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Copies of ℓ_∞ in the space of Pettis integrable functions with integrals of finite variation

by

JUAN CARLOS FERRANDO (Elche)

Abstract. Let (Ω, Σ, μ) be a complete finite measure space and X a Banach space. We show that the space of all weakly μ-measurable (classes of scalarly equivalent) X-valued Pettis integrable functions with integrals of finite variation, equipped with the variation norm, contains a copy of ℓ_∞ if and only if X does.

1. Preliminaries. Along this paper X will be a Banach space over the field $\mathbb{K}$ of real or complex numbers. If (Ω, Σ) is a measurable space, we denote by $ca(\Sigma, X)$ the Banach space over $\mathbb{K}$ of all X-valued countably additive measures F on Σ equipped with the semivariation norm $\|F\|$, and by $bvca(\Sigma, X)$ the Banach space of all X-valued countably additive measures F of bounded variation on Σ with the variation norm $|F|$. Let $ca^+(\Sigma)$ denote the set of all positive and finite countably additive measures defined on Σ.

Let us recall some useful facts. If (Ω, Σ, μ) is a finite measure space, a weakly μ-measurable function $f : \Omega \to X$ is said to be *Dunford integrable* if $x^* f \in \mathcal{L}_1(\mu)$ for every $x^* \in X^*$, and if f is Dunford integrable and $E \in \Sigma$ the map $x^* \mapsto \int_E x^* f \, d\mu$, denoted by $(D) \int_E f \, d\mu$, is a continuous linear form on X^*. If $(D) \int_E f \, d\mu \in X$ for each $E \in \Sigma$ then f is said to be *Pettis integrable* and one writes $(P) \int_E f \, d\mu$ instead of $(D) \int_E f \, d\mu$. The *Pettis space* of all weakly measurable (classes of scalarly equivalent) Pettis integrable functions $f : \Omega \to X$ will be denoted by $\mathcal{P}_1(\mu, X)$ and the subspace of all those strongly measurable (classes of) functions by $P_1(\mu, X)$. These spaces are provided with the semivariation norm

$$\|f\|_{\mathcal{P}_1(\mu,X)} = \sup\Big\{ \int_\Omega |x^* f(\omega)| \, d\mu(\omega) : x^* \in X^*, \ \|x^*\| \le 1 \Big\}.$$

Neither $\mathcal{P}_1(\mu, X)$ nor $P_1(\mu, X)$ is in general a Banach space, although they are barrelled normed spaces [4]. According to a result of Pettis, if $f : \Omega \to X$

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is (weakly measurable and) Pettis integrable, the map $F : \Sigma \to X$ defined by $F(E) = (P)\int_E f(\omega)\,d\mu(\omega)$ is a μ-continuous countably additive X-valued measure, that is, $F \in ca_\mu(\Sigma, X)$. Moreover the linear operator $S : \mathcal{P}_1(\mu, X) \to ca(\Sigma, X)$ defined by $Sf = F$ is a linear isometry from $\mathcal{P}_1(\mu, X)$ into $ca(\Sigma, X)$, i.e. $\|Sf\| = \|f\|_{\mathcal{P}_1(\mu,X)}$. In addition, if f is strongly measurable, i.e. if $f \in P_1(\mu, X)$, then $Sf(\Sigma)$ is a relatively compact subset of X [1, Chapter VIII], so that $Sf \in cca(\Sigma, X)$. A finite measure space (Ω, Σ, μ) is called *perfect* if for each measurable function $f : \Omega \to \mathbb{R}$ and each set A in $\mathbb{R}$ with $f^{-1}(A) \in \Sigma$ there exists a Borel set $B \subseteq A$ such that $\mu(f^{-1}(B)) = \mu(f^{-1}(A))$. If (Ω, Σ, μ) is a perfect finite measure space and $f \in \mathcal{P}_1(\mu, X)$ then the linear operator $S_f : L_\infty(\mu) \to X$ defined by $S_f(\chi_E) = Sf(E)$ is compact and consequently $Sf(\Sigma)$ is a relatively compact subset of X.

If each $\mu \in ca^+(\Sigma)$ is purely atomic, then $ca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does [2]. Assuming that X has the Radon–Nikodým property with respect to each $\mu \in ca^+(\Sigma)$, the space $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does [5]. As a consequence, if each $\mu \in ca^+(\Sigma)$ is purely atomic, the space $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does. If there exists a nonzero atomless measure $\mu \in ca^+(\Sigma)$, the latter statement is no longer true [16]. However, if the range space of the measures is a dual Banach space X^*, then $bvca(\Sigma, X^*)$ contains a copy of c_0 or ℓ_∞ if and only if X^* does [15]. On the other hand, according to [6] and [9, 10] it is known that the Pettis space $P_1(\mu, X)$ contains a copy of c_0 if and only if X does.

