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Some properties of packing measure with doubling gauge

by

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Abstract. Let g be a doubling gauge. We consider the packing measure $\mathcal{P}^g$ and the packing premeasure $\mathcal{P}_0^g$ in a metric space X. We first show that if $\mathcal{P}_0^g(X)$ is finite, then as a function of X, $\mathcal{P}_0^g$ has a kind of "outer regularity". Then we prove that if X is complete separable, then $\lambda \sup \mathcal{P}_0^g(F) \le \mathcal{P}^g(B) \le \sup \mathcal{P}_0^g(F)$ for every Borel subset B of X, where the supremum is taken over all compact subsets of B having finite $\mathcal{P}_0^g$-premeasure, and λ is a positive number depending only on the doubling gauge g. As an application, we show that for every doubling gauge function, there is a compact metric space of finite positive packing measure.

1. Introduction. Let g: $[0,\infty) \to [0,\infty)$ be a *gauge*, i.e., a function which is non-decreasing for $t \ge 0$, right-continuous at $t = 0$, and $g(t) = 0$ if and only if $t = 0$. A gauge g is said to be *doubling* if there are numbers $c, \delta > 0$ such that $g(2t) \le cg(t)$ for all $t \in (0,\delta)$. For a doubling gauge g we introduce a non-decreasing function $g_*(x)$ as follows:

$$g_*(x) = \liminf_{t \downarrow 0} \frac{g(xt)}{g(t)}, \qquad x \in [0,\infty). \tag{1}$$

We write $g_*(1-0)$ for the left limit of $g_*(x)$ at $x = 1$.

Let X be a metric space. Let $E \subset X$ and $\delta > 0$. A *δ-packing* of E is defined to be a countable family $\{B(x_i, r_i)\}$ of disjoint closed balls with $2r_i \le \delta$ and centers $x_i \in E$. The *packing premeasure* of E with respect to the gauge g is defined by

$$\mathcal{P}_0^g(E) = \lim_{\delta \to 0} \mathcal{P}_\delta^g(E),$$

where $\mathcal{P}_\delta^g(E) := \sup \sum g(2r_i)$, the supremum being taken over all δ-packings

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of E. The *packing measure* of E with respect to the gauge g is defined by

$$\mathcal{P}^g(E) = \inf\Big\{\sum_{i=1}^{\infty}\mathcal{P}_0^g(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\Big\}.$$

The packing premeasure and measure with respect to the gauge t^s $(s \geq 0)$, which we denote by $\mathcal{P}_0^s$ and $\mathcal{P}^s$ respectively, are the ordinary s-dimensional packing premeasure and measure. From the definitions above, $\mathcal{P}_0^g$ is monotonic and finitely subadditive, and $\mathcal{P}^g$ is an outer measure of $\mathcal{P}_0^g$; for more details, we refer to [3], [8], [9].

Let $K \subset \mathbb{R}^n$ be a compact set and $0 \leq s \leq n$. Feng, Hua and Wen [4] proved that if $\mathcal{P}_0^s(K) < \infty$, then

$$\mathcal{P}_0^s(K) = \mathcal{P}^s(K). \tag{2}$$

The above conclusion may fail for doubling gauges. M. Csörnyei [1] constructed a compact set $K \subset \mathbb{R}^1$ and a doubling gauge g such that

$$\mathcal{P}^g(K) < \mathcal{P}_0^g(K) < \infty. \tag{3}$$

Motivated by this fact, we discuss some measure-theoretic properties of the packing measure $\mathcal{P}^g$ and the premeasure $\mathcal{P}_0^g$ with respect to a doubling gauge g in a metric space X. It will be shown that if $\mathcal{P}_0^g(X) < \infty$ then $\mathcal{P}_0^g$ is "outer regular-like", meaning that

$$g_*(1-0)\inf \mathcal{P}_0^g(U) \leq \mathcal{P}_0^g(F) \leq \inf \mathcal{P}_0^g(U)$$

