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Some properties of N-supercyclic operators

by

P. S. Bourdon (Lexington, VA), N. S. Feldman (Lexington, VA) and
J. H. Shapiro (East Lansing, MI)

Abstract. Let T be a continuous linear operator on a Hausdorff topological vector space $\mathcal{X}$ over the field $\mathbb{C}$. We show that if T is N-supercyclic, i.e., if $\mathcal{X}$ has an N-dimensional subspace whose orbit under T is dense in $\mathcal{X}$, then T^* has at most N eigenvalues (counting geometric multiplicity). We then show that N-supercyclicity cannot occur nontrivially in the finite-dimensional setting: the orbit of an N-dimensional subspace cannot be dense in an $(N+1)$-dimensional space. Finally, we show that a subnormal operator on an infinite-dimensional Hilbert space can never be N-supercyclic.

1. Introduction. Let E be a subset of a complex, Hausdorff topological vector space $\mathcal{X}$ and let $T : \mathcal{X} \to \mathcal{X}$ be a continuous linear operator. The orbit of E under T, denoted $\operatorname{orb}_T(E)$, is the subset of $\mathcal{X}$ given by

$$\operatorname{orb}_T(E) = \bigcup_{k=0}^{\infty} T^k(E).$$

If $E = \{x\}$ is a singleton and $\operatorname{orb}_T(E)$ is dense in $\mathcal{X}$, then T is said to be *hypercyclic* and x is a *hypercyclic vector* for T. If $E = \{x\}$ is a singleton and the linear span of $\operatorname{orb}_T(E)$ is dense in $\mathcal{X}$, then T is *cyclic* with *cyclic vector* x. Finally, if E is a subspace of dimension N and $\operatorname{orb}_T(E)$ is dense in $\mathcal{X}$, then T is said to be N-*supercyclic* and E is a called a *supercyclic subspace* for T.

The study of density of orbits of singletons, i.e., the study of what is now called "hypercyclicity", dates back at least to a 1929 paper by G. D. Birkhoff [4]. The study of cyclicity dates back even further. In 1974 Hilden and Wallen [12] introduced the notion of "supercyclicity", defining $T : \mathcal{X} \to \mathcal{X}$ to be *supercyclic* provided there is a vector $x \in \mathcal{X}$ such that $\{\zeta T^k x : \zeta \in \mathbb{C},\ k \geq 0\}$ is dense in $\mathcal{X}$. More recently, Feldman [9] observed that T is supercyclic provided there is a one-dimensional subspace E of $\mathcal{X}$ with dense orbit and

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introduced the concept of N-supercyclicity, with, of course, 1-supercyclicity being equivalent to supercyclicity.

For a survey of results related to hypercyclicity, the reader may see [10]. We now set the stage for our study of some properties of N-supercyclic operators by discussing pertinent results concerning these operators. In [12], Hilden and Wallen showed that the backward shift on ℓ^2 is a supercyclic operator and proved that no normal operator on a Hilbert space of dimension > 1 can be supercyclic. They also established that no linear operator on $\mathbb{C}^n$ can be supercyclic ($1 < n < \infty$). Herrero [11] proved that the adjoint of a supercyclic operator on Hilbert space may have at most one simple eigenvalue, a result that was generalized to the Banach-space setting by Ansari and Bourdon [2]. Working in the infinite-dimensional setting, Feldman [9] provided examples of operators that are N-supercyclic but not $(N-1)$-supercyclic and proved several necessary conditions and several sufficient conditions for an operator on Hilbert space to be N-supercyclic. In particular, he showed that the class of N-supercyclic operators has interesting spectral structure, establishing the following:

N-Circles Theorem. *Suppose that T is N-supercyclic. Then there are N circles $\{\Gamma_1, \ldots, \Gamma_N\}$ centered at the origin such that for every invariant subspace $\mathcal{M}$ of T^*, the spectrum of $T^*|_{\mathcal{M}}$ must intersect $\bigcup_{j=1}^N \Gamma_j$. In particular, every component of the spectrum of T must intersect $\bigcup_{j=1}^N \Gamma_j$.*

In [9], Feldman also generalized one of Hilden and Wallen's results by establishing that no normal operator on an infinite-dimensional Hilbert space may be N-supercyclic.

Observe that if an operator T is hypercyclic or supercyclic, then it is clearly cyclic. However, an N-supercyclic operator need not be cyclic—trivial examples are furnished by the observation that if $\dim(\mathcal{X}) = N$, then every linear operator on $\mathcal{X}$ is N-supercyclic (here, $\mathcal{X}$ itself is a supercyclic subspace). We now describe a more interesting example featuring an operator that is nontrivially N-supercyclic. Take a hypercyclic operator T, say twice the backward shift on ℓ^2 (see [13] for the original proof that this operator is hypercyclic), then form $T \oplus I : \ell^2 \oplus \mathbb{C} \to \ell^2 \oplus \mathbb{C}$. It is not difficult to prove that $T \oplus I$ is supercyclic with supercyclic vector $h \oplus 1$, where h is any hypercyclic vector for T (see [11, Lemma 3.2]). Note that 1 is a simple eigenvalue for $T \oplus I$ with corresponding eigenvector $0 \oplus 1$. (Thus, the adjoint of a supercyclic operator on an infinite-dimensional space may have a simple eigenvalue.) Now, consider $T \oplus I \oplus I : \ell^2 \oplus \mathbb{C} \oplus \mathbb{C} \to \ell^2 \oplus \mathbb{C} \oplus \mathbb{C}$. We claim that $S := T \oplus I \oplus I$ is 2-supercyclic with supercyclic subspace spanned by $\{h \oplus 1 \oplus 0, 0 \oplus 0 \oplus 1\}$, where, as above, h is hypercyclic for T. Because S^* has a multiple eigenvalue (namely 1), S cannot be cyclic (or supercyclic). Thus if our claim that S is 2-supercyclic is valid, we have the desired example of an N-supercyclic operator that is not cyclic. We now prove the claim. Let $v \oplus \alpha \oplus \beta$ be an arbitrary vector in $\ell^2 \oplus \mathbb{C} \oplus \mathbb{C}$ and let $\varepsilon > 0$. Because $h \oplus 1$ is a supercyclic vector for $T \oplus I$, there is a scalar ζ and a positive integer k such that $\|(T \oplus I \oplus I)^k(\zeta(h \oplus 1 \oplus 0)) - v \oplus \alpha \oplus 0\| < \varepsilon$. Hence

$$\|(T \oplus I \oplus I)^k(\zeta(h \oplus 1 \oplus 0) + \beta(0 \oplus 0 \oplus 1)) - v \oplus \alpha \oplus \beta\| < \varepsilon$$

and the claim follows.

It is easy to modify the construction of the preceding paragraph to produce an N-supercyclic operator S (on the ℓ^2 direct sum of N copies of $\mathbb{C}$) such that S^* has any desired collection of N nonzero eigenvalues (counting multiplicity). On the other hand, note that the N-Circles Theorem shows that S^* cannot have $N+1$ eigenvalues having different moduli. In the next section, we strengthen this result by showing that the adjoint of an N-supercyclic operator can never have more than N eigenvalues counting multiplicity. This generalizes Herrero's "simple eigenvalue" result for 1-supercyclic operators. In Section 3, we prove that in finite dimensions, N-supercyclicity cannot occur nontrivially, generalizing Hilden and Wallen's result for 1-supercyclicity. Specifically, we show that a linear mapping $T : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ cannot be N-supercyclic (which obviously implies T cannot be J-supercyclic for any $J \le N-1$). The methods employed in Sections 2 and 3 are not simply routine generalizations of methods that yield the corresponding results for 1-supercyclicity. For example, when considering 1-supercyclicity in the finite-dimensional setting, one may focus on the action of a single Jordan block. We are unable to take advantage of this reduction when analyzing N-supercyclicity for $N > 1$.

In Section 4 of the paper, we prove that no subnormal operator on infinite-dimensional Hilbert space can be N-supercyclic, answering a question raised in [9]. We present two proofs, the first of which takes advantage of our work in the finite-dimensional setting. The second proof relies on our exploiting and enhancing an intertwining relationship between subnormals and normals described in [8].

In the final section of the paper, we pose some natural questions for further investigation of the properties of N-supercyclic operators.

We conclude the Introduction with a simple lemma that will often be used in what follows.

