STUDIA MATHEMATICA 210 (1) (2012)

A remark on the div-curl lemma

by

Pierre Gilles Lemarié-Rieusset (Évry)

Abstract. We prove the div-curl lemma for a general class of function spaces, stable under the action of Calderón–Zygmund operators. The proof is based on a variant of the renormalization of the product introduced by S. Dobyinsky, and on the use of divergence-free wavelet bases.

1. Introduction. In 1992, Coifman, Lions, Meyer and Semmes [COIL] gave a new interpretation of the compensated compactness introduced by Murat and Tartar [MUR]. They showed that the functions considered by Murat and Tartar had a greater regularity than expected: they belonged to the Hardy space $\mathcal{H}^1$.

Moreover, they gave a new version of the div-curl lemma of Murat and Tartar:

Theorem 1.1. *If $1 < p < \infty$, $q = p/(p-1)$, $\vec{f} \in (L^p(\mathbb{R}^d))^d$ and $\vec{g} \in (L^q(\mathbb{R}^d))^d$, then*

$$\mathbf{div}\, \vec{f} = 0 \ \textit{and}\ \mathbf{curl}\, \vec{g} = \vec{0} \ \Rightarrow \vec{f} \cdot \vec{g} \in \mathcal{H}^1.$$

There are many proofs of this result. We shall rely mainly on the proof by S. Dobyinsky, based on the renormalization of the product introduced in [DOB].

As pointed out to me by Grzegorz Karch, it is easy to see that this result can be extended to a large class of function spaces. For instance, we have the straightforward consequence of Theorem 1.1 for weak Lebesgue spaces $L^{p,*}$ (better viewed as Lorentz spaces $L^{p,\infty}$) and their preduals $L^{q,1}$:

Corollary 1.2. *If $1 < p < \infty$, $q = p/(p-1)$, $\vec{f} \in (L^{p,\infty}(\mathbb{R}^d))^d$ and $\vec{g} \in (L^{q,1}(\mathbb{R}^d))^d$, then*

$$\mathbf{div}\, \vec{f} = 0 \ \textit{and}\ \mathbf{curl}\, \vec{g} = \vec{0} \ \Rightarrow \vec{f} \cdot \vec{g} \in \mathcal{H}^1,$$

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 46E99.
Key words and phrases: div-curl lemma, singular integrals, divergence-free wavelets.

DOI: 10.4064/sm210-1-5

and

$$\mathbf{div}\,\vec{g} = 0 \ \mathit{and}\ \ \mathbf{curl}\,\vec{f} = \vec{0} \ \Rightarrow \vec{f}\cdot\vec{g} \in \mathcal{H}^1.$$

Proof. All we need is the projection operators that lead to the Helmhotz decomposition of a vector field: $\mathrm{Id} = \mathbb{P} + \mathbb{Q}$ where $\mathbb{Q}$ is the projection onto irrotational vector fields,

$$\mathbb{Q}\,\vec{h} = \vec{\nabla}\frac{1}{\Delta}\,\mathbf{div}\,\vec{h},$$

and $\mathbb{P}$ the projection operator onto solenoidal vector fields. Those projection operators are matrices of singular integral operators and thus are bounded on Lebesgue spaces L^r, $1 < r < \infty$, and, by interpolation, on Lorentz spaces $L^{r,t}$, $1 < r < \infty$, $1 \le t \le \infty$.

Let $\epsilon > 0$ be such that $\epsilon < \min(1/p, 1/q)$. We write $1/p_+ = 1/p + \epsilon$, $1/p_- = 1/p - \epsilon$, $1/q_+ = 1/q + \epsilon$ and $1/q_- = 1/q - \epsilon$. If $\vec{f} \in (L^{p,\infty}(\mathbb{R}^d))^d$, we can write, for every $A > 0$, $\vec{f} = \vec{\alpha}_A + \vec{\beta}_A$ with $\|\vec{\alpha}_A\|_{L^{p_-}} \le CA\|\vec{f}\|_{L^{p,\infty}}$ and $\|\vec{\beta}_A\|_{L^{p_+}} \le CA^{-1}\|\vec{f}\|_{L^{p,\infty}}$. If $\mathbf{div}\,\vec{f} = 0$, we have moreover $\vec{f} = \mathbb{P}\vec{f} = \mathbb{P}\vec{\alpha}_A + \mathbb{P}\vec{\beta}_A$. On the other hand, if $\vec{g} \in (L^{q,1})^d$, we can write $\vec{g} = \sum_{j\in\mathbb{N}} \lambda_j \vec{g}_j$ with $\|\vec{g}_j\|_{L^{q_-}}\|\vec{g}\|_{L^{q_+}} \le 1$ and $\sum_{j\in\mathbb{N}} |\lambda_j| \le C\|\vec{g}\|_{L^{q,1}}$. If $\mathbf{curl}\,\vec{g} = 0$, we have moreover $\vec{g} = \mathbb{Q}\vec{g} = \sum_{j\in\mathbb{N}} \lambda_j\ \mathbb{Q}\vec{g}_j$. Let $A_j = \|\vec{g}_j\|_{L^{q_-}}^{1/2}\,\|\vec{g}_j\|_{L^{q_+}}^{-1/2}$. We then write

$$\vec{f}\cdot\vec{g} = \sum_{j\in\mathbb{N}} \lambda_j(\mathbb{P}\vec{\alpha}_{A_j}\cdot\mathbb{Q}\vec{g}_j + \mathbb{P}\vec{\beta}_{A_j}\cdot\mathbb{Q}\vec{g}_j)$$

and get (from the div-curl theorem of Coifman, Lions, Meyer and Semmes)

$$\begin{aligned}
\|\vec{f}\cdot\vec{g}\|_{\mathcal{H}^1} &\le C\sum_{j\in\mathbb{N}} |\lambda_j|(\|\mathbb{P}\vec{\alpha}_{A_j}\|_{L^{p_-}}\|\mathbb{Q}\vec{g}_j\|_{L^{q_+}} + \|\mathbb{P}\vec{\beta}_{A_j}\|_{L^{p_+}}\|\mathbb{Q}\vec{g}_j\|_{L^{q_-}}) \\
&\le C'\|\vec{f}\|_{L^{p,\infty}}\sum_{j\in\mathbb{N}} |\lambda_j|(A_j\|\vec{g}_j\|_{L^{q_+}} + {A_j}^{-1}\|\vec{g}_j\|_{L^{q_-}}) \\
&= C'\|\vec{f}\|_{L^{p,\infty}}\sum_{j\in\mathbb{N}} |\lambda_j| \le C''\|\vec{f}\|_{L^{p,\infty}}\|\vec{g}\|_{L^{q,1}}.
\end{aligned}$$

The proof for the case $\mathbf{div}\,\vec{g} = 0$ and $\mathbf{curl}\,\vec{f} = \vec{0}$ is similar. ∎

In this paper, we aim to find a general class of function spaces for which the div-curl lemma still holds. As we can see from the proof of Corollary 1.2, singular integral operators will play a key role in our result. In Section 2, we shall introduce Calderón–Zygmund pairs of function spaces which will allow us to prove such a general result. In Section 3, we recall the basics of divergence-free wavelet bases (as described in the book [LEMc]). In Section 4, we prove our main theorem. Then, in Section 4, we give examples of Calderón–Zygmund pairs of function spaces.

2. Calderón–Zygmund pairs of Banach spaces. We begin by recalling the definition of a Calderón–Zygmund operator:

DEFINITION 2.1. (A) A *singular integral operator* is a continuous linear mapping from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ whose distribution kernel $K(x,y) \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ (defined formally by the formula $Tf(x) = \int K(x,y)f(y)\,dy$) has its restriction outside the diagonal $x = y$ defined by a locally Lipschitz function with the following size estimates:

(i) $\sup_{x\neq y} |K(x,y)|\,|x-y|^d < \infty$,
(ii) $\sup_{x\neq y} |\vec{\nabla}_x K(x,y)|\,|x-y|^{d+1} < \infty$,
(iii) $\sup_{x\neq y} |\vec{\nabla}_y K(x,y)|\,|x-y|^{d+1} < \infty$.

