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Weak-type $(1,1)$ bounds for oscillatory singular integrals with rational phases

by

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Abstract. We consider singular integral operators on $\mathbb{R}$ given by convolution with a principal value distribution defined by integrating against oscillating kernels of the form $e^{iR(x)}/x$ where $R(x) = P(x)/Q(x)$ is a general rational function with real coefficients. We establish weak-type $(1,1)$ bounds for such operators which are uniform in the coefficients, depending only on the degrees of P and Q. It is not always the case that these operators map the Hardy space $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ and we will characterise those rational phases $R(x) = P(x)/Q(x)$ which do map H^1 to L^1 (and even H^1 to H^1).

1. Introduction. There has been considerable attention given to the study of the mapping properties of oscillatory integral operators of the form

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}} \frac{e^{iR(y)}}{y} f(x-y)\, dy, \tag{1.1}$$

as well as their nonconvolution and higher-dimensional analogues. See, for example, [9], [4], [5], [10], [2], [8] and [3]. Various L^p, weak-type $(1,1)$ and Hardy space estimates have been proved when $R(x)$ is a polynomial or behaves like a power $|x|^a$ for positive or negative exponents a. Here we would like to consider the class of rational functions which unifies in some sense previous known results while giving uniform estimates on L^1. Our main result is the following.

THEOREM 1.1. *Let $R(x) = P(x)/Q(x)$ be a rational function with real coefficients and consider the associated operator T given in* (1.1). *Then T is weak-type $(1,1)$ with bounds depending only on the degrees of P and Q. More precisely,*

$$\alpha\, |\{x \in \mathbb{R} : |Tf(x)| \geq \alpha\}| \leq C\|f\|_{L^1(\mathbb{R})} \tag{1.2}$$

with a constant C depending only on the degrees of P and Q, and in particular, C can be taken to be independent of the coefficients.

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From Theorem 1.1 in [6] one easily deduces that T is bounded on $L^2(\mathbb{R})$ with bounds which are uniform in the coefficients. Therefore by duality and interpolation with (1.2), we obtain uniform L^p, $1 < p < \infty$, estimates for T.

We now state a result on the classical Hardy space $H^1(\mathbb{R})$. It is well known that when $R(x) = bx$ for some $b \in \mathbb{R} \setminus \{0\}$, then the associated operator T does not map H^1 to L^1, and even more, $T : H^1 \to L^{1,q}$ only for $q = \infty$. In fact if f is a smooth H^1 atom supported on $(-1, 1)$ such that the Fourier transform $\hat{f}(b)$ does not vanish, then for large x,

$$Tf(x) = \int_{\mathbb{R}} \frac{e^{ib(x-y)}}{x-y} f(y)\,dy = \frac{e^{ibx}}{x}\hat{f}(b) + O(|x|^{-2}).$$

Therefore any positive result establishing $T : H^1 \to L^{1,q}$ for some $q < \infty$ for general rational phases will not be uniform in the coefficients. We make the following observation.

THEOREM 1.2. *Let $R(x) = P(x)/Q(x)$ be a real rational function with d equal to the degree of P and e equal to the degree of Q. Consider the associated operator T given in* (1.1).

(1) *If $d \neq e + 1$, then $T : H^1(\mathbb{R}) \to H^1(\mathbb{R})$.*
(2) *If $d = e + 1$, then $T : H^1(\mathbb{R}) \to L^{1,q}(\mathbb{R})$ if and only if $q = \infty$.*

Notation. Let A, B be positive quantities. We use the notation $A \lesssim B$ or $A = O(B)$ to denote the estimate $A \le CB$ where C depends only on the degrees of P and Q. We use $A \sim B$ to denote the estimates $A \lesssim B \lesssim A$.

2. Idea of the proof for Theorem 1.1. Here we sketch the main ideas for bounding the oscillatory singular integral operator T given by (1.1) when $R(x) = P(x)/Q(x)$ is a rational function with real coefficients. By factoring the polynomials P and Q into linear factors, it is easy to see that outside a bounded number of "dyadic" intervals, P and Q behave like various monomials on the complementary intervals (see Lemma 3.1 below). Hence we can reduce ourselves to bounding

$$T_G f(x) = \int_{|y|\in G} f(x-y)\frac{e^{iR(y)}}{y}\,dy \tag{2.1}$$

where G is an interval of $\mathbb{R}^+$ (possibly very long) on which the rational function $|R(y)| = |P(y)/Q(y)| \sim |c|\,|y|^{j-k}$ behaves like a monomial for some nonnegative integers $j, k \ge 0$.

The main effort is to ensure that various derivatives of R have the expected behaviour on G. When this is the case and when $j \ge k$, say, then

$$\int_{|y|\in G\cap[0,1]} f(x-y)\frac{e^{iR(y)}}{y}\,dy$$

is a classical Calderón–Zygmund singular integral operator. Hence we obtain weak-type $(1,1)$ bounds for this part of the operator, and using a simple scaling argument, we can ensure that the bounds are uniform in the coefficients. For the part of the operator near infinity,

$$\int_{|y|\in G\cap[1,\infty)} f(x-y)\frac{e^{iR(y)}}{y}\,dy,$$

we employ the arguments of Christ and Chanillo in [2] where weak-type $(1,1)$ estimates are obtained for general oscillatory singular integral operators with polynomial phases.

When $j<k$, the part of the operator near infinity is a classical Calderón–Zygmund singular integral operator and for the part near the origin, the operator is a strongly singular integral operator of the type treated by C. Fefferman in [4].

3. Preliminaries and reductions. The following lemmas are variants of results appearing in [1], [6] and [7]. We give the proofs for the convenience of the reader.

Lemma 3.1. *Let $P(t)=a\prod_{j=1}^{d}(t-z_j)=\sum_{k=0}^{d}p_kt^k$ be a polynomial of degree d whose roots are ordered so that $|z_1|\le\cdots\le|z_d|$. For each $A>0$, we define the following intervals (possibly empty) on $\mathbb{R}^+$: for $1\le j\le d-1$, we set $G_j=G_j(A):=[A|z_j|,A^{-1}|z_{j+1}|]$, and for $j=d$, we set $G_d:=[A|z_d|,\infty)$. Furthermore if $z_1\neq 0$, we set $G_0=G_0(A)=[0,A^{-1}|z_1|]$.*

Then there exists a constant $C=C(d)>0$ such that for any $A\ge C(d)$ and $0\le j\le d$ with G_j is nonempty,

(i) *$|P(t)|\sim|p_j|\,|t|^j$ for $|t|\in G_j$, and*
(ii) *$|p_j|\sim|a|\prod_{\ell=j+1}^{d}|z_\ell|$; in particular $p_j\neq 0$.*