LEMMA 1.1. *Suppose that $T : \mathcal{X} \to \mathcal{X}$ is nontrivially N-supercyclic (so that $N < \dim(\mathcal{X})$) with supercyclic subspace $\mathcal{S}$ of dimension N. Then given any vector $x \in \mathcal{X}$, there is a sequence (s_m) of vectors in $\mathcal{S}$ and a subsequence (n_m) of the sequence of natural numbers such that*

$$\lim_{m\to\infty} T^{n_m} s_m = x.$$

Proof. Observe that for each nonnegative integer j, $\bigcup_{k=0}^{j} T^k(\mathcal{S})$ is closed and nowhere dense in $\mathcal{X}$ (because each set this finite union comprises is a subspace of $\mathcal{X}$ having dimension at most N and $N < \dim(\mathcal{X})$). Thus, because $\mathcal{S}$ is a supercyclic subspace for T, it must be the case that for each $j \geq 0$, $\bigcup_{k=j}^{\infty} T^k(\mathcal{S})$ is dense in $\mathcal{X}$. The lemma follows. ∎

Note that Lemma 1.1 tells us that nontrivially N-supercyclic operators always have dense range.

2. Point spectrum of the adjoint. As in the Introduction, $\mathcal{X}$ denotes a Hausdorff topological vector space, and T a continuous linear operator on $\mathcal{X}$. We use $\mathcal{X}^*$ to denote the *dual space* of $\mathcal{X}$, i.e., the vector space of continuous linear functionals on $\mathcal{X}$. Since $\mathcal{X}$ is not assumed locally convex, it is possible that $\mathcal{X}^*$ may not separate points; it may even equal $\{0\}$, but this will have no bearing on the work to follow. Finally, define the adjoint operator $T^* : \mathcal{X}^* \to \mathcal{X}^*$ in the usual way:

$$(T^*\Lambda)(x) = \Lambda(Tx) \qquad (x \in \mathcal{X},\, \Lambda \in \mathcal{X}^*).$$

We will be concerned with the consequences of T^* having eigenvalues. Here is a simple one:

LEMMA 2.0. *If $\varphi \in \mathcal{X}^*$ is an eigenvector of T^*, then* $\ker \varphi$ *is a proper, closed, T-invariant subspace.*

Proof. We are assuming that φ is a nontrivial continuous linear functional on $\mathcal{X}$ with $T^*\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{C}$. Clearly $\ker \varphi$ is a proper, closed subspace of $\mathcal{X}$. As for T-invariance, suppose $x \in \ker \varphi$, i.e., $\varphi(x) = 0$. Then

$$\varphi(Tx) := (T^*\varphi)(x) = \lambda\varphi(x) = \lambda \cdot 0 = 0,$$

so also $Tx \in \ker \varphi$. ∎

In the preceding section, we observed that the adjoint of an N-supercyclic operator may have N eigenvalues, counting multiplicity. The following theorem shows that adjoints of N-supercyclic operators never have more than N eigenvalues.

THEOREM 2.1. *Suppose that $T : \mathcal{X} \to \mathcal{X}$ is a continuous linear operator and N is a positive integer. If T^* has $N+1$ linearly independent eigenvectors, then T is not N-supercyclic.*

A special case of this theorem is:

COROLLARY 2.2. *No diagonalizable linear operator on $\mathbb{C}^{N+1}$ is N-supercyclic.*

In fact the proof of Theorem 2.1 reduces to this apparently innocent corollary!

Corollary 2.2 $\Rightarrow$ *Theorem* 2.1.

Proof. Suppose we have proved Corollary 2.2, and T^* has $N+1$ linearly independent eigenvectors $\varphi_1, \ldots, \varphi_{N+1} \in \mathcal{X}^*$. By Lemma 2.0, $\mathcal{K} := \bigcap_{j=1}^{N+1} \ker \varphi_j$ is a closed T-invariant subspace of $\mathcal{X}$, hence the quotient $\mathcal{X}/\mathcal{K}$ is a Hausdorff topological vector space on which the quotient operator $T/\mathcal{K}$, defined by

$$(T/\mathcal{K})(x+\mathcal{K}) := Tx + \mathcal{K} \quad (x \in \mathcal{X}),$$

is a "well defined" continuous linear operator.

By linear independence of the set of eigenvectors $\{\varphi_j\}_{j=1}^{N+1}$, the subspace $\mathcal{K}$ has codimension $N+1$, hence $\mathcal{X}/\mathcal{K}$ has *dimension* $N+1$. It is easy to check that the set of quotient functionals $\{\varphi_j/\mathcal{K}\}_{j=1}^{N+1}$ is a linearly independent subset of $(\mathcal{X}/\mathcal{K})^*$, and that each is an eigenvector of $(T/\mathcal{K})^*$ (with eigenvalue λ_j for $\varphi_j/\mathcal{K}$). Thus $(T/\mathcal{K})^*$ can be regarded as a diagonal operator on $\mathbb{C}^{N+1}$, hence the same is true of $T/\mathcal{K}$. Since $(T/\mathcal{K})^*$ has a set of $N+1$ linearly independent eigenvectors, $T/\mathcal{K}$ cannot be N-supercyclic by Corollary 2.2.

Finally, N-supercyclicity is preserved by taking quotients—in fact the "degree of supercyclicity" may even be improved. More precisely, it is easy to check that if $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ with dense T-orbit, and $\mathcal{K}$ is any T-invariant closed subspace of $\mathcal{X}$, then the image of $\mathcal{M}$ under the quotient map (a closed subspace of $\mathcal{X}/\mathcal{K}$ of dimension possibly smaller than that of $\mathcal{M}$) has dense $T/\mathcal{K}$-orbit in $\mathcal{X}/\mathcal{K}$.

Thus the "non-N-supercyclicity" of $T/\mathcal{K}$ guarantees the same for T. ∎

Observe that the argument of the next-to-last paragraph of the preceding proof establishes the following.

PROPOSITION 2.3. *Suppose that $T : \mathcal{X} \to \mathcal{X}$ is continuous and $\mathcal{K}$ is a closed subspace of $\mathcal{X}$ invariant under T. If T is N-supercyclic with supercyclic subspace $\mathcal{M}$, then the quotient map $T/\mathcal{K} : \mathcal{X}/\mathcal{K} \to \mathcal{X}/\mathcal{K}$ has supercyclic subspace $\mathcal{M}/\mathcal{K}$ and thus is J-supercyclic, where $J = \dim(\mathcal{M}/\mathcal{K})$.*

We now turn to the proof of Corollary 2.2, showing no diagonalizable operator on $\mathbb{C}^{N+1}$ is N-supercyclic. In the next section, we prove that *no linear operator on $\mathbb{C}^{N+1}$ can be N-supercyclic.* There Corollary 2.2 plays the role of a lemma, allowing us to focus on Jordan-form matrices that are not diagonal.

Proof of Corollary 2.2. Suppose that T is a diagonal operator on $\mathbb{C}^{N+1}$ having eigenvalues $\lambda_1, \ldots, \lambda_{N+1}$ (counting multiplicity). We write

$$T = \operatorname{diag}[\lambda_1, \ldots, \lambda_{N+1}], \tag{2.1}$$

and view T both as an operator on $\mathbb{C}^{N+1}$ and as an $(N+1)\times(N+1)$ matrix. We aim to show that T is not N-supercyclic. This has been done for the case $N=1$ by Hilden and Wallen [12]. The proof below is an induction, using this special case as its starting point. However, curiously, the proof for $N = 1$ is built into the induction-step argument, as we will point out when we come to it. So let us assume, for purposes of induction, that $N > 1$, and that we have proven that no diagonalizable operator on $\mathbb{C}^N$ is $(N-1)$-supercyclic. We suppose, in order to obtain a contradiction, that the diagonal operator T given by (2.1) is N-supercyclic. If one of the eigenvalues λ_j is zero, the range of T lies entirely in the subspace of vectors whose jth coordinate is zero, so clearly T cannot be N-supercyclic. Thus we may assume for the remainder of the argument that *no* λ_j *is zero*. We may also assume, without loss of generality, that

$$|\lambda_1| = \min\{|\lambda_j| : 1 \le j \le N+1\}.$$

Let us view the elements of $\mathbb{C}^{N+1}$ as column vectors, using the following notation for $x \in \mathbb{C}^{N+1}$:

$$x = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad (\xi_1 \in \mathbb{C},\ \xi_2 \in \mathbb{C}^N), \tag{2.2}$$

which effects a canonical decomposition of $\mathbb{C}^{N+1}$ into the orthogonal direct sum of $\mathbb{C}$ and $\mathbb{C}^N$.