Musiał [14, Section 13] considered the linear subspace of $\mathcal{P}_1(\mu, X)$, which he denoted by $\mathbb{P}V(\mu, X)$, formed by all those functions $f : \Omega \to X$ whose (indefinite) Pettis integral Sf has bounded variation, endowed with the variation norm. We shall denote this space by $\mathcal{M}(\Sigma, \mu, X)$. Like $\mathcal{P}_1(\mu, X)$, in general $\mathcal{M}(\Sigma, \mu, X)$ is not a complete normed space, although it can be shown as in [4] that it is barrelled. Since $\mathcal{M}(\Sigma, \mu, X)$ embeds in $bvca(\Sigma, X)$, by the previous results $\mathcal{M}(\Sigma, \mu, X^*)$ contains a copy of c_0 or ℓ_∞ if and only X^* does. The general case is not so easy due to the lack of a general criterion concerning the X-inheritance of copies of c_0 or ℓ_∞ in $bvca(\Sigma, X)$. However, in [8] we have shown that if the Pettis integral Sf of each $f \in \mathcal{M}(\Sigma, \mu, X)$ has separable range then the Musiał space $\mathcal{M}(\Sigma, \mu, X)$ contains a copy of c_0 if and only if X does. In particular, if the measure space (Ω, Σ, μ) is perfect, then $\mathcal{M}(\Sigma, \mu, X)$ contains a copy of c_0 if and only if X does. Finally, let us point out that for a general finite measure space (Ω, Σ, μ) the subspace $M(\Sigma, \mu, X)$ of $\mathcal{M}(\Sigma, \mu, X)$ consisting of all strongly measurable functions coincides with $L_1(\mu, X)$, so $M(\Sigma,\mu, X)$ always contains a copy of c_0 or ℓ_∞ if and only if X does (by [12] and [13], respectively).

In this paper we complete the research started in [8] by showing that $\mathcal{M}(\Sigma,\mu,X)$ contains a copy of ℓ_∞ if and only if X does. Nonetheless, our approach differs from that of [8] and it is closer (but not identical) to the strategy developed in [7].

2. Main theorem. In what follows, (Ω,Σ,μ) will be a finite measure space and as above $\mathcal{M}(\Sigma,\mu,X)$ will stand for the Musiał space of all those functions $f\in\mathcal{P}_1(\mu,X)$ whose associated measure F has bounded variation, endowed with the variation norm, which we shall denote by $|\ |_\Sigma$. If S is the canonical isometric embedding of $\mathcal{P}_1(\mu,X)$ into $bvca(\Sigma,X)$, defined by $(Sf)(E)=\int_E f\,d\mu$ for all $E\in\Sigma$, we shall denote by $|f|_\Sigma$ the norm of $f\in\mathcal{M}(\Sigma,\mu,X)$ on $\mathcal{M}(\Sigma,\mu,X)$, so by definition $|f|_\Sigma=|Sf|_\Sigma$.

Lemma 2.1. *If $\mathcal{M}(\Sigma,\mu,X)$ contains an isomorphic copy of ℓ_∞ then there exists a countably generated sub-σ-algebra Γ of Σ and a closed and separable linear subspace Y of X such that $\mathcal{M}(\Gamma,\mu|_\Gamma,Y)$ contains an isomorphic copy of ℓ_∞.*

Proof. Let K be an isomorphism from ℓ_∞ into $\mathcal{M}(\Sigma,\mu,X)$. Denote by $\{e_n:n\in\mathbb{N}\}$ the canonical unit sequence of ℓ_∞ and set $J:=S\circ K$. For each $m,n\in\mathbb{N}$ let $\{E^m_{n,i}:1\le i\le k(m,n)\}$ be a finite partition of Ω by elements of Σ such that

$$|Je_n|_\Sigma\le\sum_{i=1}^{k(m,n)}\|Je_n(E^m_{n,i})\|+\frac{1}{m},$$

and denote by Λ the algebra generated by the countable family

$$\{E^m_{n,i}:1\le i\le k(m,n),\ m,n\in\mathbb{N}\}.$$

Observe that Λ is also countable [11, 1.5 Theorem C], and denote by Γ the σ-algebra generated by Λ. Since clearly $\Omega\in\Gamma$, Γ is a sub-σ-algebra of Σ.