Proof. From the factorisation $P(t)=a\prod(t-z_j)$, we see that for $|t|\in G_j$ (and any $A>1$),

$$(1-1/A)^d|a|\Big[\prod_{\ell=j+1}^{d}|z_\ell|\Big]\le|P(t)|/|t|^j\le(1+1/A)^d|a|\Big[\prod_{\ell=j+1}^{d}|z_\ell|\Big],$$

which shows that (i) follows from (ii). To establish (ii) we write

$$\begin{aligned}p_j&=(-1)^j a\sum_{\ell_1<\cdots<\ell_{d-j}} z_{\ell_1}\cdots z_{\ell_{d-j}}\\&=(-1)^j a\sum_{\substack{\ell_1<\cdots<\ell_{d-j}\\ \ell_1\le j}} z_{\ell_1}\cdots z_{\ell_{d-j}}+(-1)^j a z_{j+1}\cdots z_d=\mathrm{I}+\mathrm{II}\end{aligned}$$

and hence since $|z_\ell| \le (1/A)|z_{\ell'}|$ whenever $\ell \le j \le \ell' - 1$,

$$A|I| \lesssim |a|\,|z_{j+1}|\cdots|z_d| = |\mathrm{II}|.$$

Therefore if $A \ge 1$ is large enough,

$$|p_j| \sim |\mathrm{II}| = |a| \prod_{\ell=j+1}^{d} |z_\ell|,$$

establishing (ii) and hence (i). ∎

Remark 3.2. Lemma 3.1(i) shows that with respect to P, $\mathbb{R}^+$ can be decomposed into disjoint intervals:

$$\mathbb{R}^+ = \bigcup_{\ell=0}^{M} G_\ell \cup \bigcup_{\ell=1}^{M-1} D_\ell$$

($M = O(1)$), which depend on the choice of A, where the D_ℓ are dyadic in the sense that if $D_\ell = [a,b)$, then $b/a = O(1)$. On the complementary intervals G_ℓ (which we call *gaps*), if $|t| \in G_\ell$, then $|P(t)| \sim |p_{j_\ell}|\,|t|^{j_\ell}$ for some $j_\ell \ge 0$ (and of course $p_{j_\ell} \ne 0$). See [1].

For a rational function $R = P/Q$, where $P(t) = a\prod_{\ell=1}^{d}(t - z_\ell)$, $Q(t) = b\prod_{\ell=1}^{e}(t - w_\ell)$ with $|z_1| \le \cdots \le |z_d|$ and $|w_1| \le \cdots \le |w_e|$, Lemma 3.1 tells us that $|R(t)| \sim |p_j/q_k|\,|t|^{j-k}$ on a gap $G = [A|z_j|, A^{-1}|z_{j+1}|] \cap [A|w_k|, A^{-1}|w_{k+1}|]$, if $A \ge 1$ is large enough. We now examine derivatives of R on G in the following two lemmas. We begin with the case $j \ge k$.

Lemma 3.3. *Let $R = P/Q$ be a rational function and G a gap as described above. Then for any integer $n \ge 0$, $A \ge C_n$ can be chosen large enough so that on G, if $j \ge k$,*

$$R^{(n)}(t) = R(t)\Big[\sum_{k+1 \le \ell_1 \ne \cdots \ne \ell_n \le j} \prod_{m=1}^{n} \frac{1}{t - z_{\ell_m}} + E_n(t)\Big] \tag{3.1}$$

where $|(d/dt)^r E_n(t)| \lesssim C_{n,r} A^{-1}|t|^{-n-r}$ on G for all $r \ge 0$.

Proof. We begin with the case $n = 1$:

$$R'(t) = R(t)[P'(t)/P(t) - Q'(t)/Q(t)] = R(t)\Big[\sum_{\ell=1}^{d} \frac{1}{t - z_\ell} - \sum_{\ell=1}^{e} \frac{1}{t - w_\ell}\Big].$$

We make the following two simple observations on G:

$$\Big|\frac{1}{t - z_\ell}\Big|, \Big|\frac{1}{t - w_{\ell'}}\Big| \le C[A|t|]^{-1}, \qquad \ell > j,\ \ell' > k, \tag{3.2}$$

and

$$\Big|\frac{1}{t - z_\ell} - \frac{1}{t - w_{\ell'}}\Big| = \frac{|z_\ell - w_{\ell'}|}{|t - z_\ell|\,|t - w_{\ell'}|} \le C[A|t|]^{-1}, \qquad \ell, \ell' \le k. \tag{3.3}$$

Hence

$$R' = R\Big[\sum_{\ell=k+1}^{j} \frac{1}{t-z_\ell} + E_1(t)\Big]$$

where

$$E_1(t) = \sum_{\ell=1}^{k}\Big[\frac{1}{t-z_\ell} - \frac{1}{t-w_\ell}\Big] + \sum_{\ell=j+1}^{d} \frac{1}{t-z_\ell} - \sum_{\ell=k+1}^{e} \frac{1}{t-w_\ell}$$

satisfies $|E_1^{(r)}(t)| \leq CA^{-1}|t|^{-r-1}$ for all $r \geq 0$ on G by (3.2) and (3.3), establishing (3.1) when $n = 1$.

The proof now proceeds by induction on n; if (3.1) holds for derivatives up to order $n-1$, then

$$R^{(n)}(t) = R'(t)\Big[\sum_{k+1\leq \ell_1 \neq \cdots \neq \ell_{n-1} \leq j} \prod_{m=1}^{n-1} \frac{1}{t-z_{\ell_m}} + E_{n-1}(t)\Big]$$

$$+R(t)\Big[-\sum_{r=1}^{n-1} \sum_{k+1\leq \ell_1 \neq \cdots \neq \ell_{n-1} \leq j} \frac{1}{(t-z_{\ell_1})} \cdots \frac{1}{(t-z_{\ell_r})^2} \cdots \frac{1}{(t-z_{\ell_{n-1}})} + E'_{n-1}(t)\Big]$$

$$= R(t)\Big[\sum_{\substack{k+1\leq \ell_1 \neq \cdots \neq \ell_{n-1} \leq j \\ k+1 \leq \ell \leq j}} \frac{1}{t-z_\ell} \prod_{m=1}^{n-1} \frac{1}{t-z_{\ell_m}}$$

$$- \sum_{r=1}^{n-1} \sum_{k+1\leq \ell_1 \neq \cdots \neq \ell_{n-1} \leq j} \frac{1}{(t-z_{\ell_r})^2} \prod_{\substack{1\leq m \leq n-1 \\ m \neq r}} \frac{1}{(t-z_{\ell_m})} + E_n(t)\Big]$$

where

$$E_n(t) = E_{n-1}(t)\Big(\sum_{\ell=k+1}^{j} \frac{1}{t-z_\ell} + E_1(t)\Big)$$

$$+ E_1(t) \sum_{k+1\leq \ell_1 \neq \cdots \neq \ell_{n-1} \leq j} \prod_{m=1}^{n-1} \frac{1}{t-z_\ell} + E'_{n-1}(t)$$

is easily seen to satisfy the derivative bounds on G, proving (3.1) for general n. ■

Remarks 3.4.

- It will be important for us to keep track of the number of terms in the sum

$$\sum_{k+1\leq \ell_1 \neq \cdots \neq \ell_n \leq j} \prod_{m=1}^{n} \frac{1}{t-z_{\ell_m}} \tag{3.4}$$

appearing in (3.1). The number of terms is

$$(j-k)(j-k-1)\cdots(j-k-n+1).$$

This shows in particular that (3.4) is empty (and hence equal to zero) if $0 \le j-k \le n-1$.

- Since the sum (3.4) is zero when $0 \le j-k \le n-1$, Lemma 3.3 only gives a bound from above on the nth derivative of R; namely, $|R^{(n)}(t)| \lesssim |R(t)|\,|t|^{-n}$ for $|t| \in G$. For $j-k \ge n$, we can in fact bound $|R^{(n)}(t)/R(t)|$ from below.

