

On James and Jordan–von Neumann constants and the normal structure coefficient of Banach spaces

by

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Abstract. Some relations between the James (or non-square) constant $J(X)$ and the Jordan–von Neumann constant $C_{\text{NJ}}(X)$, and the normal structure coefficient $N(X)$ of Banach spaces X are investigated. Relations between $J(X)$ and $J(X^*)$ are given as an answer to a problem of Gao and Lau [16]. Connections between $C_{\text{NJ}}(X)$ and $J(X)$ are also shown. The normal structure coefficient of a Banach space is estimated by the $C_{\text{NJ}}(X)$ -constant, which implies that a Banach space with $C_{\text{NJ}}(X)$ -constant less than $5/4$ has the fixed point property.

0. Introduction. Several results on the Jordan–von Neumann constant (which is usually called the von Neumann–Jordan constant) of a Banach space X , denoted by $C_{\text{NJ}}(X)$, have recently been obtained by the authors [25], [26], [38] (we also refer to [23] and [8] for the classical results). In particular, they determined or estimated $C_{\text{NJ}}(X)$ for various spaces X , and showed that some properties of X such as uniform non-squareness, super-reflexivity or type and cotype can be described in terms of the constant $C_{\text{NJ}}(X)$.

The aim of this paper is to clarify some relations between the $C_{\text{NJ}}(X)$ -constant and some other geometrical constants, especially the non-square constant $J(X)$ of James and the normal structure coefficient $N(X)$. In addition, we investigate the James constant $J(X)$ more carefully. Everything is supported by several examples of concrete Banach spaces with the calculation of these constants.

The paper is organized as follows: In Section 1 we collect necessary properties of the modulus of convexity and modulus of smoothness. In Section 2 the uniformly non-square spaces and the James constant $J(X)$ are consid-

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ered. A relation between $J(X)$ and $J(X^*)$ (X^* is the dual space of X) is given in Theorem 1, which partly answers a question of Gao and Lau [16], Question 7.3. Note that $J(X) \neq J(X^*)$ in general (cf. our Example 2). As a corollary we obtain the well known result that X is uniformly non-square if and only if X^* is uniformly non-square. Section 3 contains information about the $C_{\text{NJ}}(X)$ -constant and its connection with the $J(X)$ -constant. The estimates of the $C_{\text{NJ}}(X)$ -constant from below and above by the $J(X)$ -constant are the main result of this part (Theorem 3). We also calculate $J(X)$ and $C_{\text{NJ}}(X)$ for X being the two-dimensional Lorentz sequence space $\ell_{p,2}$ with $p \geq 2$.

In Section 4 we consider the relation between the $C_{\text{NJ}}(X)$ -constant and the normal structure coefficient of X in the context of the fixed point property. Gao and Lau [16] showed that if $J(X) < 3/2$, then X has normal structure. Prus [36] even estimated the normal structure coefficient $N(X)$ by $J(X)$. We present an estimate of $N(X)$ by the $C_{\text{NJ}}(X)$ -constant, which yields that if $C_{\text{NJ}}(X) < 5/4$, then the Banach space X as well as its dual X^* have the fixed point property. Finally, in the last Section 5, we compare the constants $J(X)$, $C_{\text{NJ}}(X)$ and $N(X)$ for X and for an isomorphic space Y by using the Banach–Mazur distance $d(X, Y)$. These estimates give the possibility to compute such constants for a Banach space with new but equivalent norms. Of course, these constants are not stable with respect to an equivalent renorming. We finish this part with the estimates of $J(X)$, $C_{\text{NJ}}(X)$ and $N(X)$ for the interesting Day–James ℓ_p - ℓ_q spaces and the Bynum $b_{p,q}$ spaces. Some open problems are also posed.

1. Preliminaries. Let $X = (X, \|\cdot\|)$ be a real Banach space. The *non-trivial Banach space* will mean later on that X is a real space and $\dim X \geq 2$ (but it can also be that X is a complex space and $\dim X \geq 1$). Geometrical properties of a Banach space X are determined by its unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ or its unit sphere $S_X = \{x \in X : \|x\| = 1\}$. The *modulus of convexity* of X (cf. [30], [34], [13], [35]) is a function $\delta_X : [0, 2] \rightarrow [0, 1]$ defined by

$$(1.1) \quad \delta_X(\varepsilon) = \inf\{1 - \|x + y\|/2 : x, y \in S_X, \|x - y\| \geq \varepsilon\}.$$

The function δ_X is continuous on $[0, 2]$, increasing on $[0, 2]$, strictly increasing on $[\varepsilon_0, 2]$, where $\varepsilon_0 = \varepsilon_0(X) = \sup\{\varepsilon \in (0, 2] : \delta_X(\varepsilon) = 0\}$ is the so-called *coefficient of convexity* of X , and $\delta_X(\varepsilon) \leq \varepsilon/2$ or even $\delta_X(\varepsilon) \leq \delta_{\ell_2}(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2}/4$. Also $\lim_{\varepsilon \rightarrow 2^-} \delta_X(\varepsilon) = 1 - \varepsilon_0(X)/2$. The function $\delta_X(\varepsilon)/\varepsilon$ is increasing on $(0, 2]$ but $\delta_X(\varepsilon)$ is not necessarily convex (Liokumovich 1973; cf. also our easier Example 2). The space X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon < 2$, i.e., $\varepsilon_0(X) = 0$. Day [9] (see also [30], [34]) proved the important result that if X is non-trivial, then

$$\begin{aligned}
 (1.2) \quad \delta_X(\varepsilon) &= \inf\{1 - \|x + y\|/2 : x, y \in B_X, \|x - y\| \geq \varepsilon\} \\
 &= \inf\{1 - \|x + y\|/2 : x, y \in S_X, \|x - y\| = \varepsilon\} \\
 &= \inf\{1 - \|x + y\|/2 : x, y \in B_X, \|x - y\| = \varepsilon\}.
 \end{aligned}$$