Let $\mathcal{M}$ be an N-dimensional subspace of $\mathbb{C}^{N+1}$ that is supercyclic for T. Fix, for the rest of this proof, a basis $\{x_1, \ldots, x_N\}$ for $\mathcal{M}$, where, in accordance with the convention of (2.2),

$$x_j = \begin{bmatrix} \xi_{j,1} \\ \xi_{j,2} \end{bmatrix} \qquad \text{with } \xi_{j,1} \in \mathbb{C} \text{ and } \xi_{j,2} \in \mathbb{C}^N.$$

So consider an arbitrary vector $y = \left[\begin{smallmatrix} \eta_1 \\ \eta_2 \end{smallmatrix}\right]$ with nonzero components $\eta_1 \in \mathbb{C}$ and $\eta_2 \in \mathbb{C}^N$.

Because y lies in the closure of $\mathrm{orb}_T(\mathcal{M})$, there exists a subsequence (n_k) of the sequence of positive integers and N scalar sequences $(a_{1,k}), \ldots, (a_{N,k})$ such that

$$T^{n_k}\Big(\sum_{j=1}^{N} a_{j,k} x_j\Big) \to y \qquad (k \to \infty). \tag{2.3}$$

We will write this as two matrix equations, employing the following cast of characters:

$$\begin{aligned}
\Lambda &:= \operatorname{diag}[\lambda_2, \ldots, \lambda_{N+1}], && \text{an } N \times N \text{ matrix},\\
X_1 &:= [\xi_{1,1}, \ldots, \xi_{N,1}], && \text{a } 1 \times N \text{ (row) matrix},\\
X_2 &:= [\xi_{1,2}, \ldots, \xi_{N,2}], && \text{an } N \times N \text{ matrix, and}\\
A_k &:= [a_{1,k}, \ldots, a_{N,k}]^t, && \text{an } N \times 1 \text{ (column) matrix},
\end{aligned}$$

where the superscript "t" stands for "matrix transpose". With this notation we can rewrite (2.3) as

$$\lambda_1^{n_k} X_1 A_k = \eta_1 + \varepsilon_k \quad (\varepsilon_k \to 0 \text{ as } k \to \infty), \tag{2.4}$$

an equation with scalars on both sides, and

$$\Lambda^{n_k} X_2 A_k = \eta_2 + \delta_k \quad (\delta_k \to 0 \text{ as } k \to \infty), \tag{2.5}$$

an equation with N-dimensional column vectors on each side.

We claim that the matrix X_2 is nonsingular. For the case $N = 1$ this is trivial: for then X_2 is just a 1×1 matrix, i.e., a scalar. If it is zero then we are just saying that the subspace $\mathcal{M}$ is spanned by the single vector x_1, which has second coordinate zero. Since T is diagonal, everything in the orbit of $\mathcal{M}$ therefore has second coordinate zero, hence that orbit has no chance of being dense. In the general case we need the induction hypothesis. Let $\mathcal{K}$ be the N-dimensional subspace of $\mathbb{C}^{N+1}$ consisting of vectors in $\mathbb{C}^{N+1}$ whose first coordinate is zero. Note that $\mathcal{K}$ is a reducing subspace for our diagonal operator T. Thus $T|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ is supercyclic with supercyclic subspace $P_{\mathcal{K}}(\mathcal{M})$, where $P_{\mathcal{K}}$ represents the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{K}$. Observe that $P_{\mathcal{K}}(\mathcal{M})$ is spanned by the columns of X_2. If X_2 were singular so that its columns form a dependent set, then $T|_{\mathcal{K}}$ would have a supercyclic subspace, $P_{\mathcal{K}}(\mathcal{M})$, of dimension less than N. This contradicts our induction hypothesis since $T|_{\mathcal{K}}$ is unitarily equivalent to a diagonal operator on $\mathbb{C}^N$.

So we assume from now on that X_2 is nonsingular, noting that in the case $N = 1$ this assumption did not need the induction hypothesis, and that the proof to follow works for any $N \geq 1$. Since Λ is also nonsingular (recall that we assumed that none of the diagonal elements of T, hence of Λ, were zero) we can solve equation (2.5) for A_k:

$$A_k = X_2^{-1} \Lambda^{-n_k} (\eta_2 + \delta_k) \quad (\delta_k \to 0 \text{ as } k \to \infty).$$

Substitute this result into (2.4):

$$\lambda_1^{n_k} X_1 X_2^{-1} \Lambda^{-n_k} (\eta_2 + \delta_k) = \eta_1 + \varepsilon_k \quad (\varepsilon_k, \delta_k \to 0 \text{ as } k \to \infty),$$

from which it follows that for each k,

$$|\eta_1 + \varepsilon_k| \leq |\lambda_1^{n_k}| \, \|X_1 X_2^{-1}\| \, \|\Lambda^{-n_k}\| \, \|\eta_2 + \delta_k\|. \tag{2.6}$$

Letting λ_{j_0} be the diagonal entry of Λ of minimum modulus, from (2.6) we obtain

$$|\eta_1 + \varepsilon_k| \leq C \left| \frac{\lambda_1}{\lambda_{j_0}} \right|^{n_k} \|\eta_2 + \delta_k\|, \tag{2.7}$$

where $C = \|X_1 X_2^{-1}\|$. Recall that λ_1 was selected to be the eigenvalue of T of minimum modulus so that $|\lambda_1| \leq |\lambda_{j_0}|$. Because η_1 was chosen to be nonzero and both δ_k and ε_k approach 0 as $k \to \infty$, inequality (2.7) tells us

$|\lambda_1| = |\lambda_{j_0}|$ and hence that

$$|\eta_1| \le C\|\eta_2\|. \tag{2.8}$$

This contradicts the choice of y as an arbitrary vector in $\mathbb{C}^{N+1}$ with nonzero components. ∎

REMARK. We have chosen to present a self-contained proof of Corollary 2.2. We could have presented a shorter proof based on Proposition 4.4 of [8], a result that, e.g., implies the columns of the matrix X_2 in our argument must be dependent, which immediately yields a contradiction of the induction hypothesis.

3. N-Supercyclicity in finite dimensions. Let $T : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ be linear. In this section, we prove a fundamental result: the T-orbit of an N-dimensional subspace of $\mathbb{C}^{N+1}$ cannot be dense in $\mathbb{C}^{N+1}$. Note that in the real setting such orbits can be dense. For example, let P be the xy-plane, let θ be an irrational multiple of π, let A be the linear mapping on $\mathbb{R}^3$ that rotates $\mathbb{R}^3$ by θ radians about the x-axis. Then $\mathrm{orb}_A(P)$ is dense in $\mathbb{R}^3$.

Our proof that a linear mapping T on $\mathbb{C}^{N+1}$ cannot be N-supercyclic is inductive, taking as its starting point Hilden and Wallen's result that no linear operator on $\mathbb{C}^N$ can be 1-supercyclic for $N > 1$ ([12]). We break the proof up into two pieces, first showing that no $N \times N$ Jordan-block matrix represents an $(N-1)$-supercyclic operator and then taking care of the general argument. The general argument uses ideas and techniques from the Jordan-block subsection, as well as its key lemma (Lemma 3.1) and its cousins, Lemmas 3.2 and 3.3.

Jordan-block matrices. We will show that no $N \times N$ Jordan-block matrix represents an $(N-1)$-supercyclic operator. The idea of the proof is well illustrated by the 3×3 case. We identify the linear mapping $\Lambda : \mathbb{C}^3 \to \mathbb{C}^3$ with its matrix representation:

$$\Lambda = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Note that

$$\Lambda^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & C(n,2)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix},$$

where $C(n,k)$ denotes n choose k. Now suppose that Λ is 2-supercyclic having supercyclic subspace $\mathcal{S}$ with basis $\{v, w\}$, where $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$. Observe that at least one of v_3 and w_3 is nonzero (for otherwise every vector in $\mathrm{orb}_A(\mathcal{S})$ will have 0 as its final component, contradicting the density of the orbit in $\mathbb{C}^3$). By taking an appropriate linear combination of v and w, one obtains a basis $\{\widetilde{v}, \widetilde{w}\}$ for $\mathcal{S}$ such that $\widetilde{v}_3 \neq 0$ and $\widetilde{w}_3 = 0$. We may assume that $\widetilde{v}_3 = 1$. We drop the tildes. Thus we have a basis for $\mathcal{S}$ that has the form

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \\ 1 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix} \right\}.$$