Let Y be the closure in X of the linear cover of the countable subset $\bigcup_{n=1}^\infty Je_n(\Lambda)$ of X formed by the union of the images of the countable set Λ by the measures Je_n. Suppose that $\Lambda=\{A_n:n\in\mathbb{N}\}$. Assume that X does not contain a copy of ℓ_∞ and define $J_n:\ell_\infty\to X$ by $J_n\xi=(J\xi)(A_n)$ for each $n\in\mathbb{N}$. Since ℓ_∞ does not live in X and J_n is a bounded linear operator for each $n\in\mathbb{N}$, all the operators J_n are weakly compact. So, according to [3], there exists an infinite subset N of $\mathbb{N}$ such that

$$J_n\xi=\sum_{i=1}^\infty\xi_iJ_ne_i$$

for each $n\in\mathbb{N}$ and $\xi\in\ell_\infty(N)$. Hence

$$J\xi(A_n)=\sum_{i=1}^\infty\xi_iJe_i(A_n)$$

in X for every $\xi \in \ell_\infty(N)$ and $n \in \mathbb{N}$. But since $Je_i(A_n) \in Y$ for every $i, n \in \mathbb{N}$ and Y is closed, we see that $J\xi(A_n) \in Y$ for every $\xi \in \ell_\infty(N)$ and $n \in \mathbb{N}$, i.e. $J\xi(A) \in Y$ for every $\xi \in \ell_\infty(N)$ and $A \in \Lambda$. By the classic theorem on monotone classes [11, 1.6 Theorem B], the family $\{E \in \Sigma : J\xi(E) \in Y \ \forall \xi \in \ell_\infty(N)\}$ contains the sub-σ-algebra Γ generated by Λ. So we conclude that $J\xi(E) \in Y$ for every $\xi \in \ell_\infty(N)$ and $E \in \Gamma$. There is no loss of generality in identifying N with $\mathbb{N}$.

Define a map $T : \ell_\infty \to \mathcal{M}(\Gamma, \mu|_\Gamma, Y)$ so that

$$\langle y^*, T\xi(\omega)\rangle = \langle \widetilde{y}^*, K\xi(\omega)\rangle$$

for all $y^* \in Y^*$, where $\widetilde{y}^*$ stands for a fixed norm-preserving extension of y^* to the whole of X. Let us see that T is well defined, linear and bounded. First, T is linear, since

$$\begin{aligned}\langle y^*, T(\alpha\zeta + \beta\xi)(\omega)\rangle &= \langle \widetilde{y}^*, K(\alpha\zeta + \beta\xi)(\omega)\rangle = \alpha\langle \widetilde{y}^*, K\zeta(\omega)\rangle + \beta\langle \widetilde{y}^*, K\xi(\omega)\rangle \\ &= \alpha\langle y^*, T\zeta(\omega)\rangle + \beta\langle y^*, T\xi(\omega)\rangle = \langle y^*, (\alpha T\zeta + \beta T\xi)(\omega)\rangle\end{aligned}$$

for $\zeta, \xi \in \ell_\infty$ and $\alpha, \beta \in \mathbb{K}$. Moreover, the function $T\xi : \Omega \to Y$ is weakly measurable since $\omega \mapsto \langle \widetilde{y}^*, (K\xi)(\omega)\rangle$ is μ-measurable for each $y^* \in Y^*$. As in addition $\omega \mapsto \langle y^*, (T\xi)(\omega)\rangle$ clearly belongs to $L_1(\mu)$, it follows that $T\xi$ is Dunford integrable. To show that $T\xi \in \mathcal{P}_1(\mu, Y)$ we proceed as follows. Given $\xi \in \ell_\infty$, consider the map $G_\xi : \Gamma \to Y^{**}$ defined by

$$G_\xi(E) = (D)\int_E T\xi(\omega)\, d\mu|_\Gamma(\omega)$$

for $E \in \Gamma$. We claim that $G_\xi = J\xi|_\Gamma$, so that $G_\xi(\Gamma) \subseteq Y$ and hence $T\xi \in \mathcal{P}_1(\mu, Y)$. In fact, if $y^* \in Y^*$ then

$$\begin{aligned}\langle y^*, G_\xi(E)\rangle &= \int_E \langle y^*, T\xi(\omega)\rangle\, d\mu|_\Gamma(\omega) = \int_E \langle \widetilde{y}^*, K\xi(\omega)\rangle\, d\mu(\omega) \\ &= \Big\langle \widetilde{y}^*, (P)\int_E K\xi\, d\mu \Big\rangle = \langle \widetilde{y}^*, (J\xi)(E)\rangle = \langle y^*, (J\xi)(E)\rangle\end{aligned}$$

since, as we have shown above, $(J\xi)(E) \in Y$ whenever $E \in \Gamma$. Thus $G_\xi(E) = (J\xi)(E)$ for every $E \in \Gamma$ as claimed. Moreover G_ξ has bounded variation since

$$\begin{aligned}\|G_\xi(E)\| &= \sup_{\|y^*\|\le 1} \Big| \Big\langle y^*, (P)\int_E T\xi\, d\mu|_\Gamma \Big\rangle \Big| = \sup_{\|y^*\|\le 1} \Big| \int_E \langle y^*, T\xi(\omega)\rangle\, d\mu|_\Gamma(\omega) \Big| \\ &= \sup_{\|y^*\|\le 1} \Big| \int_E \langle \widetilde{y}^*, K\xi(\omega)\rangle\, d\mu|_\Gamma(\omega) \Big| \le \|K\xi \cdot \chi_E\|_{\mathcal{P}_1(\mu, X)} \\ &= \|J\xi\|_\Sigma(E) \le |J\xi|_\Sigma(E)\end{aligned}$$

for every $E \in \Gamma$ due to the fact that $\|\widetilde{y}^*\| = \|y^*\|$ for every $y^* \in Y^*$. So if