The *modulus of smoothness* of X (cf. [9], [30], [34], [28]) is a function $\varrho_X : [0, \infty) \rightarrow [0, \infty)$ defined by

$$(1.3) \quad \varrho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\}.$$

The function ϱ_X is increasing, continuous, convex on $[0, \infty)$, $\varrho_X(0) = 0$ and $\varrho_X(\tau) \leq \tau$. Moreover, if X is non-trivial, then $\varrho_X(\tau) \geq \varrho_{\ell_2}(\tau) = \sqrt{1 + \tau^2} - 1$ and the supremum in (1.3) can be taken over all $x, y \in B_X$. The space X is called *uniformly smooth* if $\varrho'_X(0) = \lim_{\tau \rightarrow 0} \varrho_X(\tau)/\tau = 0$. The following remarkable duality relationship with the modulus of convexity is due to Lindenstrauss [28] (see also [30], [34]): for $\tau > 0$,

$$(1.4) \quad \varrho_{X^*}(\tau) = \sup\{\tau\varepsilon/2 - \delta_X(\varepsilon) : \varepsilon \in [0, 2]\},$$

$$(1.5) \quad \varrho_X(\tau) = \sup\{\tau\varepsilon/2 - \delta_{X^*}(\varepsilon) : \varepsilon \in [0, 2]\},$$

which yields the equality $\varrho_X(\tau) = \varrho_{X^{**}}(\tau)$ and also the fact that X is uniformly convex (resp. uniformly smooth) if and only if its dual X^* is uniformly smooth (resp. uniformly convex). Moreover, if X is either uniformly convex or uniformly smooth, then X is reflexive.

REMARK 1. In the real one-dimensional case, i.e., when $X = (\mathbb{R}, |\cdot|)$, the modulus of convexity $\delta_X(\varepsilon)$ from (1.1) is $\delta_X(\varepsilon) = 1$ for $0 < \varepsilon \leq 2$, and the first one from (1.2) is $\delta_X(\varepsilon) = \varepsilon/2$ for $0 \leq \varepsilon \leq 2$; the modulus of smoothness $\varrho_X(\tau)$ from (1.3) is in this case $\varrho_X(\tau) = \max\{0, \tau - 1\}$ for $\tau \geq 0$.

2. Uniformly non-square spaces and James constant. A Banach space X is called *uniformly non-square* (James, 1964) if there exists a $\delta \in (0, 1)$ such that for any $x, y \in S_X$ either $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$. The constant

$$J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in S_X\}$$

is called the *non-square* or *James constant* of X .

PROPOSITION 1. *Let X be a non-trivial Banach space. The following conditions are equivalent.*

- (i) X is uniformly non-square.
- (ii) $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$.
- (iii) $\varepsilon_0(X) < 2$.
- (iv) $J(X) < 2$.
- (v) $\varrho_X(\tau_0) < \tau_0$ for some $\tau_0 > 0$.
- (vi) $\varrho_X(\tau) < \tau$ for all $\tau > 0$.

- (vii) $\varrho'_X(0) = \lim_{\tau \rightarrow 0} \varrho_X(\tau)/\tau < 1$.
- (viii) $\varepsilon_0(X^*) < 2$.
- (ix) $\varrho'_{X^*}(0) = \lim_{\tau \rightarrow 0} \varrho_{X^*}(\tau)/\tau < 1$.

Proof. We have the equivalences:

X is uniformly non-square

$$\begin{aligned} &\Leftrightarrow \exists \delta \in (0, 1) \text{ s.t. } x, y \in S_X, \|x - y\|/2 > 1 - \delta \Rightarrow \|x + y\|/2 \leq 1 - \delta \\ &\Leftrightarrow \exists \delta \in (0, 1) \text{ s.t. } x, y \in S_X, \|x - y\| > 2 - 2\delta \Rightarrow 1 - \|x + y\|/2 \geq \delta \\ &\Leftrightarrow \exists \delta \in (0, 1) \text{ s.t. } \delta_X(2 - 2\delta) \geq \delta. \end{aligned}$$

(i) \Rightarrow (ii). If X is uniformly non-square, then by taking $\varepsilon = 2 - 2\delta \in (0, 2)$ in the above equivalences, we obtain $\delta_X(\varepsilon) \geq 1 - \varepsilon/2 > 0$.

(ii) \Rightarrow (i). Assume that there exists $\varepsilon_0 \in (0, 2)$ such that $\delta_X(\varepsilon_0) > 0$, i.e. $\delta_X(\varepsilon_0) \geq \eta_0 > 0$ for some $\eta_0 \in (0, 1)$. If we set $2 - 2\delta = \varepsilon \in [\varepsilon_0, 2)$, then $\delta \in (0, 1 - \varepsilon_0/2]$ and

$$\delta_X(2 - 2\delta) = \delta_X(\varepsilon) \geq \delta_X(\varepsilon_0) \geq \eta_0 > 0,$$

which means that, for any $x, y \in S_X$, if $\|x - y\| \geq 2 - 2\delta$, then $1 - \|x + y\|/2 \geq \eta_0$. Let $\delta' = \min\{\delta, \eta_0\}$. Of course, $\delta' \in (0, 1)$. We want to show that either

$$\|x - y\|/2 \leq 1 - \delta' \quad \text{or} \quad \|x + y\|/2 \leq 1 - \delta'.$$

If $\|x - y\|/2 \leq 1 - \delta'$, then we are done. Assume then that $\|x - y\|/2 > 1 - \delta$. This gives $\|x - y\| > 2(1 - \delta') \geq 2(1 - \delta)$ and by the above assumption we get $1 - \|x + y\|/2 \geq \eta_0$ or $\|x + y\|/2 \leq 1 - \eta_0 \leq 1 - \delta'$. This shows that X is uniformly non-square.

(ii) \Leftrightarrow (iii) follows from the definition of $\varepsilon_0(X)$, and (i) \Leftrightarrow (iv) is again only the definition.

(v) \Leftrightarrow (vi) is a consequence of Lemma 1 below, and (v) \Leftrightarrow (vii) follows from the fact that $\varrho_X(\tau)/\tau$ is increasing.

(vii) \Leftrightarrow (viii) and (ix) \Leftrightarrow (iii) follow directly from the formulas

$$(2.1) \quad \varepsilon_0(X^*) = 2\varrho_X(0) = 2 \lim_{\tau \rightarrow 0} \frac{\varrho_X(\tau)}{\tau}, \quad \varepsilon_0(X) = 2\varrho'_{X^*}(0) = 2 \lim_{\tau \rightarrow 0} \frac{\varrho_{X^*}(\tau)}{\tau},$$

which are consequences of (1.4), (1.5), and this was observed by Turett [40] and Khamsi [27] (cf. also [1], Corollary 3.12).