for any compact set $F \subset X$, where the infimum is over all open sets containing F (Theorem 1). Furthermore we get a relationship between $\mathcal{P}^g$ and $\mathcal{P}_0^g$ when X is complete separable. Namely, for any Borel set $B \subset X$ we have

$$(g_*(1-0))^2 \sup \mathcal{P}_0^g(F) \leq \mathcal{P}^g(B) \leq \sup \mathcal{P}_0^g(F),$$

where the supremum is taken over all compact subsets contained in B with $\mathcal{P}_0^g(F) < \infty$ (Theorem 2). As a corollary, we show that for every doubling gauge function there is a compact metric space of finite positive packing measure (Theorem 3), which can be regarded as a dual to a result on the Hausdorff measure obtained by A. Dvoretzky [2].

2. The "outer regularity" of a packing premeasure. We start with some statements equivalent to the doubling condition.

LEMMA 1. *Let g be a gauge. Then the following statements are equivalent*:

(a) *g is doubling*;
(b) *$g_*(x) > 0$ for some $x \in (0,1)$*;
(c) *$g_*(x) > 0$ for all $x > 0$*;
(d) *$g_*(1-0) > 0$.*

Proof. (a)⇒(b). The doubling condition implies that there is a constant $c \in (0,\infty)$ such that $g(t) \le cg(t/2)$ for all $t > 0$ small enough, so $g_*(1/2) \ge 1/c > 0$.

(b)⇒(c). Let $a \in (0,1)$ with $g_*(a) > 0$. Then $g(at) \ge \frac{1}{2} g_*(a) g(t)$ for $t > 0$ small enough. For every $x > 0$ choose a positive integer m such that $x \ge a^m$. For all $t > 0$ small enough we get inductively

$$g(xt) \ge g(a^m t) \ge \frac{1}{2} g_*(a) g(a^{m-1}t) \ge \cdots \ge \left(\frac{1}{2} g_*(a)\right)^m g(t),$$

which yields $g_*(x) \ge \left(\frac{1}{2} g_*(a)\right)^m > 0$.

(c)⇒(d). This is trivial since g_* is non-decreasing.

(d)⇒(a). Since $g_*(1-0) > 0$, we obtain $g_*(x) > 0$ for some $x \in (0,1)$. By an argument analogous to (b)⇒(c), we get $g_*(1/2) > 0$, which implies that g is doubling. ∎

In the rest of the paper, we assume that g is a doubling gauge. The following theorem shows that if the packing premeasure is finite then it is "outer regular".

THEOREM 1. *Suppose X is a metric space with $\mathcal{P}_0^g(X) < \infty$. Then for any compact subset F of X,*

$$g_*(1-0) A_F \le \mathcal{P}_0^g(F) \le A_F, \tag{4}$$

where $A_F = \inf\{\mathcal{P}_0^g(U) : U$ open and $U \supseteq F\}$.

Proof. From the monotonicity of $\mathcal{P}_0^g$, the second inequality of (4) is evident, so we only need to prove the first. Let ϱ denote the metric of X. For $\varepsilon > 0$ denote by F_ε the open ε-neighborhood of F, i.e.

$$F_\varepsilon = \{x \in X : \varrho(x,y) < \varepsilon \text{ for some } y \in F\}. \tag{5}$$

Then $A_F = \inf_{\varepsilon>0} \mathcal{P}_0^g(F_\varepsilon)$ and $0 \le A_F < \infty$ since $\mathcal{P}_0^g(X)$ is finite. Let $\omega > 0$ be arbitrary and choose $\varepsilon > 0$ small enough such that

$$A_F \le \mathcal{P}_0^g(F_\varepsilon) \le A_F + \omega. \tag{6}$$

Then, by the definition of $\mathcal{P}_0^g$, we have

$$\mathcal{P}_0^g(F_\varepsilon) \le \mathcal{P}_\delta^g(F_\varepsilon) \le \mathcal{P}_0^g(F_\varepsilon) + \omega \tag{7}$$

for $\delta > 0$ small enough. Now let $\{B(x_i, r_i)\}_{i=1}^m$ be a δ-packing of F_ε such that

$$\mathcal{P}_\delta^g(F_\varepsilon) - \omega \le \sum_{i=1}^m g(2r_i) \le \mathcal{P}_\delta^g(F_\varepsilon). \tag{8}$$