When $j-k \ge n$, the sum (3.4) is nonempty and we have

$$\Bigg|\sum_{k+1\le \ell_1\ne\cdots\ne \ell_n\le j}\prod_{m=1}^{n}\frac{1}{t-z_{\ell_m}}\Bigg| \sim |t|^{-n}. \tag{3.5}$$

The upper bound follows easily from (3.2). For the lower bound, we use the fact $|z| \ge \mathrm{Re}(z)$ to see that the left hand side is larger than

$$\Bigg|\sum_{k+1\le \ell_1\ne\cdots\ne \ell_n\le j}\prod_{m=1}^{n}\frac{\mathrm{Re}\prod_{s=1}^{n}(t-\overline{z}_{\ell_s})}{\prod_{s=1}^{n}|t-z_{\ell_s}|^2}\Bigg| = \Bigg|\sum_{k+1\le \ell_1\ne\cdots\ne \ell_n\le j}\frac{\mathrm{Re}[t^n+O(A^{-1}t^n)]}{|t|^{2n}+O(A^{-1}t^{2n})}\Bigg|,$$

which in turn is larger than $|t|^{-n}$ on G if $A \ge 1$ is large enough, since $|z_\ell| \lesssim A^{-1}|t|$ on G whenever $\ell \le j$.

The bound (3.5), together with the error bound $|E_n(t)| \lesssim A^{-1}|t|^{-n}$, shows that if $j-k \ge n$, then

$$|R^{(n)}(t)/R(t)| \sim |t|^{-n} \tag{3.6}$$

for $|t| \in G$ if $A \ge 1$ is large enough.

We now turn to the case $j<k$, which unfortunately is somewhat more involved. As in the case $k \le j$ it will be important for us to keep track of the number of terms in various sums. To this end we associate to every strictly positive multi-index $\alpha = (\alpha_1,\ldots,\alpha_r)$, $\alpha_i > 0$, $1 \le i \le r$, a size $|\alpha| = \alpha_1+\cdots+\alpha_r$ and a length $l(\alpha) = r$.

LEMMA 3.5. *Let $R = P/Q$ and G be as in Lemma 3.3 but where now $j<k$. For any integer $n \ge 1$, $A \ge C_n$ can be chosen large enough so that on G,*

$$(3.7)\qquad R^{(n)}(t)=R(t)\Bigg[(-1)^n\sum_{m=1}^{n}\sum_{\substack{|\alpha|=n\\ l(\alpha)=m}} d(\alpha) \\ \times\sum_{j+1\le \ell_1,\dots,\ell_m\le k}\frac{1}{(t-w_{\ell_1})^{\alpha_1}}\cdots\frac{1}{(t-w_{\ell_m})^{\alpha_m}}+F_n(t)\Bigg]$$

where $|(d/dt)^r F_n(t)|\lesssim C_{n,r}A^{-1}|t|^{-n-r}$ *for all* $r\ge 0$. *Here* $\{d(\alpha)\}$ *are combinatorial numbers defined on strictly positive multi-indices* α *such that the sums*

$$c_m(n)=\sum_{|\alpha|=n,\, l(\alpha)=m} d(\alpha)$$

are the well-known Stirling numbers of the second kind, i.e., $\{c_m(n)\}_{m=1}^{n}$ *are the coefficients of the polynomial*

$$x(x+1)\cdots(x+n-1)=\sum_{m=1}^{n}c_m(n)x^m.$$

Proof. For $n=1$ we argue exactly as in Lemma 3.3, using (3.3) and (3.2) to obtain

$$(3.8)\qquad R'(t)=R(t)\Bigg[-\sum_{\ell=j+1}^{k}\frac{1}{t-w_\ell}+F_1(t)\Bigg]$$

where F_1 satisfies the appropriate derivative estimates on G. For general n we argue by induction; if (3.7) holds for all derivatives up to order n, then

$$R^{(n+1)}(t)=R'(t)\Bigg[(-1)^n\sum_{m=1}^{n}\sum_{\substack{|\alpha|=n\\ l(\alpha)=m}} d(\alpha) \\ \times\sum_{j+1\le \ell_1,\cdots,\ell_m\le k}\frac{1}{(t-w_{\ell_1})^{\alpha_1}}\cdots\frac{1}{(t-w_{\ell_m})^{\alpha_m}}+F_n(t)\Bigg] \\ +R(t)\Bigg[(-1)^{n+1}\sum_{m=1}^{n}\sum_{\substack{|\alpha|=n\\ (\alpha)=m}} d(\alpha)\sum_{r=1}^{m}\alpha_r \\ \times\sum_{j+1\le \ell_1,\dots,\ell_m\le k}\frac{1}{(t-w_{\ell_1})^{\alpha_1}}\cdots\frac{1}{(t-w_{\ell_r})^{\alpha_r+1}}\cdots\frac{1}{(t-w_{\ell_m})^{\alpha_m}}+F_n'(t)\Bigg].$$

Using (3.8) we obtain

$$
\begin{aligned}
(3.9)\qquad R^{(n+1)}(t) = R(t)\Bigg[&(-1)^{n+1}\sum_{m=1}^{n}\sum_{\substack{|\alpha|=n\\ l(\alpha)=m}} d(\alpha) \\
&\times \sum_{j+1\le \ell_1,\dots,\ell_m,\ell\le k} \frac{1}{(t-w_{\ell_1})^{\alpha_1}}\cdots\frac{1}{(t-w_{\ell_m})^{\alpha_m}}\,\frac{1}{t-w_\ell} \\
&+(-1)^{n+1}\sum_{m=1}^{n}\sum_{|\alpha|=n,\,l(\alpha)=m} d(\alpha)\sum_{r=1}^{m}\alpha_r \\
&\times \sum_{j+1\le \ell_1,\dots,\ell_m\le k}\frac{1}{(t-w_{\ell_1})^{\alpha_1}}\cdots\frac{1}{(t-w_{\ell_r})^{\alpha_r+1}}\cdots\frac{1}{(t-w_{\ell_m})^{\alpha_m}} + F_{n+1}(t)\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
F_{n+1}(t) = F_n(t)\Bigg[&-\sum_{\ell=j+1}^{k}\frac{1}{t-w_\ell} + F_n'(t)\Bigg] \\
&+\Bigg[(-1)^n F_1(t)\sum_{m=1}^{n}\sum_{|\alpha|=n,\,l(\alpha)=m} d(\alpha) \\
&\qquad\times\sum_{j+1\le \ell_1,\dots,\ell_m\le k}\frac{1}{(t-w_{\ell_1})^{\alpha_1}}\cdots\frac{1}{(t-w_{\ell_m})^{\alpha_m}}\Bigg].
\end{aligned}
$$

satisfies the required derivative estimates on G.