$\{E_1, \ldots, E_n\}$ is a partition of Ω by elements of Γ then

$$\sum_{i=1}^{n} \|G_\xi(E_i)\| \le \sum_{i=1}^{n} |J\xi|_\Sigma(E_i) = |J\xi|_\Sigma(\Omega) = |J\xi|_\Sigma,$$

which implies that $|G_\xi|_\Gamma = |J\xi|_\Gamma|_\Gamma \le |J\xi|_\Sigma = |K\xi|_\Sigma < \infty$. This also shows that the map T is bounded, since by the preceding inequality

$$|T\xi|_\Gamma = |G_\xi|_\Gamma \le |K\xi|_\Sigma \le \|K\| \, \|\xi\|_\infty.$$

Further, given $m \in \mathbb{N}$, by the definition of Γ one has

$$|Je_n|_\Sigma \le \sum_{i=1}^{k(m,n)} \|Je_n(E_{n,i}^m)\| + \frac{1}{m} \le |Je_n|_\Gamma|_\Gamma + \frac{1}{m},$$

which implies that $|Je_n|_\Sigma = |Je_n|_\Gamma|_\Gamma$ for every $n \in \mathbb{N}$. Thus

$$|Te_n|_\Gamma = |G_{e_n}|_\Gamma = |Je_n|_\Gamma|_\Gamma = |Je_n|_\Sigma = |Ke_n|_\Sigma,$$

so that $\inf_{n\in\mathbb{N}} |Te_n|_\Gamma > 0$. Hence Rosenthal's ℓ_∞ theorem ensures that there exists an infinite subset $M \subseteq \mathbb{N}$ such that the restriction R of T to $\ell_\infty(M)$ is an isomorphism from $\ell_\infty(M)$ into the completion of $\mathcal{M}(\Gamma, \mu|_\Gamma, Y)$. Now, given $\zeta \in \ell_\infty(M)$, if $\xi \in \ell_\infty$ is defined by $\xi(i) = \zeta(i)$ if $i \in M$ and $\xi(i) = 0$ if $i \notin M$ then $R\zeta = T\xi \in \mathcal{M}(\Gamma, \mu|_\Gamma, Y)$, which ensures that the space $\mathcal{M}(\Gamma, \mu|_\Gamma, Y)$ contains a copy of ℓ_∞. ∎

THEOREM 2.2. *$\mathcal{M}(\Sigma, \mu, X)$ contains a copy of ℓ_∞ if and only if X does.*

Proof. If $\mathcal{M}(\Sigma, \mu, X)$ contains a copy of ℓ_∞, according to Lemma 2.1 there exists a countably generated sub-σ-algebra Γ of Σ and a closed and separable linear subspace Y of X such that $\mathcal{M}(\Gamma, \mu|_\Gamma, Y)$ contains a copy of ℓ_∞.

If $f \in \mathcal{M}(\Gamma,\mu|_\Gamma, Y)$ then $f \in \mathcal{P}_1(\mu|_\Gamma, Y)$, so that f is weakly $\mu|_\Gamma$-measurable. But since Y is separable, f is strongly $\mu|_\Gamma$-measurable, that is, $f \in P_1(\mu|_\Gamma, Y)$. This ensures that $\|f(\ )\|$ is $\mu|_\Gamma$-measurable and consequently

$$\int_\Omega \|f(\omega)\| \, d\mu|_\Gamma = |f|_\Gamma < \infty,$$

so f is Bochner integrable. This shows that $\mathcal{M}(\Gamma, \mu|_\Gamma, Y)$ coincides with $L_1(\Gamma, \mu|_\Gamma, Y)$. Thus $L_1(\Gamma, \mu|_\Gamma, Y)$ contains a copy of ℓ_∞, a contradiction since $L_1(\Gamma, \mu|_\Gamma, Y)$ is separable. Hence X must contain an isomorphic copy of ℓ_∞. ∎

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Juan Carlos Ferrando
Centro de Investigación Operativa
Edificio Torretamarit, Avda de la Universidad s/n
Universidad Miguel Hernández
E-03202 Elche (Alicante), Spain
E-mail: jc.ferrando@umh.es

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