We will now show that w_2 is nonzero. Suppose not. Then $\mathcal{S}$ has a basis one of whose elements can be taken to be $w = (1, 0, 0)$. Note w is an eigenvector for A with corresponding eigenvalue λ. Because $\langle w \rangle$, the 1-dimensional subspace of $\mathbb{C}^3$ spanned by w, is invariant for A, Proposition 2.3 tells us that $A/\langle w \rangle : \mathbb{C}^3/\langle w \rangle \to \mathbb{C}^3/\langle w \rangle$ is J-supercyclic, where $J = \dim(\mathcal{S}/\langle w \rangle) = 1$. Thus $A/\langle w \rangle$ is a 1-supercyclic operator on a 2-dimensional space, contradicting Hilden and Wallen's result. Thus, since w_2 is nonzero, we can assume that the supercyclic subspace $\mathcal{S}$ for A has a basis of the form $\{v, w\}$, where

$$v = \begin{bmatrix} v_1 \\ 0 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ 1 \\ 0 \end{bmatrix}. \tag{3.1}$$

Because the orbit of $\mathcal{S}$ with basis $\{v, w\}$ of (3.1) is dense in $\mathbb{C}^3$, there is a sequence $s_j = \alpha_{0,j} v + \alpha_{1,j} w$ of linear combinations of v and w and a subsequence (n_j) of the sequence of natural numbers such that

$$\lim_{j \to \infty} A^{n_j} s_j = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.2}$$

We will show that (3.2) leads to a contradiction. Projecting onto the third component, we deduce from (3.2) that

$$\lim_j \lambda^{n_j} \alpha_{0,j} = 1. \tag{3.3}$$

Similarly projection onto the second component yields

$$n_j \lambda^{n_j - 1} \alpha_{0,j} + \lambda^{n_j} \alpha_{1,j} \to 0 \quad \text{as } j \to \infty.$$

Divide the quantity on the left of the preceding line by n_j and note the result tends to zero (even faster); using (3.3), conclude that

$$\frac{\lambda^{n_j} \alpha_{1,j}}{n_j} \to \frac{-1}{\lambda}.$$

Finally, because we must have convergence in the first component, (3.1) gives

$$\lambda^{n_j}\alpha_{0,j}v_1+\lambda^{n_j}\alpha_{1,j}w_1+n_j\lambda^{n_j-1}\alpha_{1,j}+\frac{n_j(n_j-1)}{2}\lambda^{n_j-2}\alpha_{0,j}\to 0 \quad \text{as } j\to\infty.$$

Now divide the quantity on left-hand side of the preceding line by n_j^2 and take the limit as $j\to\infty$ to get

$$0+0-\frac{1}{\lambda^2}+\frac{1}{2\lambda^2}=\frac{-1}{2\lambda^2};$$

but we should have gotten 0. This contradiction proves that no linear operator represented by a 3×3 Jordan-block matrix can be 2 supercyclic.

Now we handle the general Jordan-block case. Let $N\geq 3$. Suppose we know that for all J less than N, a $J\times J$ Jordan-block matrix cannot be $(J-1)$-supercyclic. We wish to show that an $N\times N$ Jordan-block matrix cannot be $(N-1)$-supercyclic.

Let Λ denote our $N\times N$ Jordan-block matrix so that

$$\Lambda^n=\begin{bmatrix} \lambda^n & n\lambda^{n-1} & C(n,2)\lambda^{n-2} & \cdots & C(n,N-1)\lambda^{n-N+1} \\ 0 & \lambda^n & n\lambda^{n-1} & \cdots & C(n,N-2)\lambda^{n-N+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix};$$

that is,

$$[\Lambda^n]_{ij}=\begin{cases} C(n,j-i)\lambda^{n-(j-i)} & \text{if } j\geq i, \\ 0 & \text{if } j<i. \end{cases}$$

Let w be the vector $(1,0,\dots,0)$ in $\mathbb{C}^N$ and note that the quotient operator $\Lambda/\langle w\rangle : \mathbb{C}^N/\langle w\rangle \to \mathbb{C}^N/\langle w\rangle$ may be represented as an $(N-1)\times(N-1)$ Jordan-block matrix (with respect to the basis $\{(0,1,0,\dots,0)+\langle w\rangle,(0,0,1,0,\dots,0)+\langle w\rangle,\dots,(0,\dots,0,1)+\langle w\rangle\}$ of $\mathbb{C}^N/\langle w\rangle$). Suppose, in order to obtain a contradiction, that Λ is $(N-1)$-supercyclic. Then $\lambda\neq 0$ because Λ must have dense range. Let $B=\{v_1,\dots,v_{N-1}\}$ be a basis for a supercyclic subspace $\mathcal{S}$ for Λ. We view the elements of B as column vectors. At least one of the basis vectors in B must have a nonzero final entry (otherwise every element of the orbit of $\mathcal{S}$ under Λ would have a zero final entry and the orbit would not be dense). Thus we assume that the final entry (the entry in row N) of v_1 is nonzero, in fact 1. By taking appropriate linear combinations of basis elements of B, we can assume that the final entries in the vectors v_2 through v_{N-1} are all zeros. Now suppose, in order to obtain a contradiction, that the entries of v_2 through v_{N-1} in row $N-1$ are also all zero. This means that the $N-2$ vectors $\widetilde{v}_2$ through $\widetilde{v}_{N-1}$ in $\mathbb{C}^{N-2}$ formed, respectively, by eliminating the last two entries of v_2 through v_{N-1} must be independent (because $\{v_2,v_3,\dots,v_{N-1}\}$ is independent). That is,

$\{\widetilde{v}_2, \ldots, \widetilde{v}_{N-1}\}$ is a basis of $\mathbb{C}^{N-2}$. Thus an appropriate linear combination of these vectors will be 1 followed by zeros. It follows that $\mathcal{S}$ contains the vector $w := (1, 0, \ldots, 0)$, which is an eigenvector for Λ with corresponding eigenvalue λ. By Proposition 2.3, $\Lambda/\langle w\rangle : \mathbb{C}^N/\langle w\rangle \to \mathbb{C}^N/\langle w\rangle$ is J-supercyclic with $J = \dim(\mathcal{S}/\langle w\rangle) \le N-2$.

We have already observed that $\Lambda/\langle w\rangle$ may be represented as a Jordan-block matrix and, as we have just seen, $\Lambda/\langle w\rangle$ is $(N-2)$-supercyclic on a space of dimension $N-1$; this contradicts our induction hypothesis. Thus, we may assume that at least one of the entries in row $N-1$ of v_2 through v_{N-1} is nonzero. Upon reordering, relabeling, and taking appropriate linear combinations, we may assume that our supercyclic subspace $\mathcal{S}$ has a basis $\{v_1, \ldots, v_n\}$ where the final entry of v_1 is 1 and all other final entries are 0 and where the entry in v_2 in row $N-1$ is 1 and all other basis vectors have zero as their entry in row $N-1$. Now, the assumption that none of the vectors $v_3, v_4, \ldots, v_n$ has a nonzero entry in row $N-2$ leads once again to the conclusion that $w = (1, 0, \ldots, 0)$ belongs to $\mathcal{S}$, which in turn contradicts the induction hypothesis just as before. Repeating this process allows us to assume that $\mathcal{S}$ has a basis of the form

$$B = \{v_1, v_2, \ldots, v_{N-1}\}$$

with

$$v_1 = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} b_2 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \ldots, \quad v_{N-1} = \begin{bmatrix} b_{N-1} \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

where $b_1, b_2, \ldots, b_{N-1}$ are complex constants.

Because Λ is supercyclic with supercyclic subspace $\mathcal{S}$ with basis B displayed above, there are sequences of scalars $(\alpha_{0,j}), (\alpha_{1,j}), \ldots, (\alpha_{N-2,j})$ and a subsequence (n_j) of the sequence of natural numbers such that

$$\lim_{j\to\infty} \Lambda^{n_j}\Big(\sum_{k=0}^{N-2} \alpha_{k,j} v_{k+1}\Big) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3.4}$$

LEMMA 3.1. *For each $q \in \{0, 1, \ldots, N-2\}$,*

$$\lim_{j\to\infty} \frac{\lambda^{n_j}\alpha_{q,j}}{n_j^q} = \frac{(-1)^q}{\lambda^q q!}.$$

Proof. To see that the limit has the advertised value for $q = 0$, consider the information (3.4) gives in its last row, keeping in mind the form of the v_j's and the formula for Λ^{n_j}. Suppose that the limit has been shown to be valid for $q = 0$ to $q = k$, where $0 \le k \le N-3$. We complete the proof by establishing its validity for $q = k+1$. Consider the information provided by row $N-(k+1)$ of (3.4):

$$\sum_{m=0}^{k+1} C(n_j, k+1-m)\lambda^{n_j-(k+1-m)}\alpha_{m,j} \to 0 \quad \text{as } j \to \infty.$$