Expressing $R^{(n+1)}(t)$ in the form (3.7) we see from (3.9) that the coefficients

$$c_m(n+1) = \sum_{\substack{|\alpha|=n+1\\ l(\alpha)=m}} d(\alpha)$$

satisfy the recursive formulae

$$c_{n+1}(n+1) = 1, \qquad c_k(n+1) = nc_k(n) + c_{k-1}(n), \quad k = 1,\dots,n,$$

where $c_0(n) = 0$. These are the defining formulae for Stirling numbers of the second kind; the equivalent property for these numbers as the coefficients of the polynomial with roots at consecutive negative integers can be easily derived by induction:

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} c_k(n)x^k$$

and so

$$x(x+1)\cdots(x+n) = \sum_{k=1}^{n} c_k(n)x^{k+1} + \sum_{k=1}^{n} nc_k(n)x^k$$

$$= nc_1(n)x + \sum_{k=2}^{n}(nc_k(n) + c_{k-1}(n))x^k + x^{n+1} = \sum_{k=1}^{n+1} c_k(n+1)x^k$$

by the above recursive formulae, completing the proof of Lemma 3.5. ■

REMARK 3.6. The number of terms in the sum occurring in (3.7) is

$$\sum_{m=1}^{n} \sum_{\substack{|\alpha|=n \\ l(\alpha)=m}} d(\alpha)(k-j)^m = \sum_{m=1}^{n} c_m(n)(k-j)^m$$
$$= (k-j)(k-j+1)\cdots(k-j+n-1).$$

Suppose that $\mathcal{A} = \{\alpha\}$ is an $O(1)$ collection of strictly positive multi-indices of size n; that is, $|\alpha| = n$ for every $\alpha \in \mathcal{A}$. Furthermore, suppose for each $\alpha \in \mathcal{A}$ there is an associated collection $\{z_{\ell_1}, \ldots, z_{\ell_m}\}$ where $m = l(\alpha)$ and $\ell_1, \ldots, \ell_m \le j$. Then the argument establishing (3.5) shows

$$\Big|\sum_{\alpha\in\mathcal{A}} (t - z_{\ell_1})^{-\alpha_1} \cdots (t - z_{\ell_m})^{-\alpha_m}\Big| \sim |t|^{-n}.$$

Hence in the case $j < k$, Lemma 3.5 implies the bound

$$|R^{(n)}(t)/R(t)| \sim |t|^{-n} \tag{3.10}$$

for $|t| \in G$ if $A \ge 1$ is large enough with no further restriction on $j < k$.

4. Proof of Theorem 1.1. Here we give the details of the weak-type estimate (1.2) for

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}} f(x-t)e^{iP(t)/Q(t)}\,\frac{dt}{t}.$$

We first apply Lemma 3.1 to Q and decompose $\mathbb{R}^+$ into an $O(1)$ collection of gaps and dyadic intervals with respect to Q. On a dyadic interval $D = [a, b]$,

$$T_D f(x) := \int_{|t|\in D} f(x-t)e^{iP(t)/Q(t)}\,\frac{dt}{t}$$

is bounded on L^1 with $O(1)$ bounds since $b/a = O(1)$. Therefore we are reduced to establishing (1.2) for

$$T_G f(x) := \int_{|t|\in G} f(x-t)e^{iP(t)/Q(t)}\,\frac{dt}{t} \tag{4.1}$$

where G is a gap on which $|Q(t)| \sim |q_k|\,|t|^k$ for some $k \ge 0$. At this point we could use Lemma 3.1 again and decompose G into gaps and dyadic intervals with respect to P, reducing matters to an interval where $|P(t)| \sim |p_j|\,|t|^j$ for some $j \ge 0$ as well and hence $|P(t)/Q(t)| \sim |p_j q_k^{-1}|\,|t|^{j-k}$. It will be essential that we bound the second derivative of P/Q from below on this interval. However, in order to do this when $k \le j$, Lemma 3.3 requires that $j \ne k$ and $j \ne k+1$; see (3.6). Fortunately it turns out that we can prevent the situations $j = k$ and $j = k+1$ from arising by observing that the weak-type estimate (1.2) is unaffected if we perturb our rational phase $P(t)/Q(t) - at - b$ by any linear polynomial. In fact

$$T_G f(x) = e^{-iax} e^{ib} \int_{|t| \in G} e^{ia(x-t)} f(x-t) e^{i[P(t)/Q(t) - at - b]} \frac{dt}{t}$$

and the L^1 norm of f is not affected by modulations of the form $e^{iay} f(y)$. We note that this is not the case for the Hardy space H^1 norm.

The idea is to choose a and b appropriately so that in the difference

$$P(t)/Q(t) - a - bt = N(t)/Q(t)$$

the numerator $N(t) = P(t) - (a+bt)Q(t) = \sum_j n_j t^j$ has vanishing kth and $(k+1)$th coefficients; that is, $n_k = n_{k+1} = 0$, putting us in a position to use Lemma 3.1 with respect to $N(t)$, decomposing the interval G further into gaps and dyadic intervals so that on a gap, $|N(t)| \sim |t|^j$ for some $j \ne k$ or $k+1$. Hence we will be able to bound from below the second derivative of $N(t)/Q(t)$ on such an interval; see (3.6) and (3.10). A little linear algebra shows that we can choose a and b so that $n_k = n_{k+1} = 0$ if and only if $q_k^2 \ne q_{k-1} q_{k+1}$. In the cases $k = 0$ and $k = \text{degree}(Q)$, we interpret the right hand side as zero and so $q_k^2 \ne q_{k-1}q_{k+1}$ holds in these trivial cases. One easily checks that an appropriate choice for a and b can be made to guarantee $n_k = n_{k+1} = 0$ when either $k = 0$ or $k = \text{degree}(Q)$.

We will now see that $q_k^2 \ne q_{k-1}q_{k+1}$ is indeed the case since G is a gap on which $|Q(t)| \sim |q_k|\,|t|^k$. Specifically we will use the fact that the kth root w_k of Q is separated from the $(k+1)$th root w_{k+1} if G is nonempty. The kth coefficient q_k of Q is related to the roots of Q by $|q_k| \sim |b| \prod_{\ell=k+1}^n |w_\ell|$; this is the content of part (ii) of Lemma 3.1. Hence

$$|q_k^2| \sim |b|^2 \prod_{\ell=k+1}^{n} |w_\ell|^2. \tag{4.2}$$

For $q_{k-1}q_{k+1}$ we have

$$\begin{aligned} q_{k-1}q_{k+1} &= b^2 \Big[\sum_{\ell_1 < \cdots < \ell_{n-k+1}} w_{\ell_1} \cdots w_{\ell_{n-k+1}} \Big] \Big[\sum_{\ell_1 < \cdots < \ell_{n-k-1}} w_{\ell_1} \cdots w_{\ell_{n-k-1}} \Big] \\ &= b^2 \sum_{\substack{\ell_1 < \cdots < \ell_{n-k+1} \\ \ell'_1 < \cdots < \ell'_{n-k-1}}} w_{\ell_1} \cdots w_{\ell_{n-k+1}} w_{\ell'_1} \cdots w_{\ell'_{n-k-1}}. \end{aligned}$$

Since in each summand defining q_{k-1}, $|w_{\ell_1}| \le |w_k| \le (1/A)|w_{k+1}|$, we have

$$|w_{\ell_1} \cdots w_{\ell_{n-k+1}} w_{\ell'_1} \cdots w_{\ell'_{n-k-1}}| \le \frac{1}{A} \prod_{\ell=k+1}^{n} |w_\ell|^2.$$

Therefore by (4.2),

$$|q_{k-1}q_{k+1}| \le \frac{1}{A}|b|^2 \prod_{\ell=k+1}^{n} |w_\ell|^2 \lesssim \frac{1}{A}|q_k|^2$$

and hence $q_k^2 \ne q_{k-1}q_{k+1}$ if $A \ge 1$ is chosen large enough.