For such an operator T, we define

$$\|T\|_{\mathrm{SIO}} = \|K(x,y)|x-y|^d\|_{L^\infty(\Omega)} + \|\vec{\nabla}_x K(x,y)|x-y|^{d+1}\|_{L^\infty(\Omega)} + \|\vec{\nabla}_y K(x,y)|x-y|^{d+1}\|_{L^\infty(\Omega)}$$

where K is the distribution kernel of T and $\Omega = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid x \neq y\}$

(B) A *Calderón–Zygmund operator* is a singular integral operator T which may be extended as a bounded operator on L^2:

$$\sup_{\varphi\in\mathcal{D},\, \|\varphi\|_2\le 1} \|T\varphi\|_2 < \infty.$$

We define CZO to be the space of Calderón–Zygmund operators, endowed with the norm

$$\|T\|_{\mathrm{CZO}} = \|T\|_{\mathcal{L}(L^2,L^2)} + \|T\|_{\mathrm{SIO}}.$$

We can now define our main tool:

DEFINITION 2.2. A *Calderón–Zygmund pair* is a pair (X,Y) of Banach spaces such that:

(i) We have the continuous embeddings $\mathcal{D}(\mathbb{R}^d) \subset X \subset \mathcal{D}'(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d) \subset Y \subset \mathcal{D}'(\mathbb{R}^d)$.
(ii) Let X_0 be the closure of $\mathcal{D}$ in X; then the dual X_0^* of X_0 (i.e. the space of bounded linear forms on X_0) coincides with Y with equivalence of norms: a distribution T belongs to Y if and only if there exists a constant C_T such that $|\langle T \mid \varphi\rangle_{\mathcal{D}',\mathcal{D}}| \le C_T\|\varphi\|_X$ for all $\varphi \in \mathcal{D}$.
(iii) Let Y_0 be the closure of $\mathcal{D}$ in Y. Then the dual Y_0^* coincides with X with equivalence of norms.
(iv) Every Calderón–Zygmund operator may be extended to a bounded operator on X_0 and on Y_0: there exists a constant C_0 such that, for every $T \in \mathrm{CZO}$ and every $\varphi \in \mathcal{D}$, we have $T(\varphi) \in X_0 \cap Y_0$ and

$$\|T\varphi\|_X \le C_0\|T\|_{\mathrm{CZO}}\|\varphi\|_X \quad\text{and}\quad \|T\varphi\|_Y \le C_0\|T\|_{\mathrm{CZO}}\|\varphi\|_Y.$$

By duality, we find that every Calderón–Zygmund operator extends to a bounded operator on X and Y: if T^* is defined by the formula

$$\langle T\varphi \,|\, \psi\rangle_{\mathcal{D}',\mathcal{D}} = \langle \varphi \,|\, T^*\psi\rangle_{\mathcal{D},\mathcal{D}'},$$

then $T \in \mathrm{CZO}$ implies $T^* \in \mathrm{CZO}$ and we may define $T(f)$ on X as the distribution $\varphi \mapsto \langle f \,|\, T^*(\varphi)\rangle_{Y_0^*,Y_0}$. The two definitions of T coincide on X_0.

For $m \in L^\infty$, the operator $T_m : \varphi \mapsto m\varphi$ belongs to CZO (with kernel $K(x,y) = m(x)\delta(x-y)$). The stability of X and Y under multiplication by bounded smooth functions (with the inequalities $\|mf\|_X \le C_0\|m\|_\infty\|f\|_X$ and $\|mf\|_Y \le C_0\|m\|_\infty\|f\|_Y$) shows that elements of X and Y are (complex) local measures and that X_0 and Y_0 are embedded into L^1_{loc}.

Our main result (to be proved in Section 4) is the following:

THEOREM 2.3. *Let (X,Y) be a Calderón–Zygmund pair of Banach spaces. If $\vec f \in X_0^d$ and $\vec g \in Y^d$, then*

$$\mathbf{div}\, \vec f = 0 \textit{ and } \mathbf{curl}\, \vec g = \vec 0 \;\Rightarrow\; \vec f \cdot \vec g \in \mathcal{H}^1$$

and

$$\mathbf{div}\, \vec g = 0 \textit{ and } \mathbf{curl}\, \vec f = \vec 0 \;\Rightarrow\; \vec f \cdot \vec g \in \mathcal{H}^1.$$

REMARK. The distribution $\vec f \cdot \vec g$ is well-defined, since $\vec f \in X_0^d$: if $\varphi \in \mathcal{D}$, then $\varphi \vec f \in X_0^d$ and $\vec g \in (X_0^*)^d$.

3. Divergence-free wavelet bases. In this section, we give a short review of properties of divergence-free wavelet bases. Wavelet theory was introduced in the 1980's as an efficient tool for signal analysis. Orthonormal wavelet bases were first constructed by Y. Meyer [LEMM], G. Battle [BAT] and P. G. Lemarié-Rieusset; a major advance was the construction of compactly supported orthonormal wavelets by I. Daubechies [DAU]. Then bi-orthogonal bases were introduced by A. Cohen, I. Daubechies and J.-C. Feauveau [COH]. Divergence-free wavelets were introduced by Battle and Federbush [BATF]. Compactly divergence-free wavelets were introduced by P. G. Lemarié-Rieusset [LEMa]; they are not orthogonal wavelets [LEMb], but have been explored for the numerical analysis of the Navier–Stokes equations [DER, URB].

Let $H_{\mathbf{div}=0}$ and $H_{\mathbf{curl}=0}$ be defined as

$$H_{\mathbf{div}=0} = \{\vec f \in (L^2)^d \mid \mathbf{div}\, \vec f = 0\} \text{ and } H_{\mathbf{curl}=0} = \{\vec f \in (L^2)^d \mid \mathbf{curl}\, \vec f = 0\}.$$

For $\vec f \in (L^2)^d$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, we define $\vec f_{j,k}$ as $\vec f_{j,k}(x) = 2^{jd/2}\vec f(2^j x - k)$. Let us recall the main results of [LEMa] (described as well in the book [LEMc]). The idea is to begin with a Hilbertian basis of compactly supported wavelets, associated to a multi-resolution analysis $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$. Associated to this multi-resolution analysis (with orthogonal projection operator Π_j onto V_j), there is a bi-orthogonal multi-resolution analysis (V_j^+),

(V_j^-) with projection $\Pi_{(j)}$ onto V_j^- orthogonally to V_j^+ such that $\frac{d}{dx} \circ \Pi_j = \Pi_{(j)} \circ \frac{d}{dx}$.

Starting from this one-dimensional setting, we now consider a bi-orthogonal multi-resolution analysis of $(L^2(\mathbb{R}^d))^d$, $(V_{j,1}, \dots, V_{j,d})$ and $(V_{j,1}^*, \dots, V_{j,d}^*)$, where $V_{j,k} = V_{j,k,1} \otimes \dots \otimes V_{j,k,d}$ with $V_{j,k,l} = V_j$ for $k \neq l$ and $V_{j,k,k} = V_j^-$ and $V_{j,k}^* = V_{j,k,1}^* \otimes \dots \otimes V_{j,k,d}^*$ with $V_{j,k,l}^* = V_j$ for $k \neq l$ and $V_{j,k,k}^* = V_j^+$. Let P_j be the projection operator onto $(V_{j,1}, \dots, V_{j,d})$ orthogonally to $(V_{j,1}^*, \dots, V_{j,d}^*)$. Its adjoint P_j^* is the projection operator onto $(V_{j,1}^*, \dots, V_{j,d}^*)$ orthogonally to $(V_{j,1}, \dots, V_{j,d})$. The point is that we have $P_j(\vec{\nabla} f) = \vec{\nabla}(\Pi_j f)$ and $\mathbf{div}(P_j^* \vec{f}) = \Pi_j^*(\mathbf{div}\, \vec{f})$.