LEMMA 1. *The modulus of smoothness of any Banach space X satisfies either the equality $\varrho_X(\tau) = \tau$ for all $\tau > 0$ or the strict inequality $\varrho_X(\tau) < \tau$ for all $\tau > 0$.*

Proof. It is enough to show that if $\varrho_X(\tau_0) = \tau_0$ for a certain $\tau_0 > 0$, then $\varrho_X(\tau) = \tau$ for all $\tau > 0$. Since $\varrho_X(\tau)/\tau$ is increasing and $\varrho_X(\tau) \leq \tau$, it follows that for $\tau \geq \tau_0$ we have $1 = \varrho_X(\tau_0)/\tau_0 \leq \varrho_X(\tau)/\tau \leq 1$, i.e.,

$\varrho_X(\tau) = \tau$ for $\tau \geq \tau_0$. Let $0 < \tau < \tau_0$ and assume that $\varrho_X(\tau) < \tau$. Then, by the convexity of ϱ_X , we have, for $\tau_1 > \tau_0$,

$$\begin{aligned} \tau_0 = \varrho_X(\tau_0) &= \varrho_X\left(\frac{\tau_0 - \tau}{\tau_1 - \tau}\tau_1 + \frac{\tau_1 - \tau_0}{\tau_1 - \tau}\tau\right) \\ &\leq \frac{\tau_0 - \tau}{\tau_1 - \tau}\varrho_X(\tau_1) + \frac{\tau_1 - \tau_0}{\tau_1 - \tau}\varrho_X(\tau) \\ &< \frac{\tau_0 - \tau}{\tau_1 - \tau}\tau_1 + \frac{\tau_1 - \tau_0}{\tau_1 - \tau}\tau = \tau_0, \end{aligned}$$

a contradiction. Therefore $\varrho_X(\tau) = \tau$.

REMARK 2. A uniformly non-square Banach space is super-reflexive (James [20], cf. also [3]), i.e., there exists an equivalent norm $\|\cdot\|$ on X in which $(X, \|\cdot\|)$ is uniformly convex. The converse implication is not true in the sense that there exists an equivalent norm $|\cdot|$ on X for which $\delta_{(X,|\cdot|)}(\varepsilon) = 0$ for all $0 < \varepsilon < 2$.

EXAMPLE 1. Let $1 < p < \infty$ and consider the new norms in ℓ_p :

$$\begin{aligned} \|x\| &= \max\left\{|x_1| + |x_2|, \left(\sum_{k=3}^{\infty} |x_k|^p\right)^{1/p}\right\}, \\ \|x\|' &= \max\left\{|x_1|, |x_2|, \left(\sum_{k=3}^{\infty} |x_k|^p\right)^{1/p}\right\}. \end{aligned}$$

Then $\|x\|_p \leq 2^{1/p}\|x\| \leq 2^{1/p+1}\|x\|' \leq 2^{1/p+1}\|x\| \leq 4\|x\|_p$, and since ℓ_p is uniformly convex it follows that the spaces $X = (\ell_p, \|\cdot\|)$ and $X = (\ell_p, \|\cdot\|')$ are super-reflexive but not uniformly non-square since $\delta_X(\varepsilon) = 0$ for all $0 < \varepsilon < 2$. To show the last equality it is enough to take $x = (1, 0, 0, \dots)$ and $y = (0, 1, 0, \dots)$. This example shows that there exists a super-reflexive Banach space such that $\delta_X(\varepsilon) = 0$ for all $0 < \varepsilon < 2$. More generally, if $X = (X, \|\cdot\|)$ is a non-trivial Banach space and X_1 is a two-dimensional subspace of X , then $X = X_1 \oplus X_2$ and on X_1 we can take any equivalent norm $\|x\|_1$ (for example, $\|x\|_1 = |x_1| + |x_2|$) such that $\delta_{(X, \|\cdot\|_1)}(\varepsilon) = 0$ for all $0 < \varepsilon < 2$. Let $\|\cdot\|_2$ be the norm $\|\cdot\|$ restricted to X_2 . Then, clearly, $\|x\| = \max\{\|x\|_1, \|x\|_2\}$ is equivalent to $\|\cdot\|$ and $\delta_{(X, \|\cdot\|)}(\varepsilon) = 0$ for all $0 < \varepsilon < 2$. Therefore we can formulate a corollary:

COROLLARY 1. *For any non-trivial Banach space X there is an equivalent norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is not uniformly non-square.*

Let us collect properties of the James constant $J(X)$ for non-trivial Banach spaces (see Casini [6], Gao–Lau [15] and [16]):

- (i) $\sqrt{2} \leq J(X) \leq 2$.
- (ii) $J(X) = \sqrt{2}$ if X is a Hilbert space; the converse is not true.

(iii) If $1 \leq p \leq \infty$ and $\dim L_p(\mu) \geq 2$, then $J(L_p(\mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$. Casini [6, Prop. 2.2] and also Gao–Lau [15, Th. 3.3] and [16, Th. 5.4] proved that

$$(2.2) \quad J(X) = \sup\{\varepsilon \in (0, 2) : \delta_X(\varepsilon) \leq 1 - \varepsilon/2\}.$$

$$(iv) \quad J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in B_X\}.$$

REMARK 3. In the one-dimensional real case, i.e., when $X = (\mathbb{R}, |\cdot|)$, we have $J(X) = 0$ and $J_1(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in B_X\} = 1$.

We give an example of a two-dimensional Banach space X for which $J(X^*) \neq J(X)$.