Divide the quantity on the left of the preceding line by n_j^{k+1}, use that fact that the result still tends to 0 as $j \to \infty$, and apply the induction hypothesis to obtain

$$\frac{1}{\lambda^{k+1}} \sum_{m=0}^{k} \frac{(-1)^m}{(k+1-m)!m!} + \lim_j \frac{\lambda^{n_j}\alpha_{k+1,j}}{n_j^{k+1}} = 0. \tag{3.5}$$

Thus

$$\begin{aligned}
\lim_{j\to\infty} \frac{\lambda^{n_j}\alpha_{k+1,j}}{n_j^{k+1}} &= -\frac{1}{\lambda^{k+1}} \sum_{m=0}^{k} \frac{(-1)^m}{(k+1-m)!m!} \\
&= -\frac{1}{(k+1)!\lambda^{k+1}} \sum_{m=0}^{k} \frac{(-1)^m(k+1)!}{(k+1-m)!m!} \\
&= -\frac{1}{(k+1)!\lambda^{k+1}} \sum_{m=0}^{k} (-1)^m C(k+1,m) \\
&= -\frac{1}{(k+1)!\lambda^{k+1}} \Big[\sum_{m=0}^{k+1} (-1)^m C(k+1,m) - (-1)^{k+1}\Big] \\
&= -\frac{1}{(k+1)!\lambda^{k+1}} [(1+(-1))^{k+1} - (-1)^{k+1}] = \frac{(-1)^{k+1}}{\lambda^{k+1}(k+1)!},
\end{aligned}$$

as desired. ∎

Using the preceding lemma, we can show that the "first-row" limit of (3.4) cannot be 0, and this contradiction completes the proof that Λ cannot be $(N-1)$-supercyclic. The first-row information from (3.4) is

$$0 = \lim_{j\to\infty} \sum_{m=0}^{N-2} \alpha_{m,j}[b_{1+m}\lambda^{n_j} + C(n_j, N-1-m)\lambda^{n_j-(N-1-m)}].$$

Dividing this by n_j^{N-1}, taking the limit as $j \to \infty$, and using Lemma 3.1, we have

$$\begin{aligned} 0 &= \frac{1}{\lambda^{N-1}} \sum_{m=0}^{N-2} \frac{(-1)^m}{(N-1-m)!m!} \\ &= \frac{1}{(N-1)!\lambda^{N-1}} \Big(\sum_{m=0}^{N-2} (-1)^m C(N-1,m) + (-1)^{N-1} - (-1)^{N-1} \Big) \\ &= \frac{1}{(N-1)!\lambda^{N-1}} \left((1+(-1))^{N-1} - (-1)^{N-1} \right) = \frac{(-1)^N}{(N-1)!\lambda^{N-1}} \neq 0, \end{aligned}$$

the desired contradiction.

At this point we know that no $N \times N$ Jordan-block matrix represents an $(N-1)$-supercyclic operator. Although this fact will not be used directly to establish that no $N \times N$ matrix can represent an $(N-1)$-supercyclic operator, much of the reasoning that we used to dispense with the Jordan-block case will be needed for the general argument. In particular, we will need the following lemmas, which are essentially corollaries of the proof of Lemma 3.1.

LEMMA 3.2. *Let N be a positive integer. Suppose that Λ is an $N \times N$ Jordan-block matrix with eigenvalue λ. If there are sequences $(\alpha_{0,j})$, $(\alpha_{1,j})$, and $(\alpha_{N-1,j})$ of scalars and a subsequence (n_j) of the sequence of natural numbers such that*

$$\lim_{j\to\infty} \Lambda^{n_j} \begin{bmatrix} \alpha_{N-1,j} \\ \alpha_{N-2,j} \\ \vdots \\ \alpha_{0,j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{3.6}$$

then for each $q \in \{0,1,\dots,N-1\}$,

$$\lim_{j\to\infty} \frac{\lambda^{n_j}\alpha_{q,j}}{n_j^q} = \frac{(-1)^q}{\lambda^q q!}. \tag{3.7}$$

Proof. Suppose that (3.6) holds. Then equation (3.4) is valid in rows 2 through N, which is what we needed to obtain Lemma 3.1. Thus, (3.7) holds for $q \in \{0,1,\dots,N-2\}$. Now, the information provided by the first row of (3.6) is

$$\sum_{m=0}^{N-1} C(n_j, N-1-m)\lambda^{n_j-(N-1-m)}\alpha_{m,j} \to 0 \quad \text{as } j \to \infty$$

and the validity of (3.7) for $q \in \{0,1,\dots,N-2\}$ implies its validity for $q = N-1$ by the argument that concludes Lemma 3.1. ∎

LEMMA 3.3. *Let N be a positive integer. Suppose that Λ is an $N \times N$ Jordan-block matrix with eigenvalue λ. If there are sequences $(\alpha_{0,j})$, $(\alpha_{1,j})$,* *and* $(\alpha_{N-1,j})$ *of scalars and a subsequence* (n_j) *of the sequence of natural numbers such that*

$$\lim_{j\to\infty} \Lambda^{n_j} \begin{bmatrix} \alpha_{N-1,j} \\ \alpha_{N-2,j} \\ \vdots \\ \alpha_{0,j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

then for each $q \in \{0, 1, \ldots, N-1\}$,

$$\lim_{j\to\infty} \frac{\lambda^{n_j}\alpha_{q,j}}{n_j^q} = 0.$$

Proof. The argument proceeds exactly as that of Lemma 3.1. At the stage (3.5) of the proof, the sum on the left is now zero by the induction hypothesis. ∎

The general argument. We argue by induction. We know that no 2×2 matrix operator on $\mathbb{C}^2$ can be 1-supercyclic (Hilden and Wallen's result). Suppose $N \geq 3$ and that we know that for all J less than N, a $J \times J$ matrix cannot be $(J-1)$-supercyclic. We wish to show that an $N \times N$ matrix cannot be $(N-1)$-supercyclic.

Suppose, in order to obtain a contradiction, that Λ is an $N \times N$ matrix representing an operator on $\mathbb{C}^N$ that is $(N-1)$-supercyclic. Because $(N-1)$-supercyclicity is similarity invariant, we may assume that Λ is in Jordan canonical form with the largest Jordan block of Λ appearing in the upper left corner; of course, "largest block" may be ambiguous—we simply wish to arrange things so that the size of the uppermost block is greater than or equal to the size of all other blocks. We will be sloppy and continue to refer to the first block as the "largest block". By Corollary 2.2, we may assume that the largest block of Λ is at least 2×2 (that is, we may assume that Λ is not a diagonal matrix). Let $\mathcal{S}$ be an $(N-1)$-dimensional subspace of $\mathbb{C}^N$ that is supercyclic for Λ with basis $\{v_1, \ldots, v_{N-1}\}$. Just as in the Jordan-block argument, our induction hypothesis together with the fact that $w = (1, 0, \ldots, 0)$ is an eigenvector for Λ allows us to assume that $\mathcal{S}$ has a basis $B = \{v_1, v_2, \ldots, v_{N-1}\}$, where

$$v_1 = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} b_2 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \ldots, \quad v_{N-1} = \begin{bmatrix} b_{N-1} \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Now suppose that the upper left Jordan block of Λ has dimension $r_1 \times r_1$, and the next block has dimension $r_2 \times r_2$, and so on. Let k be the number of Jordan blocks that Λ comprises. (Note $k < N$ since we are assuming that Λ is not diagonal.) Let $\{\lambda_1, \dots, \lambda_k\}$ be the eigenvalues of Λ corresponding, respectively, to the k Jordan blocks of Λ.

The remainder of the argument consists of two cases with the first being much easier than the second.

