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Numerical radius inequalities for Hilbert space operators

by

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Abstract. It is shown that if A is a bounded linear operator on a complex Hilbert space, then

$$\frac{1}{4}\|A^*A + AA^*\| \le (w(A))^2 \le \frac{1}{2}\|A^*A + AA^*\|,$$

where $w(\cdot)$ and $\|\cdot\|$ are the numerical radius and the usual operator norm, respectively. These inequalities lead to a considerable improvement of the well known inequalities

$$\frac{1}{2}\|A\| \le w(A) \le \|A\|.$$

Numerical radius inequalities for products and commutators of operators are also obtained.

1. Introduction. Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, and that for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \le w(A) \le \|A\|. \tag{1}$$

Thus, the usual operator norm and the numerical radius norm are equivalent. Inequalities (1) are sharp. If $A^2 = 0$, then the first inequality in (1) becomes an equality. If A is normal, then the second inequality in (1) becomes an equality.

It has recently been shown in [10] that if $A \in B(H)$, then

$$w(A) \le \frac{1}{2}(\|A\| + \|A^2\|^{1/2}). \tag{2}$$

This inequality, which refines the second inequality in (1), has been employed in [10] to give an estimate for the numerical radius of the Frobenius companion matrix. This estimate yields new bounds for the zeros of polynomials.

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For a host of numerical radius inequalities, and for diverse applications of these inequalities, we refer to [4]–[6], [10], [11], and references therein.

In Section 2 of this paper, we establish a considerable improvement of inequalities (1). New numerical radius inequalities involving products and commutators of operators are presented in Section 3. These inequalities follow as special cases of a general numerical radius inequality.

2. An improvement of inequalities (1). Our desired improvement of inequalities (1) can be stated as follows.

THEOREM 1. *If* $A \in B(H)$, *then*

$$\frac{1}{4}\|A^*A + AA^*\| \le (w(A))^2 \le \frac{1}{2}\|A^*A + AA^*\|. \tag{3}$$

Proof. Let $A = B + iC$ be the Cartesian decomposition of A. Then B and C are self-adjoint, and $A^*A + AA^* = 2(B^2 + C^2)$. Let x be any vector in H. Then, by the convexity of the function $f(t) = t^2$, we have

$$\begin{aligned}|\langle Ax, x\rangle|^2 &= \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2 \ge \frac{1}{2}(|\langle Bx, x\rangle| + |\langle Cx, x\rangle|)^2\\ &\ge \frac{1}{2}|\langle (B \pm C)x, x\rangle|^2,\end{aligned}$$

and so

$$\begin{aligned}(w(A))^2 &= \sup\{|\langle Ax, x\rangle|^2 : x \in H,\ \|x\| = 1\}\\ &\ge \frac{1}{2}\sup\{|\langle (B \pm C)x, x\rangle|^2 : x \in H,\ \|x\| = 1\}\\ &= \frac{1}{2}\|B \pm C\|^2 = \frac{1}{2}\|(B \pm C)^2\|.\end{aligned}$$

Thus,

$$\begin{aligned}2(w(A))^2 &\ge \frac{1}{2}\|(B + C)^2\| + \frac{1}{2}\|(B - C)^2\|\\ &\ge \frac{1}{2}\|(B + C)^2 + (B - C)^2\| = \|B^2 + C^2\| = \frac{1}{2}\|A^*A + AA^*\|,\end{aligned}$$

and hence

$$(w(A))^2 \ge \frac{1}{4}\|A^*A + AA^*\|,$$

which proves the first inequality in (3).

To prove the second inequality in (3), note that for every unit vector $x \in H$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\langle Ax, x\rangle|^2 &= \langle Bx, x\rangle^2 + \langle Cx, x\rangle^2\\ &\le \|Bx\|^2 + \|Cx\|^2 = \langle B^2x, x\rangle + \langle C^2x, x\rangle = \langle (B^2 + C^2)x, x\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}(w(A))^2 &= \sup\{|\langle Ax, x\rangle|^2 : x \in H,\ \|x\| = 1\}\\ &\le \sup\{\langle (B^2 + C^2)x, x\rangle : x \in H,\ \|x\| = 1\}\\ &= \|B^2 + C^2\| = \frac{1}{2}\|A^*A + AA^*\|,\end{aligned}$$

which proves the second inequality in (3), and completes the proof of the theorem.

To see that inequalities (3) improve inequalities (1), consider the chain of inequalities

$$\frac{1}{4}\|A\|^2 \le \frac{1}{4}\|A^*A + AA^*\| \le (w(A))^2 \le \frac{1}{2}\|A^*A + AA^*\| \le \|A\|^2. \tag{4}$$

The first inequality in (4) is an immediate consequence of inequality (33) in [12], and the last follows by the triangle inequality and the fact that $\|A^*A\| = \|AA^*\| = \|A\|^2$. In order to appreciate this improvement, we need to discuss the equality conditions of the first and the last inequalities in (4).