EXAMPLE 2. Let $X = \mathbb{R}^2$ with the ℓ_2 - ℓ_1 norm defined by

$$\|x\| = \begin{cases} \|x\|_2 & \text{if } x_1x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1x_2 \leq 0, \end{cases}$$

where $\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}$ for $p = 1, 2$. Then $\delta_X(\varepsilon) = 0$ if $0 \leq \varepsilon \leq \sqrt{2}$, $\delta_X(\varepsilon) = 1 - \sqrt{2 - \varepsilon^2}/2$ if $\sqrt{2} \leq \varepsilon \leq \sqrt{8/3}$, and $\delta_X(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/8}$ if $\sqrt{8/3} \leq \varepsilon \leq 2$ (see Goebel–Kirk [17], Ex. 5.8). Therefore, by (2.2), $J(X) = \sqrt{8/3}$. The dual space X^* of X has the ℓ_2 - ℓ_∞ norm

$$\|x\|^* = \begin{cases} \|x\|_2 & \text{if } x_1x_2 \geq 0, \\ \|x\|_\infty & \text{if } x_1x_2 \leq 0. \end{cases}$$

For the vectors $x = (-1, 1)$ and $y = (1/\sqrt{2}, 1/\sqrt{2})$ from the unit sphere S_{X^*} we have $\|x + y\|^* = \|x - y\|^* = 1 + 1/\sqrt{2}$, and so

$$J(X^*) \geq 1 + 1/\sqrt{2} > \sqrt{8/3} = J(X).$$

We now show the relation between the James constants for a space and for its dual, which answers a question of Gao and Lau [16], Question 7.3.

THEOREM 1. *For any Banach space X we have*

$$(2.3) \quad 2J(X) - 2 \leq J(X^*) \leq J(X)/2 + 1.$$

Proof. First observe that if $a, b \in S_X$, then

$$(*) \quad \|a + b\| + \|a - b\| \leq \min\{\|a + b\|, \|a - b\|\} + 2.$$

From the definition of $J(X)$ for any $\varepsilon > 0$ we can find $x, y \in S_X$ such that

$$\min\{\|x + y\|, \|x - y\|\} \geq J(X) - \varepsilon.$$

Moreover, by using the Hahn–Banach theorem we can find $u^*, v^* \in S_{X^*}$ such that

$$u^*(x + y) = \|x + y\| \quad \text{and} \quad v^*(x - y) = \|x - y\|.$$

Then

$$\begin{aligned} J(X^*) &\geq \min\{\|u^* + v^*\|, \|u^* - v^*\|\} \\ &\geq \|u^* + v^*\| + \|u^* - v^*\| - 2 \geq (u^* + v^*)(x) + (u^* - v^*)(y) - 2 \\ &= u^*(x + y) + v^*(x - y) - 2 = \|x + y\| + \|x - y\| - 2 \\ &\geq 2 \min\{\|x + y\|, \|x - y\|\} - 2 \geq 2\{J(X) - \varepsilon\} - 2. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we obtain

$$J(X^*) \geq 2J(X) - 2,$$

and the first inequality of (2.3) is proved. To obtain the other inequality, let $u^*, v^* \in S_{X^*}$ and suppose $\varepsilon > 0$. Then there exist $x, y \in S_X$ such that

$$(u^* + v^*)(x) > \|u^* + v^*\| - \varepsilon \quad \text{and} \quad (u^* - v^*)(y) > \|u^* - v^*\| - \varepsilon.$$

Thus

$$\begin{aligned} \min\{\|u^* + v^*\|, \|u^* - v^*\|\} &\leq (\|u^* + v^*\| + \|u^* - v^*\|)/2 \\ &< [(u^* + v^*)(x) + (u^* - v^*)(y) + 2\varepsilon]/2 = [u^*(x + y) + v^*(x - y) + 2\varepsilon]/2 \\ &\leq [\|u^*\| \cdot \|x + y\| + \|v^*\| \cdot \|x - y\| + 2\varepsilon]/2 = [\|x + y\| + \|x - y\| + 2\varepsilon]/2 \\ &\leq [\min\{\|x + y\|, \|x - y\|\} + 2 + 2\varepsilon]/2 \leq [J(X) + 2 + 2\varepsilon]/2. \end{aligned}$$

From this the second inequality readily follows.

REMARK 4. If X is not uniformly non-square (for example $X = \ell_1, \ell_\infty$ or c_0), then $J(X) = 2$ and we have equalities in (2.3):

$$2J(X) - 2 = J(X^*) = J(X)/2 + 1 = 2.$$

As an immediate consequence of Proposition 1 and Theorem 1 we have a result proved by several authors (implicitly James [21] and explicitly e.g. Fitzpatrick–Reznick [14] and Takahashi–Kato [38]; cf. also our Proposition 1, the equivalence of (i) with (iii) and (viii)).

COROLLARY 2. *A Banach space X is uniformly non-square if and only if its dual X^* is uniformly non-square.*

Let us mention that Schäffer (1970) introduced another definition of uniform non-squareness. X is called *uniformly non-square* (in the sense of Schäffer) if there exists $\lambda > 1$ such that for every $x, y \in S_X$ we have either $\|x + y\| > \lambda$ or $\|x - y\| > \lambda$. The constant

$$S(X) = \inf\{\max(\|x + y\|, \|x - y\|) : x, y \in S_X\}$$

is called the *Schäffer constant* of the space X . James's and Schäffer's definitions are equivalent in the sense that $J(X) < 2$ if and only if $S(X) > 1$. This follows from the equality below (the method of proof is taken from Casini [6], Proposition 2.1):

THEOREM 2. *If X is a non-trivial Banach space, then $J(X)S(X) = 2$.*

Proof. For any $x, y \in S_X, x \neq \pm y$, let

$$u = \frac{x + y}{\|x + y\|}, \quad v = \frac{x - y}{\|x - y\|}.$$

Then $u, v \in S_X$, and

$$u \pm v = \left(\frac{1}{\|x + y\|} \pm \frac{1}{\|x - y\|} \right) x + \left(\frac{1}{\|x + y\|} \mp \frac{1}{\|x - y\|} \right) y.$$