By the compactness of F, we may choose $\{y_i\}_{i=1}^m \subset F$ such that

$$\varrho(x_i, y_i) = \varrho(x_i, F), \quad 1 \le i \le m.$$

Let

$$r_i^* = \max\{r_i - \varrho(x_i, y_i), 0\}, \qquad r_i^{**} = \min\{\varrho(x_i, y_i), r_i\}.$$

By the definitions above, we easily see that $r_i^* + r_i^{**} = r_i$. Let $I = \{i : B(x_i, r_i) \cap F \neq \emptyset,\ 1 \le i \le m\}$. It is obvious that $\{B(y_i, r_i^*)\}_{i\in I}$ is a δ-packing of F. Thus

$$P_\delta^g(F) \ge \sum_{i\in I} g(2r_i^*) = \sum_{i=1}^m g(2r_i^*). \tag{9}$$

Now let $J = \{i : x_i \notin F,\ 1 \le i \le m\}$ and $1/2 < t < 1$, and choose $0 < \varepsilon_1 \le 3\delta$ such that $F_{\varepsilon_1} \cap \bigcup_{i\in J} B(x_i, tr_i^{**}) = \emptyset$. We see that if $\{B(z_i, l_i)\}_{i=1}^\infty$ is a $3^{-1}\varepsilon_1$-packing of $F_{3^{-1}\varepsilon_1}$ then $\{B(z_i, l_i)\}_{i=1}^\infty \cup \{B(x_i, tr_i^{**})\}_{i\in J}$ is a δ-packing of F_ε. Thus, in view of (6) and (7), we have

$$\sum_{i=1}^\infty g(2l_i) + \sum_{i\in J} g(2tr_i^{**}) \le A_F + 2\omega,$$

so

$$\sum_{i\in J} g(2tr_i^{**}) \le 2\omega,$$

which together with the doubling property of g yields

$$\sum_{i=1}^m g(2r_i^{**}) = \sum_{i\in J} g(2r_i^{**}) \le c\sum_{i\in J} g(2tr_i^{**}) \le 2c\omega, \tag{10}$$

where $c > 0$ is a constant.

Now we are going to estimate the sum on the right hand side of (9). Let

$$x(\omega) = \omega + 2\sup\{x \ge 0 : g_*(x) < \omega^{1/2}\}.$$

From Lemma 1, we have $\lim_{\omega\to 0} x(\omega) = 0$ and $\lim_{\omega\to 0} \omega/g_*(x(\omega)) = 0$.

Let $I_\omega = \{i : r_i^{**} \ge r_i x(\omega)\}$ and $J_\omega = \{i : r_i^{**} < r_i x(\omega)\}$. Then, by (10), we have

$$\sum_{i\in I_\omega} g(2r_i) \le \sum_{i\in I_\omega} \frac{g(2r_i^{**})g(2r_i)}{g(2r_i x(\omega))} \le 2c\omega\left(\inf_{0<t\le\delta} \frac{g(tx(\omega))}{g(t)}\right)^{-1}, \tag{11}$$

which combined with (6), (7), (8) and (11) yields

$$\begin{aligned}\sum_{i\in J_\omega} g(2r_i) &= \sum_{i=1}^m g(2r_i) - \sum_{i\in I_\omega} g(2r_i) \\ &\ge A_F - \omega - 2c\omega\left(\inf_{0<t\le\delta} \frac{g(tx(\omega))}{g(t)}\right)^{-1}.\end{aligned} \tag{12}$$

