty$ define an operator D by

$$Dh(z) = \sum_{k=1}^{n} \frac{1}{d} \int_{\varphi^{-1}((k-1)d)}^{\varphi^{-1}(kd)} h(t)\,d\varphi(t)\chi_{A_k}.$$

Then D is positive, $\|D\|_{(\Lambda_\varphi,L^\infty)\to(L^1,L^\infty)} \le 1$ and D is monotone (cf. [Dm74, pp. 529–530]). Moreover, Df is nonincreasing and $g \prec Df$, so we can apply Theorem 3.1 to find a monotone substochastic operator T with $TDf=g$. Since T and D are monotone, also TD is monotone and we take just $S=TD$.

Now, let a sequence (g_n) consist of functions of the form (4.5) and be such that $g_n \uparrow g$ a.e. For each g_n we find a monotone operator S_n as above. Then, once again following Dmitriev's reasoning, we find the desired S as an accumulation point of the sequence (S_n) with respect to the weak operator topology Γ (see Appendix below), thanks to the result of Sedaev [Se71].

Taking a subnet (S_α) of the sequence (S_n) that tends to S we see that for each measurable $A \subset [0,\infty)$ with $\mu(A) < \infty$ and each $h \in \Lambda_\varphi + L^\infty$ we have

$$\lim_\alpha \int_A S_\alpha h\, d\mu = \int_A Sh\, d\mu.$$

Therefore, one can deduce monotonicity of S as in the proof of Theorem 3.1. ∎

Corollary 4.7. *The couple $(\widetilde{\Lambda_\varphi}, L^\infty)$ is a relative Calderón couple with respect to $(\widetilde{L^1}, L^\infty)$.*

Note that it is not necessary to follow the way described above to conclude that some couple of the form $(\widetilde{X}, \widetilde{Y})$ is a Calderón couple. In fact, a straightforward application of Theorem 14 from [AC12] to Theorem 4.5 gives such a result for $(\widetilde{L^p}, L^\infty)$, although we know nothing about monotone operators in this case. In fact, we see that $\widetilde{|f|^p} = (\tilde{f})^p$, which means that the p-convexification $(\widetilde{X})^p$ of $\widetilde{X}$ is exactly $\widetilde{X^p}$. Recall that the *p-convexification* $(p \ge 1)$ of a Banach function space X is $X^p = \{f : |f|^p \in X\}$ with the norm $\|f\|_{X^p} = \||f|^p\|_X^{1/p}$.

Theorem 4.8. *For $1 \le p < \infty$, $(\widetilde{L^p}, L^\infty)$ is a Calderón couple.*

Remark 4.9. All the above results, except Remark 4.3, remain true when the underlying measure space $(\mathbb{R}_+, \mu)$ is replaced by $([0,1], \mu)$ with the Lebesgue measure μ.

According to the above considerations the following question seems to be of interest.

Problem 4.10. *Let X, Y be symmetric spaces such that (X, Y) is a Calderón couple. Let $0 \le f, g \in X + Y$ be both nonincreasing and such that*

$$K(t, g; X, Y) \le K(t, f; X, Y) \quad \text{for all } t > 0.$$

Does there exist a monotone operator T acting on the couple (X, Y) with $Tf = g$, or is $(\widetilde{X}, \widetilde{Y})$ a Calderón couple? As we have already seen, $(\widetilde{L^p}, L^\infty)$ is a Calderón couple but we do not know if there is a monotone operator as above. Notice that the relevant operators from the papers of Lorentz–Shimogaki, Cwikel, and Arazy–Cwikel are not monotone. Also the proof of Theorem 14 from [AC12] says nothing about this, because it is based on the lattice version of the Hahn–Banach extension theorem.

Appendix. The paper of Sedaev [Se71] is not easy to acquire and is in Russian; moreover, Dmitriev's explanation is quite brief. Therefore, just for the sake of convenience, we recall Sedaev's result and explain how it is applied in Theorem 4.6.