So in order to prove the weak-type $(1,1)$ estimate in Theorem 1.1 it suffices to bound the integral operator T_G defined in (4.1) where on G we have $|R(t)| = |P(t)/Q(t)| \sim |p_j q_k^{-1}|\,|t|^{j-k}$ and $j \ne k,\ k+1$. Before proceeding, we make a simple scaling by changing variables $t \mapsto ct$ in (4.1) (the bound in the weak-type estimate (1.2) is unaffected). Choosing the constant c appropriately, we may assume that $|R(t)| \sim |t|^{j-k}$ for $|t| \in G$. Hence on G we have $|R^{(n)}(t)| \lesssim |t|^{j-k-n}$ for any $n \ge 0$; furthermore if $n = 0$, 1 or 2, then $|R^{(n)}(t)| \sim |t|^{j-k-n}$ if $|t| \in G$. For ease of notation later on, we rewrite the exponents j, k as $r := j - k$ and so $r \ne 0, 1$. We record that for $|t| \in G$ and for every $n \ge 0$,

$$|R^{(n)}(t)| \lesssim |t|^{r-n} \quad \text{whereas} \quad |R^{(n)}(t)| \sim |t|^{r-n} \quad \text{for } n = 0, 1, 2. \tag{4.3}$$

We split the operator $T_G = T_G^1 + T_G^2$ into two pieces where

$$T_G^1 f(x) = \int_{|t| \in G \cap [0,1]} f(x-t) e^{iR(t)}\, dt/t,$$

$$T_G^2 f(x) = \int_{|t| \in G \cap [1,\infty)} f(x-t) e^{iR(t)}\, dt/t.$$

From Theorem 1.1 in [6], it follows that both T_G^1 and T_G^2 are bounded on L^2 with bounds uniform in the coefficients of R. We now break up the analysis into two cases: (I) $r \ge 0$ (and hence $r \ge 2$ in fact) and (II) $r < 0$.

4.1. Case (I): when $r \ge 0$. In this case, the kernel $K(t) := e^{iR(t)}/t$ is a Calderón–Zygmund kernel on $G \cap [0,1]$, satisfying the bounds $|K(t)| \lesssim |t|^{-1}$ and $|K'(t)| \lesssim |t|^{-2}$ for $|t| \in G \cap [0,1]$. Since T_G^1 is bounded on L^2, it is a classical Calderón–Zygmund singular integral operator satisfying the weak-type estimate (1.2) with bounds uniform in the coefficients of R. For T_G^2 we follow the arguments in [2], using the classical Calderón–Zygmund decomposition lemma to decompose our L^1 function $f = g + b$ at level α, into a good function g (with nice L^∞ bounds $\|g\|_\infty \lesssim \alpha$ so that the L^2 theory of T_G^2 controls the bound (1.2) on g), and a bad function $b = \sum_I b_I$ where the collection of disjoint dyadic intervals $\{I\}$ with side lengths $2^{L(I)}$ have the property that if $\operatorname{dist}(I, J) \le 2^{L(I)}$, then $|L(I) - L(J)| \lesssim 1$ for any two such intervals and $\sum_I |I| \lesssim \|f\|_1/\alpha$. Furthermore, $\|b_I\|_1 \lesssim \alpha|I|$ and $\sum_I \|b_I\|_1 \lesssim \|f\|_1$ for each I.

Matters are then reduced to establishing the weak-type bound (1.2) for the bad function b off the exceptional set $\bigcup_I I$. Here we deviate from the classical Calderón–Zygmund paradigm where one uses L^1 estimates off the exceptional set. Instead we use L^2 estimates exploiting the oscillation of the phase R and so we do not need any cancellation between positive and negative values of t in T_G. Thus we concentrate on establishing (1.2) for

$$T^+ b(x) = \int_{t\in G\cap[1,\infty)} b(x-t)e^{iR(t)}\,dt/t.$$

Furthermore we split the bad function $b = b_{\text{small}} + b_{\text{large}}$ into two parts where $b_{\text{small}} = \sum_{I:\, L(I)<0} b_I$ and $b_{\text{large}} = \sum_{I:\, L(I)\ge 0} b_I$, and apply T^+ separately to these functions. We concentrate first on the more difficult b_{large} and establish

$$\alpha\Big|\Big\{x \notin \bigcup_I I^* : |T^+ b_{\text{large}}(x)| \ge \alpha\Big\}\Big| \lesssim \|f\|_1 \tag{4.4}$$

where I^* is the 2-fold dilate of I. For $x \notin \bigcup_I I^*$, we observe that $T^+b(x) = \sum_I T^{L(I)} b_I(x)$ where $T^L g(x) = \sum_{k\ge L} T_k g(x)$,

$$T_k g(x) = 2^{-k} \int_{t\in G} g(x-t)\psi(2^{-k}t)e^{iR(t)}\,dt$$

and $\psi \in C_0^\infty(\mathbb{R})$ is an appropriate function supported in $[1,2]$.

According to [2], the bound (4.4) will follow from certain estimates on the kernel

$$\begin{aligned} &L_{\ell,m}(x,y) \\ &\quad= \sum_{j\ge\ell}\sum_{k\ge m} 2^{-k-j} \int_{x-z,y-z\in G} \psi(2^{-k}(x-z))\psi(2^{-j}(y-z))e^{i(R(x-z)-R(y-z))}\,dz \end{aligned}$$

of $(T^\ell)^* T^m$; namely, if $0 \le m \le \ell$,

$$|L_{\ell,m}(x,y)| \lesssim \min((1+\ell-m)2^{-\ell}, |x-y|^{-2}). \tag{4.5}$$

Before establishing (4.5), we recall how (4.4) follows from it. Using the identity $T^+b(x) = \sum_I T^{L(I)} b_I(x)$, valid off the exceptional set $\bigcup_I I^*$, we apply Chebyshev's inequality to bound the left side of (4.4) by

$$\alpha^{-2}\Big\|\sum_{I:\, L(I)\ge 0} T^{L(I)} b_I\Big\|_2^2 = \alpha^{-2} \sum\sum_{I,J:\, L(I),L(J)\ge 0} \langle (T^{L(I)})^* T^{L(J)} b_J, b_I\rangle.$$

We split the double sum into two parts, depending on the relative sizes of $L(I)$ and $L(J)$; we will consider only that part of the sum where $L(J) \le L(I)$, without loss of generality. We fix I and show that the sum in J has the bound

$$\sum_{J:\, L(J)\le L(I)} \langle (T^{L(I)})^* T^{L(J)} b_J, b_I\rangle \lesssim \alpha \|b_I\|_1, \tag{4.6}$$

which gives us the desired estimate (4.4) after summing over the dyadic intervals I.