Those projection operators P_j and P_j^* yield an accurate description of $H_{\mathbf{div}=0}$ and $H_{\mathbf{curl}=0}$:

Proposition 3.1 (Multi-resolution analysis for divergence-free or irrotational vector fields). *Let $N \in \mathbb{N}$. Then there exists a compact set $K_N \subset \mathbb{R}^d$ such that:*

(A) Multi-resolution analysis: *There exist*

- *functions $\vec{\varphi}_\xi$ and $\vec{\varphi}_\xi^*$ in $(L^2)^d$, $1 \le \xi \le d$,*
- *functions $\vec{\psi}_\chi$ and $\vec{\psi}_\chi^*$ in $(L^2)^d$, $1 \le \chi \le d(2^d - 1)$,*

such that:

(i) *$\vec{\varphi}_\xi$, $\vec{\varphi}_\xi^*$, $\vec{\psi}_\chi$ and $\vec{\psi}_\chi^*$ are supported in K_N.*

(ii) *$\vec{\varphi}_\xi$, $\vec{\varphi}_\xi^*$, $\vec{\psi}_\chi$ and $\vec{\psi}_\chi^*$ are of class $\mathcal{C}^N$.*

(iii) *For $l \in \mathbb{N}^d$ with $\sum_{i=1}^d l_i \le N$, we have $\int x^l \vec{\psi}_\chi \, dx = \int x^l \vec{\psi}_\chi^* \, dx = 0$.*

(iv) *For j, j' in $\mathbb{Z}$, k, k' in $\mathbb{Z}^d$, ξ, ξ' in $\{1, \dots, d\}$, and χ, χ' in $\{1, \dots, d(2^d - 1)\}$,*

$$\int \vec{\varphi}_{\xi,j,k} \cdot \vec{\varphi}^*_{\xi',j,k'} \, dx = \delta_{k,k'} \delta_{\xi,\xi'} \quad \text{and} \quad \int \vec{\psi}_{\chi,j,k} \cdot \vec{\psi}^*_{\chi',j',k'} \, dx = \delta_{j,j'} \delta_{k,k'} \delta_{\chi,\chi'}.$$

(v) *The operators P_j defined on $(L^2)^d$ by*

$$P_j \vec{f} = \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \xi \le d} \langle \vec{f} \mid \vec{\varphi}^*_{\xi,j,k} \rangle \vec{\varphi}_{\xi,j,k}$$

are bounded projections and satisfy

$$P_j \circ P_{j+1} = P_{j+1} \circ P_j = P_j, \lim_{j \to -\infty} \|P_j \vec{f}\|_2 = 0 = \lim_{j \to \infty} \|\vec{f} - P_j \vec{f}\|_2 = 0.$$

(vi) *The operators Q_j defined on $(L^2)^d$ by*

$$Q_j \vec{f} = \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \chi \le d(2^d - 1)} \langle \vec{f} \mid \vec{\psi}^*_{\chi,j,k} \rangle \vec{\psi}_{\chi,j,k}$$

are bounded and satisfy

$$Q_j = P_{j+1} - P_j$$

and

$$\|\vec{f}\|_2 \approx \sqrt{\sum_{j\in\mathbb{Z}} \|Q_j\vec{f}\|_2^2} \approx \sqrt{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\chi\le d(2^d-1)} |\langle \vec{f}\,|\,\vec{\psi}^*_{\chi,j,k}\rangle|^2}.$$

(B) Irrotational vector fields: *The projection operators P_j satisfy*

$$\vec{f} \in (L^2)^d \text{ and } \mathbf{curl}\,\vec{f} = 0 \;\Rightarrow\; \mathbf{curl}\,P_j(\vec{f}) = 0.$$

Moreover, there exist

- $2^d - 1$ *functions* $\vec{\gamma}_\eta \in (L^2)^d$, $1 \le \eta \le 2^d - 1$, *with* $\mathbf{curl}\,\vec{\gamma}_\eta = 0$,
- $2^d - 1$ *functions* $\vec{\gamma}^*_\eta \in (L^2)^d$, $1 \le \eta \le 2^d - 1$,

such that:

(i) $\vec{\gamma}_\eta$ *and* $\vec{\gamma}^*_\eta$ *are supported in* K_N.
(ii) $\vec{\gamma}_\eta$ *and* $\vec{\gamma}^*_\eta$ *are of class* $\mathcal{C}^N$.
(iii) *For* $l \in \mathbb{N}^d$ *with* $\sum_{i=1}^d l_i \le N$, *we have* $\int x^l \vec{\gamma}_\eta\,dx = \int x^l \vec{\gamma}^*_\eta\,dx = 0$.
(iv) *For* j, j' *in* $\mathbb{Z}$, k, k' *in* $\mathbb{Z}^d$, *and* η, η' *in* $\{1,\dots,2^d-1\}$,

$$\int \vec{\gamma}_{\eta,j,k}\cdot\vec{\gamma}^*_{\eta',j',k'}\,dx = \delta_{j,j'}\delta_{k,k'}\delta_{\eta,\eta'}.$$

(v) *The operators S_j defined on $(L^2)^d$ by*

$$S_j\vec{f} = \sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1} \langle \vec{f}\,|\,\vec{\gamma}^*_{\eta,j,k}\rangle \vec{\gamma}_{\eta,j,k}$$

are bounded and satisfy

$$\forall \vec{f} \in H_{\mathbf{curl}=0} \qquad S_j\vec{f} = Q_j\vec{f}$$

and

$$\forall \vec{f} \in H_{\mathbf{curl}=0} \qquad \|\vec{f}\|_2 \approx \sqrt{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1} |\langle \vec{f}\,|\,\vec{\gamma}^*_{\eta,j,k}\rangle|^2}.$$

(C) Divergence-free vector fields: *The projection operators P_j satisfy*

$$\vec{f} \in (L^2)^d \text{ and } \mathbf{div}\,\vec{f} = 0 \;\Rightarrow\; \mathbf{div}\,P^*_j\vec{f} = 0.$$

Moreover, there exist

- $(d-1)(2^d-1)$ *functions* $\vec{\alpha}_\epsilon \in (L^2)^d$, $1 \le \epsilon \le (d-1)(2^d-1)$, *with* $\mathbf{div}\,\vec{\alpha}_\epsilon = 0$,
- $(d-1)(2^d-1)$ *functions* $\vec{\alpha}^*_\epsilon \in (L^2)^d$, $1 \le \epsilon \le (d-1)(2^d-1)$,

such that:

(i) $\vec{\alpha}_\epsilon$ *and* $\vec{\alpha}^*_\epsilon$ *are supported in* K_N.
(ii) $\vec{\alpha}_\epsilon$ *and* $\vec{\alpha}^*_\epsilon$ *are of class* $\mathcal{C}^N$.

(iii) *For* $l \in \mathbb{N}^d$ *with* $\sum_{i=1}^d l_i \le N$, *we have* $\int x^l \vec{\alpha}_\epsilon \, dx = \int x^l \vec{\alpha}_\epsilon^* \, dx = 0$.

(iv) *For* j, j' *in* $\mathbb{Z}$, k, k' *in* $\mathbb{Z}^d$ *and* ϵ, ϵ' *in* $\{1, \ldots, (d-1)(2^d-1)\}$,

$$\int \vec{\alpha}_{\epsilon,j,k} \cdot \vec{\alpha}^*_{\epsilon',j',k'} \, dx = \delta_{j,j'} \delta_{k,k'} \delta_{\epsilon,\epsilon'}.$$

(v) *The operators* R_j *defined on* $(L^2)^d$ *by*

$$R_j \vec{f} = \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \epsilon \le (d-1)(2^d-1)} \langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,k} \rangle \vec{\alpha}_{\epsilon,j,k}$$

are bounded and satisfy

$$\forall \vec{f} \in H_{\mathbf{div}=0} \qquad R_j \vec{f} = Q_j^* \vec{f}$$

and

$$\forall \vec{f} \in H_{\mathbf{div}=0} \qquad \|\vec{f}\|_2 \approx \sqrt{\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \epsilon \le (d-1)(2^d-1)} |\langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,k} \rangle|^2}.$$

We would like now to use those special functions for our spaces X and Y. We begin with the following lemma:

Lemma 3.2.

(a) *If* $\vec{f} \in X_0^d$, *then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge strongly to* 0 *in* X^d *as* $j \to -\infty$ *and converge strongly to* $\vec{f}$ *in* X^d *as* $j \to \infty$.