From (9) and (12),

$$\mathcal{P}_\delta^g(F) \ge \sum_{i\in J_\omega} g(2r_i^*) \ge \sum_{i\in J_\omega} \frac{g(2r_i(1 - x(\omega)))g(2r_i)}{g(2r_i)}$$

$$\geq \inf_{0<t\leq\delta} \frac{g(t(1-x(\omega)))}{g(t)} \sum_{i\in J_\omega} g(2r_i)$$

$$\geq \inf_{0<t\leq\delta} \frac{g(t(1-x(\omega)))}{g(t)} \left(A_F - \omega - 2c\omega\left(\inf_{0<t\leq\delta} \frac{g(tx(\omega))}{g(t)}\right)^{-1}\right).$$

Thus by letting $\delta \to 0$ we get

$$\mathcal{P}_0^g(F) \geq g_*(1-x(\omega))\left(A_F - \omega - \frac{2c\omega}{g_*(x(\omega))}\right).$$

Letting $\omega \to 0$, finally we obtain $\mathcal{P}_0^g(F) \geq g_*(1-0)A_F$. ∎

3. The relationship between $\mathcal{P}^g$ and $\mathcal{P}_0^g$. In this section, we will investigate the relation between the packing measure and the premeasure with respect to a doubling gauge in a complete separable metric space.

LEMMA 2. *Let X be a metric space and let g be a gauge. Then*:

(a) *for any subset F of X, we have*

$$g_*(1-0)\mathcal{P}_0^g(\mathrm{cl}(F)) \leq \mathcal{P}_0^g(F) \leq \mathcal{P}_0^g(\mathrm{cl}(F)), \tag{13}$$

where $\mathrm{cl}(F)$ *denotes the closure of F*;

(b) *if g is left-continuous for $t > 0$, then $\mathcal{P}_0^g(F) = \mathcal{P}_0^g(\mathrm{cl}(F))$.*

Proof. (a) To prove (13), it suffices to prove the first inequality. Let $\varepsilon, \delta \in (0,1)$. For every δ-packing $\{B(x_i,r_i)\}$ of $\mathrm{cl}(F)$, choose $\{y_i\} \subset F$ such that $\{B(y_i,(1-\varepsilon)r_i)\}$ is a δ-packing of F. Then we have

$$\mathcal{P}_\delta^g(F) \geq \sum g(2(1-\varepsilon)r_i) \geq \inf_{0<t\leq\delta} \frac{g((1-\varepsilon)t)}{g(t)} \sum g(2r_i),$$

so

$$\mathcal{P}_\delta^g(F) \geq \inf_{0<t\leq\delta} \frac{g((1-\varepsilon)t)}{g(t)}\, \mathcal{P}_\delta^g(\mathrm{cl}(F)).$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, we immediately get the desired inequality.

(b) Now assume that g is left-continuous for $t > 0$. In this case, from a δ-packing $\{B(x_i,r_i)\}$ of $\mathrm{cl}(F)$ we may construct a δ-packing $\{B(y_i,r_i^*)\}$ of F such that for every i,

$$g(2r_i^*) \geq g(2r_i) - \varepsilon/2^i.$$

From this we get $\mathcal{P}_0^g(F) \geq \mathcal{P}_0^g(\mathrm{cl}(F))$, which yields the required equality immediately. ∎

LEMMA 3. *Let X be a metric space. Then for any compact subset K with $\mathcal{P}_0^g(K) < \infty$, we have*

$$(g_*(1-0))^2\mathcal{P}_0^g(K) \leq \mathcal{P}^g(K) \leq \mathcal{P}_0^g(K). \tag{14}$$