We split the sum in (4.6) into two parts, where $\mathrm{dist}(J,I) \le 2^{L(I)}$ and where $\mathrm{dist}(J,I) \ge 2^{L(I)}$. For those dyadic intervals J with $\mathrm{dist}(J,I) \le 2^{L(I)}$, we have $|L(J)-L(I)| \lesssim 1$ implying that there are $O(1)$ terms in the J sum in this case. Furthermore in this case, $|L_{L(I),L(J)}(x,y)| \lesssim 2^{-L(I)}$ by (4.5) implying

$$\langle (T^{L(I)})^* T^{L(J)} b_J, b_I\rangle \lesssim |I|^{-1} \|b_J\|_1 \|b_I\|_1 \lesssim \alpha \|b_I\|_1.$$

Hence

$$\sum_{\substack{J:\, L(J)\le L(I)\\ \mathrm{dist}(J,I)\le 2^{L(I)}}} \langle (T^{L(I)})^* T^{L(J)} b_J, b_I\rangle \lesssim \alpha \|b_I\|_1,$$

which is the estimate (4.6) for this part of the sum.

We now examine the sum in J with $L(J) \le L(I)$ and $\mathrm{dist}(J,I) \ge 2^{L(I)}$. Here we will use the bound $|L_{L(I),L(J)}(x,y)| \lesssim |x-y|^{-2}$ from (4.5) implying that

$$\langle (T^{L(I)})^* T^{L(J)} b_J, b_I\rangle \lesssim \|b_I\|_1 \|b_J\|_1 \min_{(x,y)\in I\times J} |x-y|^{-2} \lesssim \alpha \|b_I\|_1 \int_J |x_I - y|^{-2}\, dy$$

where x_I denotes the centre of I. Here we used the fact that $|x-y|$ is about constant as (x,y) varies over $I \times J$ when $\mathrm{dist}(J,I) \ge 2^{L(I)}$ and $L(J) \le L(I)$. Now summing over the disjoint intervals J, we see

$$\sum_{J:\, \mathrm{dist}(J,I)\ge 2^{L(I)}} \int_J |x_I - y|^{-2}\, dy \lesssim 2^{-L(I)} \lesssim 1$$

since $L(I) \ge 0$ and so

$$\sum_{\substack{J:\, L(J)\le L(I)\\ \mathrm{dist}(J,I)\ge 2^{L(I)}}} \langle (T^{L(I)})^* T^{L(J)} b_J, b_I\rangle \lesssim \alpha \|b_I\|_1,$$

which completes the proof of (4.6) and hence (4.4) once we establish the estimate (4.5).

The estimate $|L_{\ell,m}(x,y)| \lesssim (1+\ell-m)2^{-\ell}$ in (4.5) for $0 \le m \le \ell$ follows from the size of the z integration $2^{-\min(j,k)}$ of the integral defining $L_{\ell,m}(x,y)$. Hence

$$|L_{\ell,m}(x,y)| \lesssim \sum_{j\ge \ell} \sum_{k\ge m} 2^{-j-k} 2^{-\min(j,k)} \lesssim (1+\ell-m)2^{-\ell}.$$

To see $|L_{\ell,m}(x,y)| \lesssim |x-y|^{-2}$, we will integrate by parts twice to estimate the integral

$$I_{j,k}(x,y) := 2^{-j-k} \int_{x-z,y-z\in G} \psi(2^{-k}(x-z))\psi(2^{-j}(y-z))e^{i(R(x-z)-R(y-z))}\,dz.$$

This requires a bound from below on the derivative of the phase function $\phi(z) := R(x-z)-R(y-z)$ as well as bounds from above on the first, second and third derivatives of ϕ. We write

$$\phi^{(n)}(z) = (-1)^n(x-y)\int_0^1 R^{(n+1)}(y-z+s(x-y))\,ds$$

for the nth derivative of ϕ and make the simple observation that $y-z+s(x-y)\in G$ for all $0\le s\le 1$ since $x-z, y-z\in G$ and G is an interval. Recall that by scaling and conjugating our operator with appropriate modulations from the outset, we have put ourselves in the favourable position where for every $n\ge 0$, $|R^{(n+1)}(w)| \lesssim |w|^{r-n-1}$ on G for some positive integer $r\ge 2$; furthermore, $|R''(w)| \sim |w|^{r-2}$ on G (see (4.3)). This translates into bounds for $\phi^{(n)}$; namely, for z such that $x-z$, $y-z\in G$, $x-z\sim 2^k$ and $y-z\sim 2^j$, we have

$$|\phi'(z)| \sim |x-y|2^{\max(j,k)(r-2)}, \qquad |\phi''(z)| \lesssim |x-y|\max(j,k)\,2^{\max(j,k)(r-3)}$$

and

$$|\phi'''(z)| \lesssim |x-y|\cdot\begin{cases}\max(j,k)2^{\max(j,k)(r-4)} & \text{if } r\ge 3,\\ 2^{-j-k} & \text{if } r=2.\end{cases}$$

Using the differential operator $D := [i/\phi'(z)](d/dz)$ so that $De^{i\phi(z)} = e^{i\phi(z)}$, we have, by integrating by parts twice,

$$I_{j,k}(x,y) = 2^{-j-k}\int_{x-z,y-z\in G} [D^*]^2\big(\psi(2^{-k}(x-z))\psi(2^{-j}(y-z))\big)e^{i\phi(z)}\,dz$$

where $D^*g(z) = (d/dz)[g(z)/i\phi'(z)]$ is the formal adjoint of D. Using the above derivative bounds on ϕ, we see that

$$|I_{j,k}(x,y)| \lesssim 2^{-\max(j,k)}\,|x-y|^{-2},$$

which implies the estimate $|L_{\ell,m}(x,y)| \lesssim |x-y|^{-2}$ in (4.5).

To finish Case (I) where $r\ge 0$, we need to establish (4.4) with b_{large} replaced with $b_{\text{small}} = \sum_{I:\,L(I)<0} b_I$. Instead of the original operator T^+, it suffices to apply T^0 (which differs from T^+ by an operator bounded uniformly on L^1) to b_{small} and verify (4.4). Again applying Chebyshev's inequality to bound the left side of (4.4) by

$$\alpha^{-2}\Big\|\sum_{I:\,L(I)<0} T^0 b_I\Big\|_2^2 = \alpha^{-2}\sum\sum_{I,J:\,L(I),L(J)<0}\langle (T^0)^*T^0 b_J, b_I\rangle, \tag{4.7}$$

we use the basic estimates $|L_{0,0}(x,y)| \lesssim \min(1,|x-y|^{-2})$ from (4.5) for the kernel of $(T^0)^*T^0$ to prove

$$\sum_{J:\,L(J)\le L(I)} \langle (T^0)^*T^0 b_J, b_I\rangle \lesssim \alpha \|b_I\|_1 \tag{4.8}$$

for each fixed dyadic interval I. Summing over the disjoint intervals I successfully bounds (4.7) for those intervals I,J with $L(J)\le L(I)$. Of course the symmetric sum over those intervals with $L(I)\le L(J)$ also holds. The proof of (4.8) is similar to (4.6); we split the sum into those J with $\mathrm{dist}(J,I) \le 1$ and those with $\mathrm{dist}(J,I)\ge 1$. The bound $|L_{0,0}(x,y)|\lesssim 1$ implies

$$\sum_{\substack{J:\,L(J)\le L(I)\\ \mathrm{dist}(J,I)\le 1}} \langle (T^0)^*T^0 b_J, b_I\rangle \lesssim \alpha\|b_I\|_1 \sum_{J:\,\mathrm{dist}(J,I)\le 1} |J| \lesssim \alpha\|b_I\|_1$$

by the disjointness of the intervals J and the fact that those J with $L(J)\le L(I)\le 0$ and $\mathrm{dist}(J,I)\le 1$ cover an interval of length at most 1.