(b) *If* $\vec{f} \in Y_0^d$, *then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge strongly to* 0 *in* Y^d *as* $j \to -\infty$ *and converge strongly to* $\vec{f}$ *in* Y^d *as* $j \to \infty$.

(c) *If* $\vec{f} \in X^d$, *then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge* **-weakly to* 0 *in* X^d *as* $j \to -\infty$ *and converge* **-weakly to* $\vec{f}$ *in* X^d *as* $j \to \infty$.

(d) *If* $\vec{f} \in Y^d$, *then* $P_j \vec{f}$ *and* $P_j^* \vec{f}$ *converge* **-weakly to* 0 *in* Y^d *as* $j \to -\infty$ *and converge* **-weakly to* $\vec{f}$ *in* Y^d *as* $j \to \infty$.

Proof. First, we check that the operators are well defined. If $f \in \mathcal{C}^N$ has a compact support, then we may write $f = f\theta$ with $\theta \in \mathcal{D}$ equal to 1 on a neighborhood of the support of f. Thus, $f = T_f(\theta)$ and we find that $f \in X_0 \cap Y_0$. Hence, $\langle f \,|\, g \rangle_{X_0,Y}$ is well defined for every $g \in Y$, and $\langle f \,|\, h \rangle_{Y_0,X}$ is well defined for every $h \in X$. We may thus consider the following operators on X^d:

$$P_j \vec{f} = \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \xi \le d} \langle \vec{f} \,|\, \vec{\varphi}^*_{\xi,j,k} \rangle_{X,Y_0} \vec{\varphi}_{\xi,j,k}$$

and

$$P_j^* \vec{f} = \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \xi \le d} \langle \vec{f} \,|\, \vec{\varphi}_{\xi,j,k} \rangle_{X,Y_0} \vec{\varphi}^*_{\xi,j,k}.$$

We have $\sup_{j \in \mathbb{Z}} \|P_j\|_{\mathrm{CZO}} = \sup_{j \in \mathbb{Z}} \|P_j^*\|_{\mathrm{CZO}} < \infty$. Thus, those operators are equicontinuous on X^d.

To prove (a), we need to check the limits only on a dense subspace of X_0^d. The space X_0 cannot be embedded into L^1: if $f \in \mathcal{D}$ with $\hat{f}(0) \neq 0$, then the Riesz transforms $R_j f$ are not in L^1 but belong to X_0. This means that $f \in \mathcal{D} \mapsto \|f\|_1$ is not continuous for the X_0 norm. We may thus find a sequence of functions f_n such that $\|f_n\|_X$ converges to 0 and $\|f_n\|_1 = 1$. Since $|f_n|$ is Lipschitz and compactly supported, we can regularize f_n and find a sequence of smooth compactly supported functions $f_{n,k}$ such that all the $f_{n,k}$, $k \in \mathbb{N}$, are supported in a compact neighborhood of the support of f_n and converge, as $k \to \infty$, uniformly to $|f_n|$; then, we have convergence in X (since $Y_0 \subset L^1_{\mathrm{loc}}$) and in L^1. Thus, we can find a sequence of functions f_n which are in $\mathcal{D}$, with $\int f_n\,dx = 1$ and $\lim_{n\to\infty} \|f_n\|_X = 0$. This shows that the set of functions $f \in \mathcal{D}$ with $\int f\,dx = 0$ is dense in X_0.

We now consider $Q_j = P_{j+1} - P_j$ and $Q_j^* = P_{j+1}^* - P_j^*$:

$$Q_j \vec{f} = \sum_{k\in\mathbb{Z}^d} \sum_{1\le\chi\le d(2^d-1)} \langle \vec{f} \mid \vec{\psi}^*_{\chi,j,k}\rangle_{X,Y_0} \vec{\psi}_{\chi,j,k}$$

and

$$Q_j^* \vec{f} = \sum_{k\in\mathbb{Z}^d} \sum_{1\le\chi\le d(2^d-1)} \langle \vec{f} \mid \vec{\psi}_{\chi,j,k}\rangle_{X,Y_0} \vec{\psi}^*_{\chi,j,k}.$$

If $\vec{f} \in \mathcal{D}^d$ and $\int \vec{f}\,dx = 0$, we have $\vec{f} = \sum_{1\le l\le d} \partial_l \vec{f_l}$ for some $\vec{f_l} \in \mathcal{D}^d$. Similarly, we have $\psi^*_\chi = \sum_{1\le l\le d} \partial_l \vec{\Psi}^*_{\chi,l}$ and $\psi_\chi = \sum_{1\le l\le d} \partial_l \vec{\Psi}_{\chi,l}$ for some compactly supported functions of class $\mathcal{C}^N$. Thus, we find that, for $\vec{f} \in \mathcal{D}^d$ with $\int \vec{f}\,dx = 0$,

$$\|P_{j+1}\vec{f} - P_j\vec{f}\|_X + \|P^*_{j+1}\vec{f} - P^*_j\vec{f}\|_X \le C \min\Big(\sum_{l=1^d} \|\partial_l \vec{f}\|_X 2^j, \sum_{i=1}^{d} \|\vec{f_l}\|_X 2^{-j}\Big).$$

Thus, $P_j\vec{f}$ and $P_j^*\vec{f}$ have strong limits in X_0^d as j goes to $-\infty$ or ∞. If $\vec{g} \in \mathcal{D}^d$, viewing $\vec{f}$ and $\vec{g}$ as elements of $(L^2)^d$, we see that

$$\lim_{j\to-\infty} \langle P_j\vec{f} \mid \vec{g}\rangle_{X,Y_0} = \lim_{j\to-\infty} \langle P_j^*\vec{f} \mid \vec{g}\rangle_{X,Y_0} = 0$$

and

$$\lim_{j\to\infty} \langle P_j\vec{f} \mid \vec{g}\rangle_{X,Y_0} = \lim_{j\to\infty} \langle P_j^*\vec{f} \mid \vec{g}\rangle_{X,Y_0} = \langle \vec{f} \mid \vec{g}\rangle_{X,Y_0}.$$

Thus, we have, for $\vec{f} \in \mathcal{D}^d$ with $\int \vec{f}\,dx = 0$,

$$\lim_{j\to-\infty} \|P_j\vec{f}\|_X = \lim_{j\to-\infty} \|P_j^*\vec{f}\|_X = 0$$

and

$$\lim_{j\to\infty} \|P_j\vec{f} - \vec{f}\|_X = \lim_{j\to\infty} \|P_j^*\vec{f} - \vec{f}\|_X = 0$$

Hence (a) is proved; and (b) is proved in a similar way. By duality, we get (c) and (d). ∎

We may now consider the operators

$$R_j\vec{f} = \sum_{k\in\mathbb{Z}^d} \sum_{1\le\epsilon\le(d-1)(2^d-1)} \langle \vec{f} \mid \vec{\alpha}^*_{\epsilon,j,k}\rangle_{X,Y_0}\vec{\alpha}_{\epsilon,j,k}$$

and

$$S_j\vec{f} = \sum_{k\in\mathbb{Z}^d} \sum_{1\le\eta\le 2^d-1} \langle \vec{f} \mid \vec{\gamma}^*_{\eta,j,k}\rangle_{X,Y_0}\vec{\gamma}_{\eta,j,k}.$$

From the equalities $\mathbb{P}^*R_j^* = \mathbb{P}^*Q_j$ and $\mathbb{Q}^*S_j^* = \mathbb{Q}^*Q_j^*$ of maps from $\mathcal{D}^d$ to Y_0^d, we find by duality that $R_j\mathbb{P} = Q_j^*\mathbb{P}$ and $S_j\mathbb{Q} = Q_j\mathbb{Q}$ on X^d. To be able to use those identities, we shall need the following lemma:

LEMMA 3.3. *Let $\vec{f}\in X^d$. Then:*

(i) $\mathbb{P}\vec{f} = \vec{f} \ \Leftrightarrow\ \mathbf{div}\,\vec{f} = 0$.
(ii) $\mathbb{Q}\vec{f} = \vec{f} \ \Leftrightarrow\ \mathbf{curl}\,\vec{f} = 0$.