CASE 1. Suppose that λ_1, the eigenvalue corresponding to the uppermost Jordan block of Λ, is such that

$$|\lambda_1| \le |\lambda_i| \quad \forall i \in \{2, \dots, k\}.$$

Recall that Λ comprises k Jordan blocks and that the uppermost one is $r_1 \times r_1$. Let e_{r_1} be the vector in $\mathbb{C}^N$ with 1 as its r_1 component and zeros elsewhere. Because Λ is $(N-1)$-supercyclic with supercyclic subspace $\mathcal{S}$ spanned by the basis B above, there are sequences of scalars $(\alpha_{0,j}), (\alpha_{1,j}), \dots, (\alpha_{N-1,j})$ and a subsequence (n_j) of the sequence of natural numbers such that

$$\lim_{j\to\infty} \Lambda^{n_j} \sum_{i=0}^{N-2} \alpha_{i,j} v_{i+1} = e_{r_1}. \tag{3.8}$$

For $m \in \{2, \dots, k\}$, consider the information that (3.8) provides for the rows corresponding to the mth Jordan block of Λ. Conclude from Lemma 3.3:

$$\lim_{j\to\infty} \frac{\lambda_k^{n_j} \alpha_{q,j}}{n_j^q} = 0, \qquad q = 0, \dots, r_k - 1, \tag{3.9}$$

$$\lim_{j\to\infty} \frac{\lambda_{k-1}^{n_j} \alpha_{q,j}}{n_j^{q-r_k}} = 0, \qquad q = r_k, \dots, r_k + r_{k-1} - 1, \tag{3.10}$$

$$\vdots$$

$$\lim_{j\to\infty} \frac{\lambda_2^{n_j} \alpha_{q,j}}{n_j^{q-\sum_{u=3}^k r_u}} = 0, \qquad q = \sum_{u=3}^k r_u, \dots, \sum_{u=3}^k r_u + r_2 - 1. \tag{3.11}$$

Now let us determine what information can be gleaned from examining the limit (3.8) in its first r_1 rows. Using Lemma 3.1, for $q = 0, \dots, r_1 - 2$ we have

$$\lim_{j\to\infty} \frac{\lambda_1^{n_j} \alpha_{N-r_1+q,j}}{n_j^q} = \frac{(-1)^q}{\lambda_1^q q!}. \tag{3.12}$$

Finally, consider the first-row information provided by (3.8):

$$\begin{aligned} \lim_{j\to\infty} &\sum_{h=0}^{N-r_1-1} \lambda_1^{n_j} b_{h+1} \alpha_{h,j} \\ &+ \sum_{q=0}^{r_1-2} [b_{N-r_1+q+1} \lambda_1^{n_j} + C(n_j, r_1 - 1 - q) \lambda_1^{n_j-(r_1-1-q)}] \alpha_{N-r_1+q,j} = 0. \end{aligned} \tag{3.13}$$

We can, of course, divide on the left by $n_j^{r_1-1}$ and still have zero limit. Using (3.9) through (3.11), $r_1 - 1 \geq r_j - 1$ for $j = 2, \ldots, k$ (remember the uppermost Jordan block is the largest), and the fact that $|\lambda_1| \leq |\lambda_j|$ for $j = 2, \ldots, k$, we see that

$$\lim_{j\to\infty} \sum_{h=0}^{N-r_1-1} \frac{\lambda_1^{n_j} b_{h+1}\alpha_{h,j}}{n_j^{r_1-1}} = 0. \tag{3.14}$$

However, using (3.12) just as we did in the Jordan-block subsection, we deduce

$$\lim_{j\to\infty} \sum_{q=0}^{r_1-2} \frac{[b_{N-r_1+1+q}\lambda_1^{n_j} + C(n_j, r_1-1-q)\lambda^{n_j-(r_1-1-q)}]\alpha_{N-r_1+q,j}}{n_j^{r_1-1}} \tag{3.15}$$
$$= \frac{1}{\lambda_1^{r_1-1}} \sum_{q=0}^{r_1-2} \frac{(-1)^q}{(r_1-1-q)!q!} = \frac{(-1)^{r_1}}{(r_1-1)!\lambda_1^{r_1-1}} \neq 0.$$

Note that (3.14) and (3.15) combine to contradict (3.13).

Case 2. We are assuming that Λ has k Jordan blocks $\{J_1, \ldots, J_k\}$ and that there is at least one $i \in \{2, \ldots, k\}$ such that

$$|\lambda_1| > |\lambda_i|,$$

where, as before, λ_i is the eigenvalue of the block J_i. Let $\{i_1, \ldots, i_m\}$ be those indices in $\{2, \ldots, k\}$ such that

$$|\lambda_1| > |\lambda_{i_j}|, \quad j = 1, \ldots, m.$$

Thus the indices $\{i_1, \ldots, i_m\}$ identify Jordan blocks of Λ in which the corresponding eigenvalue has modulus less than $|\lambda_1|$. We need to build a correspondence between the 2nd through kth Jordan blocks of Λ, $\{J_2, \ldots, J_k\}$, and the numbers $\{b_1, \ldots, b_{N-1}\}$ which appear as initial entries of the basis vectors of B. Consider the jth basis vector of B, which begins with b_j; note b_j has below it a 1 in row $N - j + 1$. If row $N - j + 1$ of Λ is one of the rows that the Jordan block J_t occupies, then associate b_j with J_t. In this way, we associate with J_t precisely r_t of the b_j's (recall J_t has dimension $r_t \times r_t$ and $2 \leq t \leq k$). For example, if

$$\Lambda = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

then b_1 and b_2 are associated with the lowest Jordan block "J_3" of Λ while b_3, b_4, and b_5 are associated with Λ's middle Jordan block J_2. If all of the b_j's associated with the Jordan blocks $\{J_{i_1}, \dots, J_{i_m}\}$ are zero, then the argument of Case 1 yields a contradiction: the calculation of (3.15) remains as before and the limit of (3.14) remains zero as well because the only nonzero terms in the sum would have zero limit even if λ_1 were replaced by the eigenvalue associated with the Jordan block corresponding to b_{h+1} (that eigenvalue has modulus $\geq |\lambda_1|$). Thus we may assume that at least one of the blocks $\{J_{i_1}, \dots, J_{i_m}\}$ has associated with it a nonzero b_j. Let s be the minimum modulus of eigenvalues from those Jordan blocks in $\{J_{i_1}, \dots, J_{i_m}\}$ having at least one nonzero b_j in association. Let $\{h_1, \dots, h_u\}$ be those indices from $\{i_1, \dots, i_m\}$ such that for $n = 1, \dots, u$,

(a) the eigenvalue associated with J_{h_n} has modulus s,
(b) at least one nonzero b_j is associated with J_{h_n}.

At this point in the argument, we have identified an important list of blocks:

$$\mathcal{J}_s := \{J_{h_1}, \dots, J_{h_u}\},$$

where each of these blocks has eigenvalue of modulus s, and s is the minimum modulus of the eigenvalues of those Jordan blocks in $\{J_{i_1}, \dots, J_{i_m}\}$ to which correspond at least one nonzero b_j. Note that s is positive: our $(N-1)$-supercyclic operator Λ on $\mathbb{C}^N$ has dense range by Lemma 1.1 and hence must be invertible so that Λ cannot have 0 as an eigenvalue. The b_j's associated with each of the blocks in $\mathcal{J}_s$ are naturally ordered by their indices. For example, if J_{h_1} occupies rows g_1 through g_2 of Λ ($g_1 \leq g_2$), then its list will be $(b_{N-g_2+1}, \dots, b_{N-g_1+1})$. We know that at least one number in each "b_j" listing will be nonzero. Eliminate any trailing zeros in these b_j lists, and measure the lengths of the resulting lists $\{L_1, \dots, L_u\}$ so that L_i counts the number of entries in the list for J_{h_i} from the first through the last nonzero entry. Choose J_* to be an element of $\mathcal{J}_s$ whose associated list length L_* satisfies $L_* \geq L_i$ for $i = 1, \dots, u$. Let λ_* be the eigenvalue of J_* so that, in particular, $|\lambda_*| = s$. Assume that J_* occupies rows g_1 through g_2 ($g_1 \leq g_2$) of Λ and let e_{g_2} be the vector in $\mathbb{C}^N$ having 1 in position g_2 and zeros elsewhere. Because Λ is $(N-1)$-supercyclic with supercyclic subspace $\mathcal{S}$ spanned by the basis B above, there are sequences of scalars $(\alpha_{0,j}), (\alpha_{1,j}), \dots, (\alpha_{N-1,j})$ and a subsequence (n_j) of the sequence of natural numbers such that

$$\lim_{j\to\infty} \Lambda^{n_j} \sum_{i=0}^{N-1} \alpha_{i,j} v_{i+1} = e_{g_2}. \tag{3.16}$$

To complete the proof, we show that this equation leads to a contradiction.