We now examine the sum (4.8) when $L(J)\le L(I)$ and $\mathrm{dist}(J,I)\ge 1$. Here we will use the bound $|L_{0,0}(x,y)|\lesssim |x-y|^{-2}$ implying that

$$\langle (T^0)^*T^0 b_J, b_I\rangle \lesssim \|b_I\|_1\|b_J\|_1 \min_{(x,y)\in I\times J} |x-y|^{-2} \lesssim \alpha\|b_I\|_1 \int_J |x_I-y|^{-2}\,dy$$

as before. Now summing over the disjoint intervals J, we see

$$\sum_{J:\,\mathrm{dist}(J,I)\ge 1} \int_J |x_I-y|^{-2}\,dy \lesssim 1$$

since $L(I)<0$ and $\mathrm{dist}(J,I)\ge 1$. Hence

$$\sum_{\substack{J:\,L(J)\le L(I)\\ \mathrm{dist}(J,I)\ge 1}} \langle (T^0)^*T^0 b_J, b_I\rangle \lesssim \alpha\|b_I\|_1,$$

which completes the proof of (4.8) and hence Case (I).

4.2. Case (II): when $r<0$. In this case, the kernel $K(t):=e^{iR(t)}/t$ is a Calderón–Zygmund kernel on $G\cap[1,\infty)$, satisfying the bounds $|K(t)|\lesssim |t|^{-1}$ and $|K'(t)|\lesssim |t|^{-2}$ for $|t|\in G\cap[1,\infty)$. Since T_G^2 is bounded on L^2, it is a classical Calderón–Zygmund singular integral operator and so satisfies the weak-type estimate (1.2) with bounds uniform in the coefficients of R. For T_G^1 we use the following result of C. Fefferman about strongly singular integral operators (see [4]).

THEOREM 4.3. *Let K be a tempered distribution on $\mathbb{R}$, agreeing with a locally integrable function away from the origin with compact support. Suppose that for all $\xi\in\mathbb{R}$, we have $|\hat{K}(\xi)|\le A(1+|\xi|)^{-\theta/2}$, and for all*

$y \in \mathbb{R}$, we have

$$\int_{|x|\geq 2|y|^{1-\theta}} |K(x-y) - K(x)|\, dx \leq A \tag{4.9}$$

for some $A > 0$ and $0 \leq \theta < 1$. Then the operator T given by convolution with K is weak-type $(1,1)$ with bounds depending only on A and θ.

We will apply this theorem to the kernel of T_G^1. Again, due to the oscilllation of the phase R, we do not need any possible cancellation between positive and negative values of t and so we treat them separately. We will verify the Fourier decay estimate $|\hat{K}(\xi)| \lesssim (1+|\xi|)^{-\theta/2}$ and (4.9) for $K(t) = \chi_{[0,1]\cap G}(t) e^{iR(t)}/t$ with $\theta = |r|/(|r|+1)$. Similar estimates hold when t is negative and so for the entire kernel of T_G^1, giving us the desired estimate (1.2) for T_G^1 in Case (II).

Since $|R^{(n)}(t)| \sim |t|^{-|r|-n}$ for $t \in G$ and every $n \geq 0$ (see (3.10)), we see that for $|x| \geq 2|y|$,

$$|K(x-y) - K(x)| \lesssim |y|/|x|^{2+|r|}$$

and so the regularity condition (4.9) holds for $\theta = |r|/(|r|+1)$ with a constant A which can be taken to be independent of the coefficients of R. Next we claim that the uniform estimate

$$\left| \int_{t\in[0,1]\cap G} e^{i[R(t)-\xi t]} \frac{dt}{t} \right| \lesssim (1+|\xi|)^{-|r|/2(|r|+1)} \tag{4.10}$$

holds which shows $|\hat{K}(\xi)| \lesssim (1+|\xi|)^{-\theta/2}$ for $\theta = |r|/(|r|+1)$. This will complete our analysis of Case (II) by Theorem 4.3. Since R'' does not vanish on G, there is at most one critical point of the phase $R(t) - \xi t$, and if such a critical point t_* exists, then $|\xi| = |R'(t_*)| \sim |t_*|^{-|r|-1}$ or $|t_*| \sim |\xi|^{-1/(|r|+1)}$. This only happens if $|\xi| \gtrsim 1$. If $|\xi| \lesssim 1$, there is no critical point and the estimate $|\hat{K}(\xi)| \lesssim 1$ follows easily from an integration by parts argument. We will assume from now on that $|\xi| \gtrsim 1$ and the critical point t_* exists. We split the integral in (4.10) into three parts

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = \int_0^{(1/B)t_*} \ldots\, dt/t + \int_{(1/B)t_*}^{Bt_*} \ldots\, dt/t + \int_{Bt_*}^{1} \ldots\, dt/t$$

for some absolute, uniform constant B.

It is understood that the integrals defining I, II and III are taken over our gap G as well so that the derivative estimates (3.10) of R hold. In particular, on $[Bt_*, 1]$ the estimate $|R'(t) - \xi| \gtrsim |\xi|$ holds if B is large enough and so integrating by parts gives the (better than desired) estimate $|III| \lesssim |\xi|^{-|r|/(|r|+1)}$. Similarly, on the interval $[0, (1/B)t_*]$ we have the bound $|R'(t) - \xi| \gtrsim |t|^{-|r|-1}$, which together with our upper bounds on

R'' gives the same estimate $|\mathrm{I}| \lesssim |\xi|^{-|r|/(|r|+1)}$ by an integration by parts argument. Finally we turn to II, which is the main contribution to $\hat{K}$. Using the bound $|R''(t)| \sim |t|^{-|r|-2} \sim |\xi|^{(|r|+2)/(|r|+1)}$ on $[[1/B]t_*, Bt_*]$, we can apply van der Corput's lemma (see for example, Proposition 2 in Chapter VIII of [11]), together with an integration by parts, to see the desired estimate $|\mathrm{II}| \lesssim |\xi|^{-|r|/2(|r|+1)}$, which completes the proof that $|\hat{K}(\xi)| \lesssim (1+|\xi|)^{-|r|/2(|r|+1)}$ and hence Case (II).