Therefore

$$\begin{aligned} \|u \pm v\| &\geq \left| \|x\| \left| \frac{1}{\|x + y\|} \pm \frac{1}{\|x - y\|} \right| - \|y\| \left| \frac{1}{\|x + y\|} \mp \frac{1}{\|x - y\|} \right| \right| \\ &= \left| \left| \frac{1}{\|x + y\|} \pm \frac{1}{\|x - y\|} \right| - \left| \frac{1}{\|x + y\|} \mp \frac{1}{\|x - y\|} \right| \right| \\ &= 2 \min \left(\frac{1}{\|x + y\|}, \frac{1}{\|x - y\|} \right) = \frac{2}{\max(\|x + y\|, \|x - y\|)}, \end{aligned}$$

and so

$$J(X) \geq \min(\|u + v\|, \|u - v\|) \geq \frac{2}{\max(\|x + y\|, \|x - y\|)}$$

for all $x, y \in S_X, x \neq \pm y$, i.e., $J(X) \geq 2/S(X)$. Also

$$\begin{aligned} \|u \pm v\| &\leq \|x\| \left| \frac{1}{\|x + y\|} \pm \frac{1}{\|x - y\|} \right| + \|y\| \left| \frac{1}{\|x + y\|} \mp \frac{1}{\|x - y\|} \right| \\ &= \left| \frac{1}{\|x + y\|} \pm \frac{1}{\|x - y\|} \right| + \left| \frac{1}{\|x + y\|} \mp \frac{1}{\|x - y\|} \right| \\ &= 2 \max \left(\frac{1}{\|x + y\|}, \frac{1}{\|x - y\|} \right) = \frac{2}{\min(\|x + y\|, \|x - y\|)}, \end{aligned}$$

and so

$$S(X) \leq \max(\|u + v\|, \|u - v\|) \leq \frac{2}{\min(\|x + y\|, \|x - y\|)}$$

for all $x, y \in S_X, x \neq \pm y$, i.e., $S(X) \leq 2/J(X)$. Combining the above inequalities, we obtain $J(X)S(X) = 2$.

REMARK 5. If $X = (\mathbb{R}, |\cdot|)$, then $J(X) = 0$ and $S(X) = 2$.

3. Jordan–von Neumann constant and James constant. The *Jordan–von Neumann constant* $C_{NJ}(X)$ of a Banach space X is defined by

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, \text{ not both zero} \right\}.$$

Since usually this constant is called the von Neumann–Jordan constant we will keep the notation $C_{NJ}(X)$. Let us collect its properties (see Jordan–von Neumann [23], Clarkson [8], Kato–Takahashi [25], [26] and [38]):

- (i) $1 \leq C_{\text{NJ}}(X) \leq 2$.
- (ii) X is a Hilbert space if and only if $C_{\text{NJ}}(X) = 1$.
- (iii) X is uniformly non-square if and only if $C_{\text{NJ}}(X) < 2$.
- (iv) $C_{\text{NJ}}(X) = C_{\text{NJ}}(X^*)$.
- (v) If $1 \leq p \leq \infty$ and $\dim L_p(\mu) \geq 2$, then $C_{\text{NJ}}(L_p(\mu)) = 2^{2/r-1}$ with $r = \min\{p, p'\}$.

THEOREM 3. For any non-trivial Banach space X ,

$$(3.1) \quad \frac{1}{2}J(X)^2 \leq C_{\text{NJ}}(X) \leq \frac{J(X)^2}{(J(X) - 1)^2 + 1}.$$

Proof. For any $x, y \in S_X$, we have

$$\begin{aligned} \min\{\|x + y\|^2, \|x - y\|^2\} &\leq \frac{\|x + y\|^2 + \|x - y\|^2}{2} \\ &= 2 \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq 2C_{\text{NJ}}(X), \end{aligned}$$

which implies the left inequality of (3.1). To prove the right inequality we first observe that

$$(3.2) \quad C_{\text{NJ}}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, \|x\| = 1, \|y\| \leq 1 \right\}.$$

In fact, if $0 \neq \|x\| \geq \|y\|$ (for $\|x\| \leq \|y\| \neq 0$ the proof is similar), then

$$\|x \pm y\| = \|x\| \left\| \frac{x}{\|x\|} \pm \frac{y}{\|x\|} \right\|$$

and so

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} = \frac{\left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\|^2 + \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\|^2}{2 \left(1 + \left\| \frac{y}{\|x\|} \right\|^2 \right)},$$

which shows that the supremum in the definition of $C_{\text{NJ}}(X)$ can be taken just over $x, y \in X$ such that $\|x\| = 1$ and $\|y\| \leq 1$.

To obtain the right estimate in (3.1) we consider two cases (of course for $\|x\| = 1$ and $\|y\| \leq 1$):

CASE 1: $\|y\| = t \geq J(X) - 1$. Then

$$\begin{aligned} A := \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} &\leq \frac{(\|x\| + \|y\|)^2 + (\min\{\|x + y\|, \|x - y\|\})^2}{2(\|x\|^2 + \|y\|^2)} \\ &\leq \frac{(1 + t)^2 + J(X)^2}{2(1 + t^2)} =: f(t). \end{aligned}$$

Since $f'(t) = (1 - tJ(X)^2 - t^2)/(1 + t^2)^2$ it follows that f is increasing on $(0, t_0)$ and decreasing on $(t_0, 1)$, where $t_0 = (-J(X)^2 + \sqrt{J(X)^4 + 4})/2$. Now $J(X) - 1 \geq t_0$ and so

$$A \leq f(t) \leq f(J(X) - 1) = \frac{J(X)^2}{(J(X) - 1)^2 + 1}.$$

CASE 2: $\|y\| = t \leq J(X) - 1$. Then

$$A \leq \frac{2(\|x\| + \|y\|)^2}{2(\|x\|^2 + \|y\|^2)} = \frac{(1 + t)^2}{1 + t^2} =: g(t).$$

Since $g'(t) = 2(1 - t^2)/(1 + t^2)^2$ it follows that g is increasing on $(0, 1]$ and so

$$A \leq g(t) \leq g(J(X) - 1) = \frac{J(X)^2}{(J(X) - 1)^2 + 1}.$$

Thus, in both cases,

$$A \leq \frac{J(X)^2}{(J(X) - 1)^2 + 1}.$$

This completes the proof.

REMARK 6. (a) Immediately from Proposition 1 and Theorem 3 we obtain the result of (iii) above: a non-trivial Banach space X is uniformly non-square if and only if $C_{\text{NJ}}(X) < 2$.

(b) Concerning equalities and strict inequalities in (3.1), we have:

(i) If X is not uniformly non-square, then $J(X) = C_{\text{NJ}}(X) = 2$ and we have both equalities in (3.1): $J(X)^2/2 = C_{\text{NJ}}(X) = J(X)^2/[(J(X) - 1)^2 + 1] = 2$.

(ii) For a Hilbert space H , we have $J(H)^2/2 = C_{\text{NJ}}(H) = 1$ and so $J(H)^2/[(J(H) - 1)^2 + 1] \approx 1.7$.