PROPOSITION 1. *If $A \in B(H)$ is such that $A^2 = 0$, then*

$$\|A\|^2 = \|A^*A + AA^*\| = \|A^*A - AA^*\|.$$

Proof. The desired result follows from the chain of inequalities

$$\|A\|^2 - \|A^2\| \le \|A^*A - AA^*\| \le \|A\|^2 \le \|A^*A + AA^*\| \le \|A\|^2 + \|A^2\|, \tag{5}$$

which can be found in [9].

It should be mentioned here that the converse of Proposition 1 is not true if $\dim H > 2$. To see this, consider

$$A = \begin{bmatrix} 0 & 2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Then $\|A\|^2 = \|A^*A + AA^*\| = \|A^*A - AA^*\| = 4$, but $A^2 \ne 0$.

For the case $\dim H = 2$, elementary computations reveal that if $\|A\|^2 = \|A^*A + AA^*\|$, then $A^2 = 0$. However, when $\dim H > 2$, we have the following result.

PROPOSITION 2. *If $A \in B(H)$ is such that either $\|A\|^2 = \|A^*A + AA^*\|$ or $\|A\|^2 = \|A^*A - AA^*\|$, then A is not invertible.*

Proof. It has recently been shown in [12] that if A is invertible, then

$$\|A\|^2 + \|A^{-1}\|^{-2} \le \|A^*A + AA^*\| \tag{6}$$

and

$$\|A^*A - AA^*\| \le \|A\|^2 - \|A^{-1}\|^{-2}, \tag{7}$$

from which the desired result follows.

We have seen that when $\dim H > 2$, the sufficient condition for equality to hold in the first inequality in (4), which is given in Proposition 1, is different from the necessary condition, which is given in Proposition 2. In the following result, we give a necessary and sufficient condition for equality to hold in the last inequality in (4).

PROPOSITION 3. *For $A \in B(H)$, we have $\|A^*A + AA^*\| = 2\|A\|^2$ if and only if $\|A^2\| = \|A\|^2$.*

Proof. The desired result follows from the chain of inequalities

$$2\|A^2\| \leq \|A^*A + AA^*\| \leq \|A^2\| + \|A\|^2 \leq 2\|A\|^2, \tag{8}$$

which can be found in [9].

3. A general numerical radius inequality. In spite of the pleasant properties of the numerical radius, it is known not to be submultiplicative, even for commuting operators. In fact, if

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then

$$w(A) = \cos\frac{\pi}{5}, \qquad w(A^2) = w(A^3) = \frac{1}{2}.$$

Thus, if $B = A^2$, then

$$w(AB) = \frac{1}{2} > \frac{1}{2}\cos\frac{\pi}{5} = w(A)w(B)$$

(see [5, p. 37] or [6, p. 118]).

In view of this example, it is desirable to establish numerical radius inequalities for products of operators. In addition to the obvious fact that the numerical radius is submultiplicative for normal operators, the best known result in the affirmative direction is the power inequality, which asserts that if $A \in B(H)$, then

$$w(A^n) \leq (w(A))^n \tag{9}$$

for $1, 2, \ldots$. In the same mould, it is evident from inequalities (1) that if $A, B \in B(H)$, then

$$w(AB) \leq 4w(A)w(B). \tag{10}$$

Moreover, if $AB = BA$, then

$$w(AB) \leq 2w(A)w(B), \tag{11}$$

and if, in addition, $AB^* = B^*A$ (i.e., if A and B doubly commute), then

$$w(AB) \leq w(B)\|A\|. \tag{12}$$

Also, if A is an isometry such that $AB = BA$, then

$$w(AB) \le w(B). \tag{13}$$

These inequalities, among other related ones concerning the submultiplicativity of the numerical radius, can be found in [2], [4]–[7], and references therein.

The related question of the best constant c for which $w(AB) \le cw(A)\|B\|$ for commuting operators $A, B \in B(H)$ has been considered by several authors. It is known that $1.064 < c < 1.169$ (see [3], [15], and [16]). Akin to this problem, it has been shown in [4] that if $A, B \in B(H)$, then

$$w(AB + BA) \le 2\sqrt{2}\, w(A)\|B\|. \tag{14}$$

In this section we establish a general numerical radius inequality, from which numerical radius inequalities for products and commutators of operators follow as special cases. To achieve this, we need the following generalized mixed Schwarz inequality (see, e.g., [8]).

Lemma 1. *If $T \in B(H)$, then*

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle \tag{15}$$

for all $x, y \in H$ and for all α with $0 \le \alpha \le 1$. Here $|X| = (X^*X)^{1/2}$ is the absolute value of X.

Theorem 2. *If $A, B, C, D, S, T \in B(H)$, then*

$$\begin{aligned} &w(ATB + CSD) \\ &\quad \le \frac{1}{2}\|A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D\| \end{aligned} \tag{16}$$

for all α with $0 \le \alpha \le 1$.