Proof. First, we check that $[f\in X \text{ and } \Delta f = 0] \Rightarrow f = 0$. Take $\theta\in\mathcal{D}$ such that $\theta\ge 0$ and $\theta\ne 0$, and define

$$\gamma = \frac{1}{(1+x^2)^{(n+1)/2}} * \theta.$$

Convolution with the kernel $(1+x^2)^{-(n+1)/2}$ is a Calderón–Zygmund operator, so $\gamma\in Y_0$. Moreover, if g is a function such that $(1+x^2)^{(n+1)/2}g\in L^\infty$, we find that $g = \gamma^{-1}g\gamma = T_{\gamma^{-1}g}(\gamma)$, where the pointwise multiplication operator $T_{\gamma^{-1}g}$ is a Calderón–Zygmund operator, so $g\in Y_0$. This proves that $X\subset\mathcal{S}'$. Thus, if $f\in X$ and $\Delta f = 0$, we find that f is a harmonic polynomial. Moreover $\int |f|\gamma\,dx = \langle f \mid T_{f/|f|}(\gamma)\rangle_{X,Y_0}$, hence the integral $\int|f|\gamma\,dx$ must be finite, and f must be constant. As the smooth functions with vanishing integral are dense in Y_0, we find that the constant is equal to 0.

Now, for a distribution $\vec{f}$ we have

$$\mathbf{div}\,\vec{f} = 0 \ \Leftrightarrow\ \forall\vec{\varphi}\in\mathcal{D}^d \text{ with } \mathbf{curl}\,\vec{\varphi} = 0,\ \langle\vec{f}\mid\vec{\varphi}\rangle = 0;$$

thus, $\mathbf{div}\,\mathbb{P}\vec{f} = 0$ on X^d. Similarly, for a distribution $\vec{f}$ we have

$$\mathbf{curl}\,\vec{f} = 0 \ \Leftrightarrow\ \forall\vec{\varphi}\in\mathcal{D}^d \text{ with } \mathbf{div}\,\vec{\varphi} = 0,\ \langle\vec{f}\mid\vec{\varphi}\rangle = 0;$$

thus, $\mathbf{curl}\,\mathbb{Q}\vec{f} = 0$ on X^d.

Conversely, we start from the decomposition $\mathrm{Id} = \mathbb{P}+\mathbb{Q}$ valid on X^d. If $\mathbf{div}\,\vec{f} = 0$, then we find that $\vec{h} = \vec{f} - \mathbb{P}\vec{f} = \mathbb{Q}\vec{f}$ satisfies $\mathbf{div}\,\vec{h} = 0$ and $\mathbf{curl}\,\vec{h} = 0$. But this implies $\Delta\vec{h} = 0$, hence $\vec{h} = 0$. We prove similarly that $\mathbf{curl}\,\vec{f} = 0$ implies $f = \mathbb{Q}\vec{f}$. ∎

4. The proof of the div-curl lemma. As in [LEMc], we prove Theorem 2.3 by adapting the proof given by Dobyinsky [DOB]. This proof uses the renormalization of the product through wavelet bases.

If $\vec{f} \in X^d$, $\vec{g} \in Y^d$ and if moreover $\vec{f} \in X_0^d$ or $\vec{g} \in Y_0^d$, we use Lemma 3.2 to find that, in the distribution sense, we have

$$\vec{f} \cdot \vec{g} = \lim_{j \to \infty} (P_j^* \vec{f} \cdot P_j \vec{g} - P_{-j}^* \vec{f} \cdot P_{-j} \vec{g})$$

and thus

$$\vec{f} \cdot \vec{g} = \sum_{j \in \mathbb{Z}} (P_j^* \vec{f} \cdot Q_j \vec{g} + Q_j^* \vec{f} \cdot P_j \vec{g} + Q_j \vec{f} \cdot Q_j^* \vec{g}).$$

If moreover $\mathbf{div}\, \vec{f} = 0$ and $\mathbf{curl}\, \vec{g} = 0$, we use Lemma 3.3 to get

$$\vec{f} \cdot \vec{g} = \sum_{j \in \mathbb{Z}} (P_j^* \vec{f} \cdot S_j \vec{g} + R_j \vec{f} \cdot P_j \vec{g} + R_j \vec{f} \cdot S_j \vec{g}).$$

We shall prove that the three terms

$$A(\vec{f}, \vec{g}) = \sum_{j \in \mathbb{Z}} P_j^* \vec{f} \cdot S_j \vec{g}, \quad B(\vec{f}, \vec{g}) = \sum_{j \in \mathbb{Z}} R_j \vec{f} \cdot P_j \vec{g}, \quad C(\vec{f}, \vec{g}) = \sum_{j \in \mathbb{Z}} R_j \vec{f} \cdot S_j \vec{g}$$

belong to $\mathcal{H}^1$.

We give the proof for $\vec{f} \in X_0^d$ (the proof for $\vec{g} \in Y_0^d$ is similar). We first check that A and B map $(X_0)^d \times Y^d$ to $\mathcal{H}^1$: we use the duality of H^1 and CMO (the closure of $\mathcal{C}_0$ in BMO) (see Coifman and Weiss [COIW] and Bourdaud [BOU]) and try to prove that the operators

$$\mathcal{A}(\vec{f}, h) = \sum_{j \in \mathbb{Z}} S_j^* (h P_j^* \vec{f}) \quad \text{and} \quad \mathcal{B}(\vec{f}, h) = \sum_{j \in \mathbb{Z}} P_j^* (h R_j \vec{f})$$

map $(X_0)^d \times \mathrm{CMO}$ to $(X_0)^d$.

To this end, we shall prove that $\mathcal{A}(\cdot, h)$ and that $\mathcal{B}(\cdot, h)$ are matrices of singular integral operators when $h \in \mathcal{D}$ and that we have the estimates $\|\mathcal{A}(\cdot, h)\|_{\mathrm{CZO}} \le C \|h\|_{\mathrm{BMO}}$ and $\|\mathcal{B}(\cdot, h)\|_{\mathrm{CZO}} \le C \|h\|_{\mathrm{BMO}}$. For $\mathcal{B}$, we may as well study the adjoint operator

$$\mathcal{B}^*(\vec{f}, h) = \sum_{j \in \mathbb{Z}} R_j^* (h P_j \vec{f}).$$

First, we estimate the size of the kernels and of their gradients. The kernels $A_h(x, y)$ of $\mathcal{A}(\cdot, h)$ and $B_h^*(x, y)$ of $\mathcal{B}(\cdot, h)^*$ are given by

$$A_h(x, y) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \xi \le d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \le \eta \le 2^d - 1} \vec{\gamma}^*_{\eta, j, l}(x) \langle h \vec{\varphi}^*_{\xi, j, k} \mid \vec{\gamma}_{\eta, j, l} \rangle \vec{\varphi}_{\xi, j, k}(y)$$

and

$$B_h^*(x, y) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \sum_{1 \le \xi \le d} \sum_{l \in \mathbb{Z}^d} \sum_{1 \le \epsilon \le (d-1)(2^d - 1)} \vec{\alpha}^*_{\epsilon, j, l}(x) \langle h \vec{\varphi}_{\xi, j, l} \mid \vec{\alpha}_{\epsilon, j, k} \rangle \vec{\varphi}^*_{\xi, j, k}(y).$$

There are only a few terms that interact, because of the localization of the supports: if $K_N \subset B(0,M)$, then $\langle h\vec{\varphi}^*_{\xi,j,k} \,|\, \vec{\gamma}_{\eta,j,l}\rangle = \langle h\vec{\varphi}_{\xi,j,l} \,|\, \vec{\alpha}_{\epsilon,j,k}\rangle = 0$ if $|l-k| > 2M$. Let

$$C(h) = \sup_{j\in\mathbb{Z},\, k\in\mathbb{Z}^d,\, 1\le\xi\le d,\, l\in\mathbb{Z}^d,\, 1\le\eta\le 2^d-1} |\langle h\vec{\varphi}^*_{\xi,j,k} \,|\, \vec{\gamma}_{\eta,j,l}\rangle|,$$
$$D(h) = \sup_{j\in\mathbb{Z},\, k\in\mathbb{Z}^d,\, 1\le\xi\le d,\, l\in\mathbb{Z}^d,\, 1\le\epsilon\le(d-1)(2^d-1)} |\langle h\vec{\varphi}_{\xi,j,l} \,|\, \vec{\alpha}_{\epsilon,j,k}\rangle|.$$