(iii) If $1 \leq p \leq \infty$ and $\dim L_p(\mu) \geq 2$, then $J(L_p(\mu)) = 2^{1/r}$ and $C_{\text{NJ}}(L_p(\mu)) = 2^{2/r-1}$, with $r = \min\{p, p'\}$, from which we obtain $J(L_p(\mu))^2/2 = C_{\text{NJ}}(L_p(\mu))$. Also for the space $X = L_p[L_q]$ (the L_q -valued L_p -space) we have the same equalities with $r = \min\{p, p', q, q'\}$.

(iv) There are Lorentz sequence spaces $X = \ell_{p,2}$ for which we have the equality $J(X)^2/2 = C_{\text{NJ}}(X)$.

EXAMPLE 3. Let $\ell_{p,2}$ ($2 \leq p < \infty$) be a two-dimensional Lorentz sequence space with the norm

$$\|x\|_{p,2} = (x_1^{*2} + 2^{2/p-1}x_2^{*2})^{1/2},$$

where (x_1^*, x_2^*) is the rearrangement of $(|x_1|, |x_2|)$ such that $x_1^* \geq x_2^*$ (cf. [29]). We have

$$(3.3) \quad J(\ell_{p,2}) = \frac{2}{(1 + 2^{2/p-1})^{1/2}} \quad \text{and} \quad C_{\text{NJ}}(\ell_{p,2}) = \frac{2}{1 + 2^{2/p-1}}.$$

Let us show (3.3). First, observe that $\|x\|_2 \leq [2/(1+2^{2/p-1})]^{1/2}\|x\|_{p,2}$. Then

$$\begin{aligned} \|x + y\|_{p,2}^2 + \|x - y\|_{p,2}^2 &\leq (\|x + y\|_2^2 + \|x - y\|_2^2) = 2(\|x\|_2^2 + \|y\|_2^2) \\ &\leq \frac{2}{1 + 2^{2/p-1}} 2(\|x\|_{p,2}^2 + \|y\|_{p,2}^2) \end{aligned}$$

or

$$C_{NJ}(\ell_{p,2}) \leq 2/(1 + 2^{2/p-1}).$$

On the other hand, for $x_0 = (\alpha, \alpha)$ and $y_0 = (\alpha, -\alpha)$ with $\alpha = 1/(1 + 2^{2/p-1})^{1/2}$, we have

$$C_{NJ}(\ell_{p,2}) \geq \frac{\|x_0 + y_0\|_{p,2}^2 + \|x_0 - y_0\|_{p,2}^2}{2(\|x_0\|_{p,2}^2 + \|y_0\|_{p,2}^2)} = 2\alpha^2 = \frac{2}{1 + 2^{2/p-1}}.$$

Also $J(\ell_{p,2}) \geq 2\alpha = 2/(1 + 2^{2/p-1})^{1/2}$, and by the first estimate in (3.1) we obtain

$$J(X) \leq \sqrt{2C_{NJ}(X)} = 2\alpha = 2/(1 + 2^{2/p-1})^{1/2},$$

and the proof of (3.3) is complete.

REMARK 7. There exists a two-dimensional Banach space X for which $J(X)^2/2 < C_{NJ}(X)$.

EXAMPLE 4 (ℓ_∞ - ℓ_1 norm). Let $X = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1x_2 \leq 0. \end{cases}$$

Then $\delta_X(\varepsilon) = \max\{0, (\varepsilon - 1)/2\}$ and $\varrho_X(\tau) = \max\{\tau/2, \tau - 1/2\}$. Therefore, by (2.2),

$$J(X) = 3/2.$$

If we take $x = (1, 1)$ and $y = (0, 1)$, then $C_{NJ}(X) \geq 5/4$ and so

$$J(X)^2/2 = 9/8 < 5/4 \leq C_{NJ}(X).$$

Note that the dual space X^* of X is given by the ℓ_1 - ℓ_∞ norm and is isometric to X .

REMARK 8. If $\dim X = 2$, then $C_{NJ}(X) = 2$ if and only if X is isometric to ℓ_∞^2 . If $\dim X < \infty$, then $C_{NJ}(X) = 2$ if and only if X contains a two-dimensional subspace isometric to ℓ_∞^2 .

4. Jordan–von Neumann constant and the normal structure coefficient. A Banach space X is said to have *normal structure* if $r(K) < \text{diam}(K)$ for every non-singleton closed bounded convex subset K of X , where $\text{diam}(K) := \sup\{\|x - y\| : x, y \in K\}$ is the diameter of K and $r(K) := \inf\{\sup\{\|x - y\| : y \in K\} : x \in K\}$ is the Chebyshev radius of K . The *normal structure coefficient* (Bynum, 1980) of X is the number $N(X) = \inf\{\text{diam}(K)/r(K) : K \subset X \text{ bounded and convex, } \text{diam}(K) > 0\}$.

Obviously, $1 \leq N(X) \leq 2$. The space X is said to have *uniform normal structure* if $N(X) > 1$. A Banach space X is said to have the *fixed point property* (FPP) (for non-expansive mappings) provided for any non-empty bounded convex subset K of X , every non-expansive mapping $T : K \rightarrow K$ has a fixed point. The following results are well known (see Goebel–Kirk [17], Th. 4.1 and 4.4):

(a) If X is a reflexive space and has normal structure, then X has FPP (Kirk, 1965).

(b) If X has uniform normal structure, then X is reflexive and hence X has FPP.

We now collect some properties of the normal structure coefficient (see Bynum [5], Maluta [32], Domínguez Benavides [11]; cf. also [17], [1] and [18]):

(i) $N(X) \geq 1/(1 - \delta_X(1))$; in particular, if X is uniformly convex, then $N(X) > 1$ and X has FPP.

(ii) If X is non-reflexive, then $N(X) = 1$.

(iii) If $\dim X = \infty$, then $N(X) \leq \sqrt{2}$.

(iv) If $1 \leq p \leq \infty$ and a σ -finite measure space (Ω, Σ, μ) is such that $\dim L_p(\mu) = \infty$, then $N(L_p(\mu)) = \min\{2^{1/p}, 2^{1-1/p}\}$.