We start by considering what rows 2 through r_1 of (3.16) tell us: a slight variant of Lemma 3.3 yields

$$\lim_{j\to\infty} \frac{\lambda_1^{n_j}\alpha_{N-r_1+q,j}}{n_j^q} = 0 \quad \text{for } q = 0,\ldots,r_1-2. \tag{3.17}$$

(Compare with (3.12) where these limits are nonzero because the r_1 entry of the limit vector is 1 rather than 0.) Now, note that the first row of (3.16) gives us the following, just as in (3.13):

$$\begin{aligned}\lim_{j\to\infty} & \sum_{h=0}^{N-r_1-1} \lambda_1^{n_j} b_{h+1}\alpha_{h,j} \\ & + \sum_{q=0}^{r_1-2} [b_{N-r_1+q+1}\lambda_1^{n_j} + C(n_j, r_1-1-q)\lambda_1^{n_j-(r_1-1-q)}]\alpha_{N-r_1+q,j} = 0.\end{aligned} \tag{3.18}$$

We divide by $n_j^{r_1-1}$ and note that the second sum of (3.18), so divided, will have limit zero because of (3.17):

$$\lim_{j\to\infty} \sum_{q=0}^{r_1-2} \frac{[b_{N-r_1+q+1}\lambda_1^{n_j} + C(n_j, r_1-1-q)\lambda_1^{n_j-(r_1-1-q)}]\alpha_{N-r_1+q,j}}{n_j^{r_1-1}} = 0. \tag{3.19}$$

What happens when the first sum of (3.18) is divided by n^{r_1-1}? We get

$$\sum_{h=0}^{N-r_1-1} \frac{\lambda_1^{n_j} b_{h+1}\alpha_{h,j}}{n_j^{r_1-1}} = \frac{1}{n_j^{r_1-L_*}}\left(\frac{\lambda_1}{\lambda_*}\right)^{n_j} \sum_{h=0}^{N-r_1-1} \frac{\lambda_*^{n_j} b_{h+1}\alpha_{h,j}}{n_j^{L_*-1}}. \tag{3.20}$$

By choice of L_*, λ_* (recall $|\lambda_*| = s$), and the limit vector e_{g_2} of (3.16), Lemma 3.2 shows that for exactly one value of h between $N-g_2$ and $N-g_1$ inclusive,

$$\lim_{j\to\infty} \frac{\lambda_*^{n_j} b_{h+1}\alpha_{h,j}}{n_j^{L_*-1}}$$

will be nonzero—namely for $h = N - g_2 + L_* - 1$, which corresponds to the rightmost nonzero entry in the list of b_j's corresponding to J_*. It is not difficult to check that all other summands in the sum on the right of (3.20) will be zero—use the choice of s, L_*, and Lemma 3.3 (in the notation of Lemma 3.3, observe that $|\lambda_*| < |\lambda_i|$ whenever q exceeds $L_* - 1$, and this provides an exponential decay factor which dominates the polynomial growth provided by a power of n_j). Because $|\lambda_1| > |\lambda_*|$ and

$$\lim_{j\to\infty} \sum_{h=0}^{N-r_1-1} \frac{\lambda_*^{n_j} b_{h+1}\alpha_{h,j}}{n_j^{L_*-1}} \neq 0$$

(exactly one summand has a nonzero limit), we have

$$\lim_{j\to\infty} \frac{1}{n_j^{r_1-L_*}}\left(\frac{\lambda_1}{\lambda_*}\right)^{n_j} \sum_{h=0}^{N-r_1-1} \frac{\lambda_*^{n_j} b_{h+1}\alpha_{h,j}}{n_j^{L_*-1}} = \infty. \tag{3.21}$$

Note (3.21) and (3.19) are inconsistent with (3.18), which is the contradiction we needed to establish that N-supercyclicity cannot occur nontrivially in finite dimensions. ∎

Knowing that no linear map on $\mathbb{C}^{N+1}$ can be N-supercyclic gives us the following generalization of Theorem 2.1.

THEOREM 3.4. *Suppose that $T : \mathcal{X} \to \mathcal{X}$ is a continuous linear operator and N is a positive integer. If T^* has an $(N+1)$-dimensional invariant subspace, then T is not N-supercyclic.*

Proof. The proof is quite similar to that showing Corollary 2.2 implies Theorem 2.1. Suppose that T^* has an $(N+1)$-dimensional invariant subspace $\mathcal{W}$. Then $\mathcal{K} := {}^{\perp}\mathcal{W}$ is easily seen to be a closed subspace of $\mathcal{X}$ of codimension $N+1$ that is invariant for T. If T were N-supercyclic, then by Proposition 2.3, $T/\mathcal{K} : \mathcal{X}/\mathcal{K} \to \mathcal{X}/\mathcal{K}$ would be an N-supercyclic operator on an $(N+1)$-dimensional space, a contradiction. ∎

4. Subnormal operators cannot be N-supercyclic. In this section, we present two proofs of the fact that a subnormal operator on an infinite-dimensional Hilbert space cannot be N-supercyclic. In the first proof, Corollary 2.2, which states that diagonal operators are never nontrivially N-supercyclic, plays a crucial role. The second proof depends upon a new intertwining relationship between subnormal operators and normal operators established in Theorem 4.4 below as well as the fact that no normal operators can be nontrivially N-supercyclic.

THEOREM 4.1. *A normal operator on an infinite-dimensional Hilbert space is never N-supercyclic, and a normal operator on an N-dimensional Hilbert space is never $(N-1)$-supercyclic.*

Proof. For the infinite-dimensional result, see [9]; the finite-dimensional result is exactly Corollary 2.2 above. ∎

Notice that in the following theorem $\mathcal{H}$ is allowed to be equal to $L^2(\mu)$, thus the arguments below may be turned into another proof, different from that in [9], that a normal operator on an infinite-dimensional space cannot be N-supercyclic.

THEOREM 4.2. *Suppose that $S = M_z$ on $\mathcal{H} \subseteq L^2(\mu)$, where μ is a compactly supported regular Borel measure in $\mathbb{C}$, that $\mathcal{H}$ is invariant for S, and that $\dim \mathcal{H} > N$. Then S is not N-supercyclic.*

Proof. Suppose that S is N-supercyclic and $\mathcal{M}$ is an N-dimensional subspace of $\mathcal{H}$ whose orbit under S is dense in $\mathcal{H}$. If $\dim \mathcal{H} < \infty$, then S

is normal and N-supercyclic, but since $\dim \mathcal{H} > N$, this contradicts Theorem 4.1. Hence we may assume that $\dim \mathcal{H} = \infty$, and that S is not normal (again, by Theorem 4.1). Thus, we may also assume that M_z on $L^2(\mu)$ is the minimal normal extension of S. It is then an easy exercise, using Conway [7, Proposition 17.14, p. 249], to show that up to unitary equivalence, we may assume that $1 \in \mathcal{H}$. Hence, all polynomials are also in $\mathcal{H}$. Let $\mathcal{P}$ be the (countable) set of all polynomials with (complex) rational coefficients.

Let $\{f_j : 1 \le j \le N\}$ be a basis for $\mathcal{M}$. Since $\mathcal{M}$ has dense orbit under S, for each $p \in \mathcal{P}$, there exist scalars $\{a_{j,k} : 1 \le j \le N,\ 1 \le k < \infty\}$ and integers $m_k \to \infty$ such that $\sum_{j=1}^{N} z^{m_k} a_{j,k} f_j \to p$ in $L^2(\mu)$ as $k \to \infty$. By passing to a subsequence we may assume that as $k \to \infty$, $\sum_{j=1}^{N} z^{m_k} a_{j,k} f_j(z) \to p(z)$ μ-almost everywhere, say for all $z \in \operatorname{supp}(\mu) \setminus \Delta_p$ where $\mu(\Delta_p) = 0$. If we let $\Delta = \bigcup_{p\in\mathcal{P}} \Delta_p$, then $\mu(\Delta) = 0$ and we have

$$(*) \qquad \sum_{j=1}^{N} z^{m_k} a_{j,k} f_j(z) \to p(z)$$

as $k \to \infty$, for all $z \in \operatorname{supp}(\mu) \setminus \Delta$. Since $\dim \mathcal{H} = \infty$, we may choose $N+1$ distinct points, $\lambda_1, \dots, \lambda_{N+1}$, in $\operatorname{supp}(\mu) \setminus \Delta$. Evaluating $(*)$ at these $N+1$ points shows that for each $p \in \mathcal{P}$,

$$(**) \qquad \begin{bmatrix} \lambda_1^{m_k} & & & \\ & \lambda_2^{m_k} & & \\ & & \ddots & \\ & & & \lambda_{N+1}^{m_k} \end{bmatrix} \left(\sum_{j=1}^{N} a_{j,k} \begin{bmatrix} f_j(\lambda_1) \\ f_j(\lambda_2) \\ \vdots \\ f_j(\lambda_{N+1}) \end{bmatrix} \right) \to \begin{bmatrix} p(\lambda_1) \\ p(\lambda_2) \\ \vdots \\ p(\lambda_{N+1}) \end{bmatrix}$$

as $k \to \infty$. Thus if

$$T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{N+1} \end{bmatrix}, \qquad \mathcal{L} = \operatorname{span} \left\{ \begin{bmatrix} f_j(\lambda_1) \\ f_j(\lambda_2) \\ \vdots \\ f_j(\lambda_{N+1}) \end{bmatrix} : 1 \le j \le N \right\},$$

then $\mathcal{L}$ is an N-dimensional subspace of $\mathbb{C}^{N+1}$. Furthermore, by $(**)$ it follows that the closure of $\operatorname{orb}_T(\mathcal{L})$ contains the set

$$\left\{ \begin{bmatrix} p(\lambda_1) \\ p(\lambda_2) \\ \vdots \\ p(\lambda_{N+1}) \end{bmatrix} : p \in \mathcal{P} \right\}.$$