Proof. It suffices to prove the first inequality of (14). Let $\varepsilon > 0$. By the definition of $\mathcal{P}^g$, there exists a countable family $\{F_i\}_{i\geq 1}$ of sets such that $\bigcup_{i\geq 1} F_i = K$ and

$$\mathcal{P}^g(K) + \varepsilon \geq \sum_{i=1}^{\infty} \mathcal{P}_0^g(F_i).$$

From Theorem 1 we see that for every i there is an open set U_i such that $U_i \supset \mathrm{cl}(F_i)$ and

$$\mathcal{P}_0^g(\mathrm{cl}(F_i)) \geq g_*(1-0)(\mathcal{P}_0^g(U_i) - \varepsilon/2^i). \tag{15}$$

Since K is compact and $\{U_i\}$ is an open covering of K, we may choose a finite subcovering, say $K \subset \bigcup_{i=1}^N U_i$. From the finite subadditivity of $\mathcal{P}_0^g$ and the inequalities (13) and (15), we get

$$\begin{aligned}\mathcal{P}^g(K) + \varepsilon &\geq \sum_{i=1}^{\infty} \mathcal{P}_0^g(F_i) \geq g_*(1-0) \sum_{i=1}^{\infty} \mathcal{P}_0^g(\mathrm{cl}(F_i)) \\ &\geq (g_*(1-0))^2 \Big(\sum_{i=1}^{\infty} \mathcal{P}_0^g(U_i) - \varepsilon \Big) \\ &\geq (g_*(1-0))^2 \Big(\sum_{i=1}^{N} \mathcal{P}_0^g(U_i) - \varepsilon \Big) \\ &\geq (g_*(1-0))^2 (\mathcal{P}_0^g(K) - \varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$, we finally obtain $\mathcal{P}^g(K) \geq (g_*(1-0))^2 \mathcal{P}_0^g(K)$. ∎

Lemma 3 implies immediately

COROLLARY 1. *For any compact set $K \subset X$ with $\mathcal{P}_0^g(K) < \infty$, we have*:

(a) $0 < \mathcal{P}^g(K) < \infty \Leftrightarrow 0 < \mathcal{P}_0^g(K) < \infty$;
(b) $g_*(1-0) = 1 \Rightarrow \mathcal{P}^g(K) = \mathcal{P}_0^g(K)$.

REMARK 1. It is known that a countable compact subset E of $\mathbb{R}^d$ may have strictly positive upper box-counting dimension. However, from the result above we see that either $\mathcal{P}_0^g(E) = 0$ or $\mathcal{P}_0^g(E) = \infty$ for every doubling gauge g.

LEMMA 4. *Let X be a complete separable metric space with $\mathcal{P}^g(X) < \infty$ and let B be a Borel subset of X. Then for any $\varepsilon > 0$, there is a compact set $F \subset B$ with $\mathcal{P}_0^g(F) < \infty$ such that*

$$\mathcal{P}^g(F) \geq \mathcal{P}^g(B) - \varepsilon. \tag{16}$$

Proof. Since $\mathcal{P}^g$ is a finite Borel measure on a complete separable metric space X, $\mathcal{P}^g$ is inner regular, thus it suffices to prove the statement under the assumption that B is compact. By the definition of $\mathcal{P}^g$, the condition

$\mathcal{P}^g(X) < \infty$ implies that there is a family $\{F_i\}$ of subsets with $B = \bigcup_{i=1}^{\infty} F_i$ such that

$$\sum_{i=1}^{\infty} \mathcal{P}_0^g(F_i) < \infty. \tag{17}$$

Since g is doubling we have $g_*(1-0) > 0$. By Lemma 2, we may assume that all F_i are compact. Let $\varepsilon > 0$ and choose a positive integer m such that

$$\sum_{i=m+1}^{\infty} \mathcal{P}_0^g(F_i) < \varepsilon. \tag{18}$$

Take $F = \bigcup_{i=1}^{m} F_i$. In view of (17) and (18), we see that F is a compact subset of B such that

$$\mathcal{P}_0^g(F) \le \sum_{i=1}^{m} \mathcal{P}_0^g(F_i) < \infty$$

and

$$\mathcal{P}^g(B) - \mathcal{P}^g(F) \le \mathcal{P}^g\Big(\bigcup_{i=m+1}^{\infty} F_i\Big) \le \sum_{i=m+1}^{\infty} \mathcal{P}^g(F_i) \le \sum_{i=m+1}^{\infty} \mathcal{P}_0^g(F_i) < \varepsilon. \ ■$$