5. Proof of Theorem 1.2. The theorem comes in two parts, depending on the relationship of the degrees of P and Q defining our rational phase $R = P/Q$. Recall that we cannot expect to obtain bounds which are uniform in the coefficients of R. We split our operator T in (1.1) into three parts $T = T_1 + T_2 + T_3$ where

$$T_1 f(x) := \int_{\mathbb{R}} f(x-t)\psi_1(t)e^{iR(t)}\,\frac{dt}{t},$$

$$T_2 f(x) = \int_{\mathbb{R}} f(x-t)\psi_2(t)e^{iR(t)}\,\frac{dt}{t},$$

$$T_3 f(x) = \int_{\mathbb{R}} f(x-t)\psi_3(t)e^{iR(t)}\,\frac{dt}{t};$$

the three smooth functions are even and satisfy $\psi_1(t)+\psi_2(t)+\psi_3(t) = 1$ for all $t \in \mathbb{R}$. The cut-off function ψ_1 is supported in a sufficiently small neighbourhood of the origin, $\psi_2(t)$ vanishes for $|t|$ small and $|t|$ large and $\psi_3(t)$ is supported for $|t|$ sufficiently large. The operator T_2 maps $H^1(\mathbb{R})$ into itself but with bounds that will depend on the coefficients of R in general. By classical Hardy space theory (see for example Theorem 4 in Chapter III of [11]), this follows from the fact that T_2 is bounded on L^2 and the kernel K_2 of T_2 satisfies the regularity estimates

$$|K(x)| \le C|x|^{-1} \quad \text{and} \quad |K(x-y)-K(x)| \le C|y|/|x|^2 \quad \text{for } |x| \ge 2|y| \tag{5.1}$$

for some constant C which depends in general on the coefficients of R. Therefore to prove Theorem 1.2, it suffices to concentrate on T_1 and T_3.

The choice of ψ_1, ψ_2 and ψ_3 will depend on the coefficients of R and be such that the $|t|$ support of ψ_1 will be contained in the gap $G_0 := [0, (1/A)s_1]$ at the origin and that of ψ_3 contained in the gap $G_\infty := [As_3, \infty)$ at infinity (here s_1 and s_3 are the smallest and largest modulus of all the roots of P and Q, respectively). Hence $|R(t)| \sim c|t|^{r_1}$ for $|t| \in G_0$ and $|R(t)| \sim d|t|^{r_3}$ for $|t| \in G_\infty$ for some $r_1, r_3 \in \mathbb{Z}$ and $c, d > 0$. We may assume that both exponents r_1 and r_3 are nonzero since we are at liberty to change the phase R by any constant $R(t) - c$ without affecting the Hardy space norm $H^1(\mathbb{R})$ (here general linear perturbations $R(t) - a - bt$ are not allowed as was the case for the weak-type $(1,1)$ estimates). In fact the $r_1 = j_1 - k_1$ exponent arises from the lowest terms $P(t) = p_d t^d + \cdots + p_{j_1} t^{j_1}$, $Q(t) = q_e t^e + \cdots + q_{k_1} t^{k_1}$ in P and Q; if $j_1 = k_1$, then choosing c such that $p_{j_1} = c q_{j_1}$ guarantees that the new rational phase $R(t) - c = [P(t) - cQ(t)]/Q(t)$ for the operator T_1 behaves like t^{r_1} with $r_1 \neq 0$. Also the exponent $r_3 = d - e$ is the difference of the degrees of P and Q; if $d = e$, then choosing c such that $p_d = c q_d$ guarantees that the new rational phase $R(t) - c$ for the operator T_3 behaves like t^{r_3} with $r_3 \neq 0$. **Important:** although we may change the phase in the operators T_1 and T_3 to guarantee that $r_1, r_3 \neq 0$, the dichotomy degree(P) = degree$(Q) + 1$ or degree$(P) \neq$ degree$(Q) + 1$ remains unchanged!

Therefore from (3.6) and (3.10), we may assume that the phase R in T_1 satisfies

$$|R(t)| \sim c|t|^{r_1} \quad \text{and} \quad |R'(t)| \sim c'|t|^{r_1 - 1} \tag{5.2}$$

for $t \in \text{support}(\psi_1)$, $c, c' > 0$ and some $r_1 \neq 0$. Also we may assume that the phase R in T_3 satisfies

$$|R(t)| \sim d\,|t|^{r_3} \quad \text{and} \quad |R'(t)| \sim d'\,|t|^{r_3 - 1} \tag{5.3}$$

for $t \in \text{support}(\psi_3)$, $d, d' > 0$ and some $r_3 \neq 0$. In both cases upper bounds $|R^{(n)}| \lesssim c_n |t|^{r-n}$ hold for every $n \geq 0$ for $r = r_1$ or $r = r_3$, respectively. Furthermore, if $r_3 \neq 1$ (which will be the case when degree$(P) \neq$ degree(Q) $+ 1$), then $|R''(t)| \sim d''|t|^{r_3 - 2}$ for $t \in \text{support}(\psi_3)$ and some $d'' \neq 0$.

Hence if $r_1 \geq 1$ in (5.2) and/or $r_3 \leq -1$ in (5.3), then the regularity condition (5.1) is satisfied by the kernels of T_1 and/or T_3, and together with the L^2 boundedness of these operators we can conclude that T_1 and/or T_3 maps H^1 into itself.

We are now in a position to give a proof of part (1) of Theorem 1.2, which assumes that degree$(P) \neq$ degree$(Q) + 1$. In particular this condition on the degrees implies that the exponent r_3 is not 1. From the remarks above it therefore suffices to prove $T_1, T_3 : H^1 \to H^1$ when $r_1 \leq -1$ and $r_3 \geq 2$. Let us consider T_1 first, where, as we have seen from the previous section, the kernel K_1 satisfies the conditions of Theorem 4.3 with $\theta = |r_1|/(|r_1| + 1)$. As shown by C. Fefferman and E. M. Stein [5], such strongly singular integral operators map H^1 into itself. For the operator T_3, we appeal to the work of D. Fan and Y. Pan [3] who proved that oscillatory singular integral operators map H^1 into itself for general phase functions which satisfy the derivative bounds $|R(t)| \sim a|t|^r$, $|R'(t)| \sim b|t|^{r-1}$, $|R''(t)| \sim c|t|^{r-2}$ and $|R'''(t)| \lesssim d|t|^{r-3}$ for some $r \neq 0, 1$; see [3].

Finally we turn to the proof of part (2) of Theorem 1.2, where we assume degree(P) = degree$(Q) + 1$ and in particular $r_3 = 1$. As before, the operators T_1 and T_2 map H^1 into itself and so it suffices to show that the operator T_3

does not map H^1 into $L^{1,q}$ for any $q < \infty$. Write

$$P(t) = p_d t^d + \cdots + p_0 \quad \text{and} \quad Q(t) = q_{d-1} t^{d-1} + \cdots + q_0$$

and $T_3 = T_3^1 + T_3^2$ where

$$T_3^1 f(x) = \int_{\mathbb{R}} f(x-t)\psi_3(t) e^{i((p_d/q_{d-1})t+c)} \frac{dt}{t},$$

$$T_3^2 f(x) = \int_{\mathbb{R}} f(x-t)\psi_3(t)\big[e^{iR(t)} - e^{i((p_d/q_{d-1})t+c)}\big] \frac{dt}{t}.$$

The constant c is chosen so that $p_d - cq_{d-1} - p_d q_{d-2}/q_{d-1} = 0$, which implies that the kernel $K_3^2(t) = \psi_3(t)[e^{iR(t)} - e^{i((p_d/q_{k-1})t+c)}]/t$ of T_3^2 is integrable, satisfying $|K_3^2(t)| \le C|t|^2$. Hence T_3^2 maps H^1 into H^1 and this leaves T_3^1, which we have already seen does not map H^1 into $L^{1,q}$ for any $q < \infty$. This completes the proof of Theorem 1.2. ∎

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