Gao and Lau [16] showed that if $J(X) < 3/2$ (or equivalently $\delta_X(3/2) > 1/4$), then X has uniform normal structure. Prus [36] gave an estimate of $N(X)$ by $J(X)$ which contains Gao–Lau’s and Bynum’s results: For any non-trivial Banach space X ,

$$(4.1) \quad N(X) \geq J(X) + 1 - \{(J(X) + 1)^2 - 4\}^{1/2}.$$

We present here the following estimate of $N(X)$ by the $C_{NJ}(X)$ -constant:

THEOREM 4. *For any non-trivial Banach space X ,*

$$(4.2) \quad N(X) \geq (C_{NJ}(X) - 1/4)^{-1/2}.$$

Proof. Let K be an arbitrary bounded convex subset of X with $d := \text{diam}(K) > 0$. Then for any $\varepsilon > 0$ there exist u and v in K such that

$$\|u - v\| > d(1 - \varepsilon).$$

Put $z = (u + v)/2 \in K$. For any $x \in K$ consider $\xi = x - u$ and $\eta = x - v$. Then since

$$\frac{\|\xi + \eta\|^2 + \|\xi - \eta\|^2}{2(\|\xi\|^2 + \|\eta\|^2)} \leq C_{NJ}(X),$$

it follows that

$$\frac{\|2x - u - v\|^2 + \|-u + v\|^2}{2(\|x - u\|^2 + \|x - v\|^2)} \leq C_{NJ}(X)$$

and hence,

$$\begin{aligned} \|x - z\|^2 &\leq \frac{1}{2}C_{\text{NJ}}(X)(\|x - u\|^2 + \|x - v\|^2) - \frac{1}{4}\| -u + v\|^2 \\ &\leq d^2C_{\text{NJ}}(X) - \frac{1}{4}d^2(1 - \varepsilon)^2. \end{aligned}$$

Thus

$$r(K) \leq \sup\{\|x - z\| : x \in K\} \leq d\{C_{\text{NJ}}(X) - \frac{1}{4}(1 - \varepsilon)^2\}^{1/2}.$$

Letting $\varepsilon \rightarrow 0$, we obtain (4.2).

Theorem 4 and the fact that $C_{\text{NJ}}(X) = C_{\text{NJ}}(X^*)$ yield in a straightforward way the corollary below. We should also mention here that the normal and uniform normal structures are not inherited by the dual spaces in general (cf. [4], [37]).

COROLLARY 3. *If $C_{\text{NJ}}(X) < 5/4$, then X as well as its dual X^* have uniform normal structure, and hence X and X^* have the fixed point property.*

REMARK 9. The assumption $C_{\text{NJ}}(X) < 5/4$ implies, by the first inequality in (3.1), that $J(X) \leq \sqrt{2C_{\text{NJ}}(X)} < \sqrt{10}/2$ and this is a little weaker than Gao–Lau’s condition $J(X) < 3/2$. Gao and Lau asked in [16], Question 7.1, whether the estimate $J(X) < 3/2$ is sharp for X to have uniform normal structure. Note that in the class of X for which $J(X)^2/2 = C_{\text{NJ}}(X)$ the constant $3/2$ is not sharp since, by Theorem 4, $J(X) < \sqrt{10}/2$ implies that X has uniform normal structure.

REMARK 10. Our estimate (4.2), like any other known estimates, is still not very sharp in certain Banach spaces. For instance, $N(\ell_2) = \sqrt{2}$ and the estimate (4.2) gives $N(\ell_2) \geq (1 - 1/4)^{-1/2} = 2/\sqrt{3}$. On the other hand, the bound (4.2) gives information about the uniform normal structure of X when $C_{\text{NJ}}(X) < 5/4$ but not when $C_{\text{NJ}}(X)$ is close to 2.

The following example shows that there exists a Banach space with uniform normal structure whose C_{NJ} -constant is arbitrarily near to 2. This space is even not strictly convex and it is also an example of a uniformly convex (Hilbert) space which has an equivalent norm without normal structure.

EXAMPLE 5. For $\lambda \geq 1$ let X_λ be the space ℓ_2 (cf. [2], Th. 2; [3], p. 213, and [1], pp. 145–146) with the norm

$$|x|_\lambda = \max\{\|x\|_2, \lambda\|x\|_\infty\}.$$

Then, since $\|x\|_2 \leq |x|_\lambda \leq \lambda\|x\|_2$ it follows that X_λ is reflexive. Moreover,

$$C_{\text{NJ}}(X_\lambda) = \min\{2, \lambda^2\}, \quad J(X_\lambda) = \min\{2, \lambda\sqrt{2}\}$$

(cf. Kato–Takahashi [25], Proposition 1) and by [5] together with the considerations in [1], p. 146,

$$N(X_\lambda) = \max\{1, \sqrt{2}/\lambda\}.$$

Thus X_λ has uniform normal structure if and only if $\lambda < \sqrt{2}$. The space X_λ has normal structure if and only if $\lambda < \sqrt{2}$ (cf. [2], Th. 2), and it has FPP for all $\lambda \geq 1$ (cf. Goebel–Kirk [17], Ex. 6.3, Bynum [5], Th. 6, and [18], p. 772). We mention here that one of the first who considered the space $X_{\sqrt{2}}$ as an example of a reflexive Banach space with FPP but without normal structure was Karlovitz in 1976 (cf. [2] and [18]). Note that X_λ is not strictly convex for $\lambda > 1$ and $C_{\text{NJ}}(X_\lambda) = \lambda^2 \rightarrow 2$ as $\lambda \rightarrow \sqrt{2}$. Moreover, X_λ is the Orlicz sequence space ℓ_{φ_λ} generated by the convex function $\varphi_\lambda(u) = u^2$ for $0 \leq u \leq 1/\lambda$ and $\varphi_\lambda(u) = \infty$ for $u > 1/\lambda$ (cf. [31]).

5. Banach–Mazur distance and stability under norm perturbations. For isomorphic Banach spaces X and Y , the *Banach–Mazur distance* between X and Y , denoted by $d(X, Y)$, is defined to be the infimum of $\|T\| \cdot \|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y (cf. [39]). Bynum [5] proved that if X and Y are isomorphic Banach spaces, then

$$(5.1) \quad \frac{N(X)}{d(X, Y)} \leq N(Y) \leq N(X)d(X, Y).$$

THEOREM 5. *If X and Y are isomorphic Banach spaces, then*

$$(5.2) \quad \frac{J(X)}{d(X, Y)} \leq J(Y) \leq J(X)d(X, Y)$$

and

$$(5.3) \quad \frac{C_{\text{NJ}}(X)}{d(X, Y)^2} \leq C_{\text{NJ}}(Y) \leq C_{\text{NJ}}(X)d(X, Y)^2.$$

In particular, if X and Y are isometric, then $J(X) = J(Y)$ and $C_{\text{NJ}}(X) = C_{\text{NJ}}(Y)$.