Then

$$|A_h(x,y)| \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d} CC(h)2^{jd}\mathbf{1}_{B(0,M)}(2^jx-k)\mathbf{1}_{B(0,3M)}(2^jy-k)$$

and thus

$$|A_h(x,y)| \le CC(h)\sum_{2^j|y-x|\le 4M} 2^{jd} \le C'C(h)|x-y|^{-d}$$

and similarly

$$|B_h(x,y)| \le CD(h)|x-y|^{-d}.$$

In the same way, we have

$$\begin{aligned}|\vec{\nabla}_x A_h(x,y)| &+ |\vec{\nabla}_y A_h(x,y)| \\ &\le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d} CC(h)2^{j(d+1)}\mathbf{1}_{B(0,M)}(2^jx-k)\mathbf{1}_{B(0,3M)}(2^jy-k)\end{aligned}$$

and thus

$$|\vec{\nabla}_x A_h(x,y)| + |\vec{\nabla}_y A_h(x,y)| \le CC(h)|x-y|^{-d-1}$$

and similarly

$$|\vec{\nabla}_x B_h(x,y)| + |\vec{\nabla}_y B_h(x,y)| \le CD(h)|x-y|^{-d-1}.$$

Moreover, $\vec{\varphi}^*_{\xi,j,k}\cdot\vec{\gamma}_{\eta,j,l}$ is supported in $B(2^{-j}k, M2^{-j})$, $\|\vec{\varphi}^*_{\xi,j,k}\cdot\vec{\gamma}_{\eta,j,l}\|_\infty \le C2^{jd}$ and $\int \vec{\varphi}^*_{\xi,j,k}\cdot\vec{\gamma}_{\eta,j,l}\,dx = 0$ (since $P^*_j\vec{\varphi}^*_{\xi,j,k} = \vec{\varphi}^*_{\xi,j,k}$ and $Q_j\vec{\gamma}_{\eta,j,l} = \vec{\gamma}_{\eta,j,l}$). Thus, we find that $\|\vec{\varphi}^*_{\xi,j,k}\cdot\vec{\gamma}_{\eta,j,l}\|_{\mathcal{H}^1} \le C$, so

$$C(h) \le C\|h\|_{BMO}.$$

We have similar estimates for $\|\vec{\varphi}_{\xi,j,l}\cdot\vec{\alpha}_{\epsilon,j,k}\|_{\mathcal{H}^1}$ (since $P_j\vec{\varphi}_{\xi,j,l} = \vec{\varphi}_{\xi,j,l}$ and $Q^*_j\vec{\alpha}_{\epsilon,j,k} = \vec{\alpha}_{\epsilon,j,k}$, and thus $\int\vec{\varphi}_{\xi,j,l}\cdot\vec{\alpha}_{\epsilon,j,k}\,dx = 0$), so

$$D(h) \le C\|h\|_{\mathrm{BMO}}.$$

Thus far, we have proved that $\mathcal{A}(\cdot,h)$ and $\mathcal{B}(\cdot,h)$ are singular integral operators. To prove L^2 boundedness, we use the $T(1)$ theorem of David and Journé [DAV]. We have to check that the operators are weakly bounded (in the sense of the WBP property), and to compute the images of the function $f=1$ under the operators and their adjoints.

Let $x_0 \in \mathbb{R}^d$, $r_0 > 0$ and let $\vec{f}$ and $\vec{g}$ be supported in $B(x_0, r_0)$. We want to estimate $\langle \mathcal{A}(\vec{f}, h) \,|\, \vec{g}\,\rangle_{\mathcal{D}',\mathcal{D}}$ and $\langle \mathcal{B}(\vec{f}, h) \,|\, \vec{g}\,\rangle_{\mathcal{D}',\mathcal{D}}$. We have $|\langle \mathcal{A}(\vec{f}, h) \,|\, \vec{g}\,\rangle_{\mathcal{D}',\mathcal{D}}| \le \sum_{j\in\mathbb{Z}} A_j$ where

$$A_j = \sum_{k\in\mathbb{Z}^d} \sum_{1\le\xi\le d} \sum_{l\in\mathbb{Z}^d} \sum_{1\le\eta\le 2^d-1} |\langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,l}\rangle \langle h\vec{\varphi}^*_{\xi,j,k} \,|\, \vec{\gamma}_{\eta,j,l}\rangle \langle \vec{f} \,|\, \vec{\varphi}_{\xi,j,k}\rangle|$$

and similarly $|\langle \mathcal{B}(\vec{f}, h) \,|\, \vec{g}\,\rangle_{\mathcal{D}',\mathcal{D}} \le \sum_{j\in\mathbb{Z}} B_j$ where

$$B_j = \sum_{k\in\mathbb{Z}^d} \sum_{1\le\xi\le d} \sum_{l\in\mathbb{Z}^d} \sum_{1\le\epsilon\le(d-1)(2^d-1)} |\langle \vec{g} \,|\, \vec{\varphi}_{\xi,j,l}\rangle \langle h\vec{\varphi}_{\xi,j,l} \,|\, \vec{\alpha}_{\epsilon,j,k}\rangle \langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,k}\rangle|.$$

We have

$$A_j \le C(h) \sum_{k\in\mathbb{Z}^d} \sum_{1\le\xi\le d} \sum_{|l-k|\le 2M} \sum_{1\le\eta\le 2^d-1} |\langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,l}\rangle \langle \vec{f} \,|\, \vec{\varphi}_{\xi,j,k}\rangle|,$$

which gives

$$\begin{aligned} A_j &\le C(h) \sum_{k\in\mathbb{Z}^d} \sum_{1\le\xi\le d} |\langle \vec{f} \,|\, \vec{\varphi}_{\xi,j,k}\rangle| \sum_{l\in\mathbb{Z}^d} \sum_{1\le\eta\le 2^d-1} |\langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,l}\rangle| \\ &\le CC(h) 2^{jd} \|\vec{f}\,\|_1 \|\vec{g}\|_1 \end{aligned}$$

and

$$A_j \le CC(h) \sqrt{\sum_{k\in\mathbb{Z}^d} \sum_{1\le\xi\le d} |\langle \vec{f} \,|\, \vec{\varphi}_{\xi,j,k}\rangle|^2} \sqrt{\sum_{l\in\mathbb{Z}^d} \sum_{1\le\eta\le 2^d-1} |\langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,l}\rangle|^2},$$

and thus

$$A_j \le C'C(h) \|S_j\vec{g}\|_2 \|P^*_j \vec{f}\,\|_2 \le C''C(h) 2^{-j} \|\vec{\nabla}\vec{g}\|_2 \|\vec{f}\,\|_2.$$

Finally, we get

$$\begin{aligned} |\langle \mathcal{A}(\vec{f}, h) \,|\, \vec{g}\,\rangle_{\mathcal{D}',\mathcal{D}}| &\le CC(h) \Big(\sum_{2^j r_0 \le 1} 2^{jd} r_0^d \|\vec{f}\,\|_2 \|\vec{g}\|_2 + \sum_{2^j r_0 > 1} 2^{-j} \|\vec{\nabla}\vec{g}\|_2 \|\vec{f}\,\|_2 \Big) \\ &\le C'C(h) (\|\vec{f}\,\|_2 + r_0 \|\vec{\nabla}\vec{f}\,\|_2)(\|\vec{g}\|_2 + r_0 \|\vec{\nabla} g\|_2). \end{aligned}$$

Similar computations (based on the inequality $\|R_j\vec{f}\,\|_2 \le C2^{-j}\|\vec{\nabla}\vec{f}\,\|_2$) give as well

$$|\langle \mathcal{B}(\vec{f}, h) \,|\, \vec{g}\,\rangle_{\mathcal{D}',\mathcal{D}}| \le CD(h)(\|\vec{f}\,\|_2 + r_0\|\vec{\nabla}\vec{f}\,\|_2)(\|\vec{g}\|_2 + r_0\|\vec{\nabla} g\|_2).$$

Thus, our operators satisfy the weak boundedness property.