We introduce some special notions following Sedaev; the remaining terminology is standard, as in the books [BL76], [BS88] or [KPS82]. Let $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ be two couples of compatible Banach spaces. Further, let

$$\begin{aligned} &\Gamma \subset (Y_0 + Y_1)^*, \\ &G = \{T : \|T\|_{(X_0,X_1)\to(Y_0,Y_1)} \le 1\}, \\ &U_i = \{y \in Y_i : \|y\|_{Y_i} \le 1\}, \quad i = 1, 2. \end{aligned}$$

Then $\sigma(Y_0 + Y_1, \Gamma)$ means the weak topology on $Y_0 + Y_1$ restricted to Γ. Similarly, the *Γ-topology* on $L(\overline{X}, \overline{Y})$ is the weak operator topology restricted to Γ. Denote also, after Sedaev,

$$a = \inf_{x\in X_0\cap X_1} \frac{\|x\|_{X_1}}{\|x\|_{X_0}}, \qquad b = \sup_{x\in X_0\cap X_1} \frac{\|x\|_{X_1}}{\|x\|_{X_0}}.$$

THEOREM A.1 (Sedaev 1971). *G is Γ-compact in $L(\overline{X}, \overline{Y})$ if and only if*

(i) *$U_0 \cap cU_1$ is $\sigma(Y_0 + Y_1, \Gamma)$-closed in $Y_0 + Y_1$ for each $a \le c \le b$,*
(ii) *if $X_0 \cap X_1$ is not dense in X_0 [X_1] then U_0 [U_1] is $\sigma(Y_0+Y_1, \Gamma)$-closed in $Y_0 + Y_1$,*
(iii) *there is a couple (X, Y) such that*
   (a) *X is an interpolation space for $\overline{X}$, and E is dense in $X_0 + X_1$,*
   (b) *Y is an intermediate space for $\overline{Y}$, and the unit ball U of Y is $\sigma(Y_0 + Y_1, \Gamma)$-compact.*

The operators S_n from the proof of Theorem 4.6 clearly belong to G, where $(X_0, X_1) = (\Lambda_\varphi, L^\infty)$ and $(Y_0, Y_1) = (L^1, L^\infty)$. Moreover, we choose

$$\Gamma = \{f \in L^\infty : \mu(\mathrm{supp}(f)) < \infty\}.$$

Of course, $\Gamma \subset L^1 \cap L^\infty = (L^1 + L^\infty)'$. We only need to show that such couples and Γ satisfy the assumptions of the above theorem.

To prove (i) and (ii) we will show that both U_0 and U_1 are $\sigma(L^1 + L^\infty, \Gamma)$-closed in $L^1 + L^\infty$. Let $f \in L^1 + L^\infty$ be such that $f \notin U_0$. This means that

$$\int_{\mathbb{R}_+} |f| \, d\mu > 1 + 3\delta$$

for some $\delta > 0$. Then there is $A \subset \mathbb{R}_+$ with $\mu(A) < \infty$ such that

$$\int_A |f| \, d\mu > 1 + 2\delta.$$

Set $g = \mathrm{sign}(f)\chi_A \in \Gamma$. Then

$$\langle g, f\rangle = \int_{\mathbb{R}_+} gf \, d\mu = \int_A |f| \, d\mu > 1 + 2\delta$$

and

$$V = \{h \in L^1 + L^\infty : |\langle g, f - h\rangle| < \delta\}$$

is a $\sigma(L^1 + L^\infty, \Gamma)$-open neighbourhood of f. Moreover, since $g \in U_1$, it follows that $|\langle g, h\rangle| \le 1$ for each $h \in U_0$, and consequently $V \cap U_0 = \emptyset$, which means that U_0 is $\sigma(L^1 + L^\infty, \Gamma)$-closed in $L^1 + L^\infty$. Consider now U_1. As before, let $f \in L^1 + L^\infty$ be such that $f \notin U_1$. This means that there is $A \subset [0, \infty)$ with $\mu(A) < \infty$ such that $|f|\chi_A > (1 + \delta)\chi_A$. This time set $g = \frac{\operatorname{sign}(f)}{\mu(A)}\chi_A \in \Gamma$. Then

$$\langle g, f\rangle = \int_{\mathbb{R}_+} gf\, d\mu > 1 + \delta$$

and we can proceed as before, because $g \in U_0$.

To prove (iii) we set $X = Y = L^\infty$. Point (a) is satisfied because simple functions are dense in Λ_φ. It remains to show that U_1 is $\sigma(L^1 + L^\infty, \Gamma)$-compact. But U_1 is $\sigma(L^\infty, L^1)$-compact in L^∞ thanks to the Alaoglu theorem. It is then also compact in the weaker topology $\sigma(L^\infty, \Gamma)$ (since $\Gamma \subset L^1$) and so also in $L^1 + L^\infty$ with the $\sigma(L^1 + L^\infty, \Gamma)$ topology.

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