THEOREM 2. *Let X be a complete separable metric space. Then for any Borel set $B \subset X$ we have*

$$(g_*(1-0))^2 \sup \mathcal{P}_0^g(F) \le \mathcal{P}^g(B) \le \sup \mathcal{P}_0^g(F), \tag{19}$$

where the supremum is taken over all compact subsets of B with $\mathcal{P}_0^g(F) < \infty$.

Proof. The first inequality in (19) follows directly from Lemma 3, and the second can be obtained immediately from Lemma 4 if $\mathcal{P}^g(X) < \infty$. To complete the proof, it suffices to prove the second inequality in the case $\mathcal{P}^g(X) = \infty$. Without loss of generality, assume $\mathcal{P}^g(B) > 0$. Let $\beta \in (0, \mathcal{P}^g(B))$. From the existence theorem of H. Joyce and D. Preiss [5], there is a compact subset $E \subset B$ such that $\beta \le \mathcal{P}^g(E) < \infty$. Applying Lemma 4 to the set E, we get $\sup \mathcal{P}_0^g(F) \ge \beta$. Since $\beta \in (0, \mathcal{P}^g(B))$ can be picked arbitrarily, we obtain $\sup \mathcal{P}_0^g(F) \ge \mathcal{P}^g(B)$ as required. ■

Theorem 2 immediately gives the following corollary.

COROLLARY 2. *Let B be a Borel subset of a complete separable space. Then*:

(a) $\mathcal{P}^g(B) > 0$ *if and only if B contains a compact subset F such that* $0 < \mathcal{P}_0^g(F) < \infty$;
(b) $\mathcal{P}^g(B) < \infty$ *if and only if there is $\lambda \in [0, \infty)$ such that for any compact subset $F \subset B$, either $\mathcal{P}_0^g(F) \le \lambda$ or $\mathcal{P}_0^g(F) = \infty$*;
(c) $g_*(1-0) = 1 \Rightarrow \mathcal{P}^g(B) = \sup \mathcal{P}_0^g(F)$.

Using the above results, we are going to prove that for every doubling gauge, there is a compact metric space which has finite positive packing measure with respect to the gauge. This result is analogous to the existence theorem proved by A. Dvoretzky for Hausdorff measures [2].

LEMMA 5. *Suppose that X is a separable metric space. Then for any subset K of X,*

$$\mathcal{H}^g(K) \le \mathcal{P}^g(K),$$

where $\mathcal{H}^g(K)$ denotes the Hausdorff measure of K with respect to the gauge g.

Proof. The proof is completely analogous to the proof of the case $g(t) = t^s$ and $X = \mathbb{R}^n$ which can be found in P. Mattila [6]. ∎

THEOREM 3. *Let g be any doubling gauge. Then there is a compact metric space X such that*

$$0 < \mathcal{P}^g(X) < \infty. \tag{20}$$

Proof. There is a compact metric space Ω such that $H^g(\Omega) > 0$ (see Theorem 36 in [7]), so $\mathcal{P}^g(\Omega) > 0$ by Lemma 5. By Corollary 2(a), there is a compact subset $X \subset \Omega$ such that $0 < \mathcal{P}_0^g(X) < \infty$. Then by Corollary 1(a), we finally get $0 < \mathcal{P}^g(X) < \infty$. ∎

4. An example. Theorem 2 states that for any Borel set G in a complete separable metric space,

$$g_*(1-0) = 1 \ \Rightarrow\ \mathcal{P}^g(G) = \sup \mathcal{P}_0^g(F),$$

where the supremum is taken over all compact subsets contained in G with $\mathcal{P}_0^g(F) < \infty$. We will show by giving a counterexample that the implication cannot be inverted, even if both $\mathcal{P}^g(G)$ and $\sup \mathcal{P}_0^g(F)$ are finite positive.