We must now compute the distributions $T(1)$ and $T^*(1)$ when T is one component of the matrix of operators $\mathcal{A}(\cdot, h)$ or $\mathcal{B}(\cdot, h)$. We must prove that if $\theta \in \mathcal{D}$ is equal to 1 on a neighborhood of 0, if $\vec{\theta}_{l,R} = (\theta_{1,l,R}, \dots, \theta_{d,l,R})$ with $\theta_{k,l,R} = \delta_{k,l}\theta(x/R)$ and if $\vec{\psi} \in \mathcal{D}^d$ with $\int \vec{\psi}\, dx = 0$, then

$$\lim_{R\to\infty} \sum_{j\in\mathbb{Z}} S^*_j(hP^*_j\vec{\theta}_{l,R}) \in (\mathrm{BMO})^d$$

(the limit is taken in $(\mathcal{D}'/\mathbb{R})^d$) and similarly

$$\lim_{R\to\infty}\sum_{j\in\mathbb{Z}} P_j(hS_j\vec{\theta}_{l,R}) \in (\mathrm{BMO})^d,$$

$$\lim_{R\to\infty}\sum_{j\in\mathbb{Z}} P_j^*(hR_j\vec{\theta}_{l,R}) \in (\mathrm{BMO})^d,$$

$$\lim_{R\to+\infty}\sum_{j\in\mathbb{Z}} R_j^*(hP_j\vec{\theta}_{l,R}) \in (\mathrm{BMO})^d.$$

To check that, we write $\vec{h}_l = (h_{1,l},\dots,h_{d,l})$ with $h_{k,l} = \delta_{k,l}h$ and we consider $\vec{\psi}\in\mathcal{D}^d$ with $\int\vec{\psi}\,dx = 0$. We have $\sum_{j\in\mathbb{Z}}\|S_j(\vec{\psi})\|_1 < \infty$ and $\|hP_j^*\vec{\theta}_{l,R}\|_\infty \le \|h\|_\infty\|\theta\|_\infty$ and thus, by dominated convergence,

$$\lim_{R\to\infty}\int\vec{\psi}\cdot\sum_{j\in\mathbb{Z}} S_j^*(hP_j^*\vec{\theta}_{l,R})\,dx = \sum_{j\in\mathbb{Z}}\int S_j\vec{\psi}\cdot\vec{h}_l\,dx.$$

$\sum_{j\in\mathbb{Z}} S_j$ is a matrix of Calderón–Zygmund operators T which satisfy $T^*(1) = 0$, hence map $\mathcal{H}^1$ to $\mathcal{H}^1$, so

$$\Big|\sum_{j\in\mathbb{Z}}\int S_j\vec{\psi}\cdot\vec{h}_l\,dx\Big| \le C\|h\|_{\mathrm{BMO}}\|\vec{\psi}\|_{\mathcal{H}^1}$$

and thus $\lim_{R\to\infty}\sum_{j\in\mathbb{Z}} S_j^*(hP_j^*\vec{\theta}_{l,R}) \in (\mathrm{BMO})^d$. Similar estimates prove that

$$\lim_{R\to\infty}\int\vec{\psi}\cdot R_j^*(hP_j\vec{\theta}_{l,R})\,dx = \sum_{j\in\mathbb{Z}}\int R_j\vec{\psi}\cdot\vec{h}_l\,dx$$

and

$$\Big|\sum_{j\in\mathbb{Z}}\int R_j\vec{\psi}\cdot\vec{h}_l\,dx\Big| \le C\|h\|_{\mathrm{BMO}}\|\vec{\psi}\|_{\mathcal{H}^1},$$

so that $\lim_{R\to\infty}\sum_{j\in\mathbb{Z}} R_j^*(hP_j\vec{\theta}_{l,R}) \in (\mathrm{BMO})^d$.

On the other hand, we have

$$\begin{aligned}\Big|\int\vec{\psi}\cdot P_j(hS_j\vec{\theta}_{l,R})\,dx\Big| &\le C\|h\|_\infty\|P_j\vec{\psi}\|_1\|S_j\vec{\theta}_{l,R}\|_\infty\\ &\le C_{\vec{\psi}}\|h\|_\infty\min(1,2^j)\min(\|\theta\|_\infty, 2^{-j}R^{-1}\|\vec{\nabla}\theta\|_\infty) = O(R^{-1/2}),\end{aligned}$$

so $\lim_{R\to\infty}\sum_{j\in\mathbb{Z}} P_j(hS_j\vec{\theta}_{l,R}) = 0$. Similarly, $\lim_{R\to\infty}\sum_{j\in\mathbb{Z}} P_j^*(hR_j\vec{\theta}_{l,R}) = 0$.

Thus, we have proved that $\mathcal{A}$ and $\mathcal{B}$ map $X_0^d\times\mathrm{CMO}$ to X_0^d, and thus A and B map $X_0^d\times Y^d$ to $\mathcal{H}^1$. We still have to deal with the term $C(\vec{f},\vec{g}) = \sum_{j\in\mathbb{Z}} R_j\vec{f}\cdot S_j\vec{g}$. We write

$$C(\vec{f},\vec{g}) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\sum_{1\le\eta\le 2^d-1}\sum_{l\in\mathbb{Z}^d}\sum_{1\le\epsilon\le(d-1)(2^d-1)}\langle\vec{g}\mid\vec{\gamma}^*_{\eta,j,k}\rangle\langle\vec{f}\mid\vec{\alpha}^*_{\epsilon,j,l}\rangle\vec{\alpha}_{\epsilon,j,l}\cdot\vec{\gamma}_{\eta,j,k}.$$

We have $\vec{\alpha}_{\epsilon,j,l} \cdot \vec{\gamma}_{\eta,j,k} = 0$ for $|k-l| > 2M$ and $\|\vec{\alpha}_{\epsilon,j,l} \cdot \vec{\gamma}_{\eta,j,k}\|_{\mathcal{H}^1} \le C$ for $|k-l| \le 2M$. Thus, we are led to prove that

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \sum_{1\le\eta\le 2^d-1} \sum_{|l-k|\le 2M} \sum_{1\le\epsilon\le(d-1)(2^d-1)} |\langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,k}\rangle| \, |\langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,l}\rangle| \le C\|\vec{f}\|_{X_0^d} \|\vec{g}\|_{Y^d}.$$

For $1 \le \eta \le 2^d - 1$, $1 \le \epsilon \le (d-1)(2^d-1)$ and $r \in \mathbb{Z}^d$ with $|r| \le 2M$, we consider a finite subset J of $\mathbb{Z} \times \mathbb{Z}^d$ and for $\epsilon_J = (\epsilon_{j,k})_{(j,k)\in J} \in \{-1,1\}^J$ and T_{ϵ_J} the operator

$$T_{\epsilon_J}(\vec{f}) = \sum_{(j,k)\in J} \epsilon_{j,k} \langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,k+r}\rangle \vec{\gamma}^*_{\eta,j,k}.$$

Using again the $T(1)$ theorem, we see that $\|T_{\epsilon_J}\|_{\mathrm{CZO}} \le C$, so that $T_{\epsilon_J}(\vec{f}) \in X_0^d$ and

$$\int T_{\epsilon_J}(\vec{f}) \cdot \vec{g}\, dx = \sum_{(j,k)\in J} \epsilon_{j,k} \langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,k+r}\rangle \langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,k}\rangle \le C\|\vec{f}\|_{X_0^d} \|\vec{g}\|_{Y^d}.$$

Now, it is enough to choose $\epsilon_{j,k}$ as the sign of $\langle \vec{f} \,|\, \vec{\alpha}^*_{\epsilon,j,k+r}\rangle \langle \vec{g} \,|\, \vec{\gamma}^*_{\eta,j,k}\rangle$ and we can conclude.

Thus, Theorem 2.3 has been proved.

5. Examples. We now give some examples of Calderón–Zygmund pairs of Banach spaces (according to Definition 2.2):

(a) *Lebesgue spaces*: $X = X_0 = L^p$ and $Y = Y_0 = L^q$ with $1 < p < \infty$ and $1/p + 1/q = 1$.