Since the latter set is dense in $\mathbb{C}^{N+1}$, it follows that T is N-supercyclic on $\mathbb{C}^{N+1}$, contradicting Theorem 4.1. Thus S cannot be N-supercyclic. ∎

THEOREM 4.3. *If S is a subnormal operator on an infinite-dimensional Hilbert space, then S is not N-supercyclic for any $N \geq 1$.*

Proof. Assume that S is a subnormal operator on an infinite-dimensional Hilbert space $\mathcal{H}$ that is N-supercyclic for some $N \geq 1$. By Theorem 4.1 we may assume that S is pure, and thus has no eigenvectors. Let $\mathcal{N}$ be the minimal normal extension of S acting on a space $\mathcal{K}$. Let $v \in \mathcal{H}$ be any nonzero vector and let $\mathcal{K}_1$ be the reducing subspace for $\mathcal{N}$ generated by v. Also let P be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_1$. Notice that $\mathcal{H}$ cannot be orthogonal to $\mathcal{K}_1$ because both $\mathcal{H}$ and $\mathcal{K}_1$ contain the nonzero vector v. Thus, let $\mathcal{H}_1 := \operatorname{cl} P(\mathcal{H})$. It follows that $\mathcal{H}_1$ is a nonzero subspace of $\mathcal{K}_1$. Furthermore since P commutes with $\mathcal{N}$, it follows that $\mathcal{H}_1$ is invariant under $\mathcal{N}$ and that $P : \mathcal{H} \to \mathcal{H}_1$ intertwines S with $\mathcal{N}|\mathcal{H}_1$.

Since $\mathcal{N}|\mathcal{K}_1$ is $*$-cyclic, it is unitarily equivalent to an operator of the form M_z on $L^2(\mu)$ for some regular Borel compactly supported measure μ in the complex plane; let $U : \mathcal{K}_1 \to L^2(\mu)$ be the unitary that conjugates $\mathcal{N}|\mathcal{K}_1$ to M_z. If we let $\mathcal{H}' := U(\mathcal{H}_1)$, then we see that $\mathcal{N}|\mathcal{H}_1$ is unitarily equivalent to $T = M_z$ on $\mathcal{H}' \subseteq L^2(\mu)$. Furthermore the map $A : \mathcal{H} \to \mathcal{H}'$ given by $A = UP$ intertwines S with T and has dense range. Since S is N-supercyclic, it follows that T is N-supercyclic. Notice that $\mathcal{H}_1$ is infinite-dimensional because $\mathcal{H}_1 \supseteq \mathcal{H} \cap \mathcal{K}_1$ and $\mathcal{H} \cap \mathcal{K}_1$ contains the invariant subspace $[v]$ of S generated by v. Furthermore, $[v]$ cannot be finite-dimensional, otherwise S would have eigenvectors, contradicting the purity of S. Thus $\mathcal{H}_1$, and hence also $\mathcal{H}'$, is infinite-dimensional. So T is N-supercyclic on $\mathcal{H}'$ and $\dim \mathcal{H}' > N$, contradicting Theorem 4.2. Thus S cannot be N-supercyclic. ∎

We now give another proof that subnormal operators are not N-supercyclic, that does not use Theorem 4.2, but uses Theorem 4.1 and the following result.

THEOREM 4.4. *If S is a subnormal operator acting on an infinite-dimensional Hilbert space $\mathcal{H}$, then there exists a normal operator $\mathcal{N}$ acting on an infinite-dimensional Hilbert space $\mathcal{K}$ and a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ with dense range satisfying $AS = \mathcal{N}A$.*

Proof. We may assume that S is pure. In the proof of Theorem 4.3 (see also [8]) it is shown that there exists a measure μ and a bounded linear operator $A_1 : \mathcal{H} \to L^2(\mu)$ such that $A_1 S = M_z A_1$. Thus A_1 intertwines S with the normal operator $\mathcal{N}_\mu = M_z$ on $L^2(\mu)$. Now let $\mathcal{H}_1$ be the closure of the range of A_1 and let $S_1 = M_z$ on $\mathcal{H}_1$. Thus $A_1 S = S_1 A_1$.

We may assume that $\mathcal{N}_\mu$ is the minimal normal extension of S_1. In this case we may replace S_1 with a unitarily equivalent operator $S_2 = M_z$ on $\mathcal{H}_2 \subseteq L^2(\nu)$ that also satisfies $1 \in \mathcal{H}_2$ and thus all polynomials are in $\mathcal{H}_2$.

Let $\varphi \in L^\infty(\nu)$ be a cyclic vector for M_z on $L^2(\nu)$ [7, p. 232]. Then let $A_2 : \mathcal{H}_2 \to L^2(\nu)$ be given by multiplication by φ. Clearly A_2 is one-to-one with dense range and satisfies $A_2S_2 = \mathcal{N}_\nu A_2$. If $U : \mathcal{H}_1 \to \mathcal{H}_2$ is the unitary that intertwines S_1 and S_2, then $A = A_2UA_1$ is the required map. ■

It now follows immediately from Theorem 4.4 that if a subnormal operator S is N-supercyclic, and A is a dense range map that intertwines S with a normal operator $\mathcal{N}$, both S and $\mathcal{N}$ acting on infinite-dimensional spaces, then $\mathcal{N}$ is also N-supercyclic, contradicting Theorem 4.1.

5. Natural questions. Question 6.4 of [9] reads:

Can a pure subnormal (hyponormal) operator be N-supercyclic?

Theorem 4.3 answers this question in the negative for subnormal operators, but the question remains open for hyponormal operators except in the $N = 1$ case (no hyponormal operator on an infinite-dimensional Hilbert space can be supercyclic [5, Theorem 3.1]).

Our proof, presented in Section 3, that N-supercyclicity cannot occur nontrivially in the finite-dimensional setting is long and quite technical.

QUESTION 5.1. Does there exist a short, nontechnical proof showing that a linear mapping $T : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ cannot be N-supercyclic?

In the Introduction, we presented examples of N-supercyclic operators T that are not cyclic, with noncyclicity being due to the existence of multiple eigenvalues of T^*.

QUESTION 5.2. Suppose that T is N-supercyclic and T^* has no multiple eigenvalues. Must T be cyclic?

Ansari [1] has shown that powers of supercyclic operators are always supercyclic.

QUESTION 5.3. If T is N-supercyclic and n is a positive integer, must T^n be N-supercyclic?

Recall that a subset E of a topological vector space $\mathcal{X}$ is *somewhere dense* in $\mathcal{X}$ provided the closure of E in $\mathcal{X}$ has nonempty interior. An affirmative answer to the following question would yield an affirmative answer to Question 5.3.

QUESTION 5.4. Suppose that there is an N-dimensional subspace $\mathcal{S}$ of $\mathcal{X}$ whose orbit under the operator $T : \mathcal{X} \to \mathcal{X}$ is somewhere dense in $\mathcal{X}$; must $\mathrm{orb}_T(\mathcal{S})$ then be everywhere dense in $\mathcal{X}$ (so that T is, in particular, N-supercyclic)?

Bourdon and Feldman [6] have shown that the answer to the preceding question is yes when $N = 1$. The argument that shows that a yes answer to

Question 5.4 yields a yes answer to Question 5.3 is essentially the same as that of [6, Corollary 2.6].

The inverse of a supercyclic operator is always supercyclic (see, e.g., [2, Section 4]).

QUESTION 5.5. If T is invertible and N-supercyclic, must T^{-1} be N-supercyclic?

We remark that Question 5.2 above is essentially the same as Question 6.2 of [9].

REMARK. After this paper was accepted for publication, the authors learned that Bayart and Matheron [3] had shown that hyponormal operators cannot be N-supercyclic.

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Washington and Lee University
Lexington, VA 24450, U.S.A.
E-mail: pbourdon@wlu.edu
feldmann@wlu.edu

Michigan State University
East Lansing, MI 48824, U.S.A.
E-mail: shapiro@math.msu.edu

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