Let $G = [0,1]$ and $g : [0,\infty) \to [0,\infty)$ be defined by

$$g(t) = \begin{cases} 2^{-n} & \text{if } 2^{-n} \le t \le (1-2^{-n})2^{-n+1},\ n \in \mathbb{N}, \\ 2^{n-1}t + 2^{-n+1} - 1 & \text{if } (1-2^{-n})2^{-n+1} \le t \le 2^{-n+1},\ n \in \mathbb{N}. \end{cases}$$

It is easy to verify that

$$\frac{1}{2} = \liminf_{t\to 0} \frac{g(t)}{t} \le \limsup_{t\to 0} \frac{g(t)}{t} = 1, \tag{21}$$

and thus g is a doubling gauge. We are going to prove that

$$\mathcal{P}^g(G) = \mathcal{P}_0^g(G) = 1, \quad \text{but} \quad g_*(1-0) = 1/2. \tag{22}$$

Let $x_k = 1 - 2^{-k}$ and $t_n = 2^{-n}$, $k, n \in \mathbb{N}$. Fix $k \in \mathbb{N}$. By the construction of g we have

$$\begin{aligned} \liminf_{t\to 0} \frac{g(tx_k)}{g(t)} &\le \liminf_{n\to\infty} \frac{g(t_n x_k)}{g(t_n)} \le \liminf_{n\to\infty} \frac{g((1-2^{-n-1})2^{-n})}{g(2^{-n})} \\ &= \lim_{n\to\infty} \frac{2^{-n-1}}{2^{-n}} = \frac{1}{2}, \end{aligned}$$

which yields $g_*(x_k) \le 1/2$. Letting $k \to \infty$, we get $g_*(1-0) \le 1/2$ since g_* is non-decreasing. On the other hand, by the inequality (21), we have

$$\liminf_{t\to 0} \frac{g(tx)}{g(t)} \ge \frac{x}{2}$$

for all $x > 0$, and thus $g_*(1-0) \ge 1/2$. We have thus proved the last equality of (22).

Notice that (21) implies that $\frac{1}{2}\mathcal{P}_0^1(F) \le \mathcal{P}_0^g(F) \le \mathcal{P}_0^1(F)$ for any Borel set F on the real line, thus

$$\tfrac{1}{2}\mathcal{P}^1 \le \mathcal{P}^g \le \mathcal{P}^1$$

for any Borel sets. Note that $\mathcal{P}^1$ is equal to the 1-dimensional Lebesgue measure and $\mathcal{P}^g$ is translation invariant and locally finite, so there is a number $c > 0$ such that $\mathcal{P}^g = c\mathcal{P}^1$. In addition, analogously to the lower density theorem for the s-dimensional packing measure (see Theorem 6.10 of [6]), we have

$$\liminf_{r\to 0} \frac{\mathcal{P}^g(G \cap B(x,r))}{g(2r)} = 1$$

for $\mathcal{P}^g$-almost all $x \in G$. Invoking the Lebesgue density theorem we get

$$c = \limsup_{r\to 0} \frac{g(t)}{t} = 1,$$

and so $\mathcal{P}^g = \mathcal{P}^1$. Since $\mathcal{P}_0^1(G) = \mathcal{P}^1(G) = 1$ we then get

$$1 = \mathcal{P}^1(G) = \mathcal{P}^g(G) \le \mathcal{P}_0^g(G) \le \mathcal{P}_0^1(G) = 1,$$

which yields the first two equalities in (22).

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