(b) *Lorentz spaces*: $X = X_0 = L^{p,r}$ and $Y = L^{q,\rho}$ with $1 < p < \infty$, $1 \le r < \infty$, $1/p + 1/q = 1$ and $1/r + 1/\rho = 1$.

(c) *Weighted Lebesgue spaces*: $X = X_0 = L^p(w\,dx)$ and $Y = Y_0 = L^q(w^{-1/(p-1)}\,dx)$ with $1 < p < \infty$ and $1/p + 1/q = 1$, when the weight w belongs to the Muckenhoupt class $\mathcal{A}_p$.

(d) *Morrey spaces*: We consider the *Morrey space* $\mathcal{L}^{\alpha,p}$ defined by

$$f \in \mathcal{L}^{\alpha,p} \Leftrightarrow \sup_{Q\in\mathcal{Q}} R_Q^\alpha \biggl(\frac{1}{|Q|} \int_Q |f(x)|^p\, dx \biggr)^{1/p} < \infty.$$

We are interested in the set of parameters $1 < p < \infty$ and $0 < \alpha \le d/p$.

The *Zorko space* $\mathcal{L}_0^{\alpha,p}$ is the closure of $\mathcal{D}$ in $\mathcal{L}^{\alpha,p}$. Adams and Xiao [ADA] have proved that $\mathcal{L}^{\alpha,p}$ is the bidual of $\mathcal{L}_0^{\alpha,p}$: $\mathcal{H}^{\alpha,q} = (\mathcal{L}_0^{\alpha,p})^*$ and $\mathcal{L}^{\alpha,p} = (\mathcal{H}^{\alpha,q})^*$ with $1/p + 1/q = 1$. One characterization of $\mathcal{H}^{\alpha,p}$ is the following: $f \in \mathcal{H}^{\alpha,q}$ if and only if there is a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^1$ and a sequence of functions f_n and of cubes Q_n such that $f_n \in L^q$, f_n is supported in Q_n and $\|f_n\|_q \le R_{Q_n}^{\alpha+d/q-d}$. The norm $\|f\|_{\mathcal{H}^{\alpha,q}}$ is then equivalent to $\inf_{(\lambda_n),(f_n),f=\sum \lambda_n f_n} \sum_{n\in\mathbb{N}} |\lambda_n|$.

Our Calderón–Zygmund pair is then $X = \mathcal{L}^{\alpha,p}$ and $Y = Y_0 = \mathcal{H}^{\alpha,q}$ with $1 < p < \infty$, $0 < \alpha \le d/p$ and $1/p + 1/q = 1$.

(e) *Multiplier spaces*: We can build new examples from the former ones. Indeed, let X be a Banach space such that:

(i) We have continuous embeddings $X_1 \subset X \subset X_2$ for some Calderón–Zygmund pairs of Banach spaces (X_1, Y_1) and (X_2, Y_2).

(ii) There is a Banach space A such that $\mathcal{D}$ is dense in A and the dual space A^* coincides with X with equivalence of norms.

(iii) Every Calderón–Zygmund operator may be extended as a bounded operator on X: $\|T(f)\|_X \le C\|T\|_{\mathrm{CZO}}\|f\|_X$.

Then, if X_0 is the closure of $\mathcal{D}$ in X and $Y = X_0^*$, (X, Y) is a Calderón–Zygmund pair of Banach spaces (and $A = Y_0$).

This is easy to prove. First, notice that every Calderón–Zygmund operator can be extended on X_2, hence be defined on X; the extra information is that it is bounded from X to X. Moreover, we have $\mathcal{D} \subset X_{1,0} \subset X_0$ with continuous embeddings, so that every Calderón–Zygmund operator maps X_0 to X_0, hence Y to Y by duality. Moreover, from $X_{1,0} \subset X_0 \subset X_{2,0}$, we get $Y_2 \subset Y \subset Y_1$. We will conclude if we prove $A = Y_0$; but we see easily (since truncation and convolution operators are Calderón–Zygmund operators) that X_0 is *-weakly dense in X and that A is embedded into Y with equivalence of norms (due to the Hahn–Banach theorem). Thus, $A = Y_0$.

We may apply this to the space $X = X^{s,p}$ of pointwise multipliers from the potential space $\dot{H}^s_p$ $(1 < p < \infty,\ 0 < s < d/p)$:

(i) We have the continuous embeddings for $p_1 > p$: $\mathcal{L}^{s,p_1} \subset X^{s,p} \subset \mathcal{L}^{s,p}$ (Fefferman–Phong inequality) [FEF].

(ii) $X^{s,p}$ is the dual space of $Y^{s,q}$ defined by: $f \in Y^{s,q}$ if and only if there is a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^1$ and a sequence of functions f_n and g_n with $f_n \in \dot{H}^s_p$, $g_n \in L^q$, $\|f_n\|_{\dot{H}^s_p} \le 1$ and $\|g_n\|_q \le 1$. The norm $\|f\|_{Y^{s,q}}$ is then equivalent to $\inf_{(\lambda_n),(f_n),(g_n),f=\sum \lambda_n f_n g_n} \sum_{n\in\mathbb{N}} |\lambda_n|$.

(iii) Every Calderón–Zygmund operator may be extended as a bounded operator on X: $\|T(f)\|_X \le C\|T\|_{\mathrm{CZO}}\|f\|_X$. This is due to a theorem of Verbitsky [MAZ].

References

[ADA] D. R. Adams and J. Xiao, *Morrey spaces in harmonic analysis*, Ark. Mat., to appear, doi: 10.1007/s11512-010-0134-0, 30 pp., 2011.

[BAT] G. Battle, *A block spin construction of ondelettes, Part I: Lemarié functions,* Comm. Math. Phys. 110 (1987), 601–615.

[BATF] G. Battle and P. Federbush, *Divergence-free vector wavelets*, Michigan Math. J. 40 (1995), 181–195.

[BOU] G. Bourdaud, *Remarques sur certains sous-espaces de $BMO(\mathbb{R}^n)$ et de $bmo(\mathbb{R}^n)$*, Ann. Inst. Fourier (Grenoble) 52 (2002), 1187–1218.

[COH] A. Cohen, I. Daubechies and J.-C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math. 45 (1992), 485–560.

[COIL] R. R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. 72 (1992), 247–286.

[COIW] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Soc. Math. 83 (1977), 569–645.

[DAU] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[DAV] G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón–Zygmund operators*, Ann. of Math. 120 (1984), 371–397.

[DER] E. Deriaz and V. Perrier, *Divergence-free and curl-free wavelets in two dimensions and three dimensions, application to turbulent flows*, J. Turbulence 7 (2006), paper 3, 37 pp.

[DOB] S. Dobyinsky, *Ondelettes, renormalisation du produit et applications à certains opérateurs bilinéaires*, Thèse, Univ. Paris IX, 1992.

[FEF] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. 9 (1983), 129–206.

[LEMa] P. G. Lemarié-Rieusset, *Analyses multi-résolutions non orthogonales, commutation entre projecteurs et dérivations et ondelettes vecteurs à divergence nulle*, Rev. Mat. Iberoamer. 8 (1992), 221–237.

[LEMb] P. G. Lemarié-Rieusset, *Un théorème d'inexistence pour les ondelettes vecteurs à divergence nulle*, C. R. Acad. Sci. Paris Sér. I 319 (1994), 811–813.

[LEMc] P. G. Lemarié-Rieusset, *Recent Developments in the Navier–Stokes Problem*, Chapman & Hall/CRC, 2002.

[LEMM] P. G. Lemarié-Rieusset et Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamer. 2 (1986), 1–18.

[MAZ] V. G. Maz'ya and I. Verbitsky, *Capacitary inequalities for fractional integrals*, Ark. Mat. 33 (1995), 81–115.

[MUR] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa 5 (1978), 489–507.

[URB] K. Urban, *Multiskalenverfahren für das Stokes-Problem und angepasste Wavelet-Basen*, Verlag der Augustinus-Buchhandlung, Aachen, 1995.

Pierre Gilles Lemarié-Rieusset
Laboratoire Analyse et Probabilités
Université d'Évry Val d'Essonne
IBGBI, 23 Bd. de France
91037 Évry Cedex, France
E-mail: plemarie@univ-evry.fr

Received December 21, 2011 (7386)