Proof. By Lemma 1 and by the arithmetic-geometric mean inequality, for every $x \in H$ we have

$$\begin{aligned} &|\langle (ATB + CSD)x, x\rangle| \\ &\quad \le |\langle ATBx, x\rangle| + |\langle CSDx, x\rangle| = |\langle TBx, A^*x\rangle| + |\langle SDx, C^*x\rangle| \\ &\quad \le \langle |T|^{2\alpha}Bx, Bx\rangle^{1/2}\langle |T^*|^{2(1-\alpha)}A^*x, A^*x\rangle^{1/2} \\ &\qquad + \langle |S|^{2\alpha}Dx, Dx\rangle^{1/2}\langle |S^*|^{2(1-\alpha)}C^*x, C^*x\rangle^{1/2} \\ &\quad \le \frac{1}{2}(\langle |T|^{2\alpha}Bx, Bx\rangle + \langle |T^*|^{2(1-\alpha)}A^*x, A^*x\rangle \\ &\qquad + \langle |S|^{2\alpha}Dx, Dx\rangle + \langle |S^*|^{2(1-\alpha)}C^*x, C^*x\rangle) \\ &\quad = \frac{1}{2}\langle (A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + D^*|S|^{2\alpha}D + C|S^*|^{2(1-\alpha)}C^*)x, x\rangle. \end{aligned}$$

Thus,

$$\begin{aligned}w(ATB+CSD) &= \sup\{|\langle (ATB+CSD)x,x\rangle| : x\in H,\ \|x\|=1\}\\ &\le \frac{1}{2}\sup\{\langle (A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + D^*|S|^{2\alpha}D\\ &\qquad\qquad + C|S^*|^{2(1-\alpha)}C^*)x,x\rangle : x\in H,\ \|x\|=1\}\\ &= \frac{1}{2}\|A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2\alpha}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2\alpha}D\|,\end{aligned}$$

as claimed.

Inequality (16) yields several numerical radius inequalities as special cases. A sample of elementary inequalities is demonstrated in the following remarks. Other inequalities of this genre are left to the interested reader.

REMARK 1. Letting $T = I$ (the identity operator) and $S = 0$ in Theorem 2, we obtain the inequality

$$w(AB) \le \frac{1}{2}\|AA^* + B^*B\|. \tag{17}$$

In addition to this, we have the related inequality

$$w(AB) \le \frac{1}{2}\|A^*A + BB^*\|. \tag{18}$$

In fact, (18) follows from the second inequality in (1) and the arithmetic-geometric mean inequality for the usual operator norm, which says that if $A, B \in B(H)$, then

$$\|AB\| \le \frac{1}{2}\|A^*A + BB^*\| \tag{19}$$

(see, e.g., [1] or [14]).

When A or B is nonnormal, inequalities (17) and (18) are not equivalent. This can be seen from the example

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\qquad B = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

In view of (17) and (18), one might conjecture that if $A, B \in B(H)$, then

$$w(AB) \le \frac{1}{2}\|A^*A + B^*B\|, \tag{20a}$$

which, by the self-adjointness of the numerical radius, is equivalent to

$$w(AB) \le \frac{1}{2}\|AA^* + BB^*\|. \tag{20b}$$

The example

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\qquad B = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},$$

however, refutes this conjecture.

Remark 2. Letting $T = S = I$, $C = B$, and $D = \pm A$ in Theorem 2, we obtain the inequality

$$w(AB \pm BA) \leq \frac{1}{2}\|A^*A + AA^* + B^*B + BB^*\|, \tag{21}$$

which furnishes an estimate for the numerical radius of the commutator $AB - BA$.

Remark 3. It has been shown in [13] that if $A, U \in B(H)$ and U is unitary, then

$$r(AU \pm UA) \leq \|A\| + \|A^2\|^{1/2}, \tag{22}$$

where $r(\cdot)$ stands for the spectral radius. This inequality has been used in [13] to derive pinching inequalities for the spectral radius.

It can be easily inferred from Theorem 2, with $\alpha = 1/2$, that if $A, B \in B(H)$, then

$$w(AB \pm B^*A) \leq \frac{1}{2}\big\||A| + |A^*| + B^*(|A| + |A^*|)B\big\|. \tag{23}$$

Using norm inequalities for sums and products of positive operators, it has been shown in the proof of Theorem 1 in [10] that if $A \in B(H)$, then

$$\big\||A| + |A^*|\big\| \leq \|A\| + \|A^2\|^{1/2}. \tag{24}$$

Employing the triangle inequality, the unitary invariance of the usual operator norm, and inequality (24), it follows from (23) that if $A, U \in B(H)$ and U is unitary, then

$$w(AU \pm U^*A) \leq \|A\| + \|A^2\|^{1/2}, \tag{25}$$

which includes a generalization of inequality (2).

Remark 4. Finally, we remark that the numerical radius in (17) cannot be replaced by the usual operator norm. To see this, consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In the same spirit, the numerical radius in (21), (23), and (25) cannot be replaced by the usual operator norm, and the spectral radius in (22) cannot be replaced by the numerical radius. This can be seen from the example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

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