Proof. Let $x, y \in S_X$. For each $\varepsilon > 0$ there exists an isomorphism T from X onto Y such that $\|T\| \cdot \|T^{-1}\| \leq (1 + \varepsilon)d(X, Y)$. Put

$$x' = \frac{Tx}{\|T\|}, \quad y' = \frac{Ty}{\|T\|}.$$

Then $x', y' \in B_Y$ since

$$\|x'\| = \frac{\|Tx\|}{\|T\|} \leq \|x\| = 1, \quad \|y'\| = \frac{\|Ty\|}{\|T\|} \leq \|y\| = 1.$$

By property (iv) we obtain

$$\begin{aligned} \min\{\|x + y\|, \|x - y\|\} &= \|T\| \min\{\|T^{-1}(x' + y')\|, \|T^{-1}(x' - y')\|\} \\ &\leq (1 + \varepsilon)d(X, Y) \min\{\|x' + y'\|, \|x' - y'\|\} \leq (1 + \varepsilon)d(X, Y)J(Y), \end{aligned}$$

and since $x, y \in S_X$ were arbitrary, $J(X) \leq (1 + \varepsilon)d(X, Y)J(Y)$, which gives the first estimate in (5.2). The second estimate follows by just interchanging X and Y . The estimates in (5.3) can be proved similarly.

COROLLARY 4. *Let $X = (X, \|\cdot\|)$ be a non-trivial Banach space and let $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$ and $x \in X$,*

$$\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|.$$

Then $\frac{\alpha}{\beta}N(X) \leq N(X_1) \leq \frac{\beta}{\alpha}N(X)$ and

$$(5.4) \quad \begin{aligned} \frac{\alpha}{\beta}J(X) &\leq J(X_1) \leq \frac{\beta}{\alpha}J(X), \\ \frac{\alpha^2}{\beta^2}C_{\text{NJ}}(X) &\leq C_{\text{NJ}}(X_1) \leq \frac{\beta^2}{\alpha^2}C_{\text{NJ}}(X). \end{aligned}$$

Proof. This follows immediately from Theorem 5 and the fact that $d(X, X_1) \leq \beta/\alpha$.

We say that a Banach space X is *finitely representable* in a Banach space Y (James [21]; cf. Beauzamy [3], p. 217) if, for any $\varepsilon > 0$ and for every finite-dimensional subspace X_0 of X , there exists a finite-dimensional subspace Y_0 of Y with $\dim Y_0 = \dim X_0$ such that $d(X_0, Y_0) \leq 1 + \varepsilon$.

COROLLARY 5. (i) *Let X be a Banach space which is finitely representable in Y . Then $J(X) \leq J(Y)$, $C_{\text{NJ}}(X) \leq C_{\text{NJ}}(Y)$ and $N(X) \leq N(Y)$.*
(ii) *$J(X^{**}) = J(X)$, $N(X^{**}) = N(X)$.*

Proof. (i) It is almost the same as the proof of Theorem 5 and (5.1) so we leave out the details.

(ii) By (i) and the principle of local reflexivity which asserts that X^{**} is finitely representable in X (cf. [41]) it follows that $J(X^{**}) \leq J(X)$. On the other hand, X is isometric to a subspace Y of X^{**} and, therefore, $J(X) = J(Y) \leq J(X^{**})$. The proof of the second equality in (ii) is similar.

REMARK 11. Let X be an infinite-dimensional Banach space and let

$$\begin{aligned} p_X &= \sup\{p \in [1, 2] : X \text{ is of type } p\}, \\ q_X &= \inf\{q \in [2, \infty] : X \text{ is of cotype } q\} \end{aligned}$$

(see [30] for details). By a theorem of Maurey–Pisier [33], ℓ_{p_X} and ℓ_{q_X} are both finitely representable in X . Thus, by using Corollary 5(i) and the values of the James and Jordan–von Neumann constants for ℓ_p -spaces, we obtain

the following estimates:

$$\begin{aligned} J(X) &\geq \max\{2^{1/p_X}, 2^{1-1/q_X}\}, \\ C_{\text{NJ}}(X) &\geq 2^{2/\min\{p_X, q'_X\}-1}, \\ N(X) &\geq \min\{2^{1/p_X}, 2^{1-1/q_X}\}. \end{aligned}$$

In particular, if $J(X) = \sqrt{2}$, then $p_X = q_X = 2$.

COROLLARY 6. *If $\dim X = 2$, then*

$$\sqrt{2}/d(X, \ell_2^2) \leq N(X) \leq \sqrt{2} d(X, \ell_2^2), \quad \sqrt{2}/d(X, \ell_2^2) \leq J(X) \leq \sqrt{2} d(X, \ell_2^2)$$

and

$$1/d(X, \ell_2^2)^2 \leq C_{\text{NJ}}(X) \leq d(X, \ell_2^2)^2.$$

REMARK 12. By using Corollary 6 we can also prove Remark 8. In fact, if $C_{\text{NJ}}(X) = 2$, then since $d(X, \ell_2^2) \leq \sqrt{2}$, it follows from Corollary 6 that

$$2 = C_{\text{NJ}}(X) \leq d(X, \ell_2^2)^2 \leq 2$$

and so $d(X, \ell_2^2) = \sqrt{2}$. Therefore a two-dimensional space X is isometric to ℓ_∞^2 (cf. [39], Proposition 37.4) and we have in fact proved the crucial part of Remark 8.

We illustrate the above remarks and results by the following examples. They will show that evidently the James constant, the Jordan–von Neumann constant and the normal structure coefficient are not stable with respect to equivalent renorming. On the other hand, the examples will also show that Theorem 5 and Corollaries 4–6 can give both precise estimates of these constants and only rough results.

EXAMPLE 6. (a) For $2 \leq p < \infty$ and $\lambda \geq 1$ let $X_{\lambda,p}$ be the space ℓ_p with the norm

$$\|x\|_{\lambda,p} = \max\{\|x\|_p, \lambda\|x\|_\infty\}$$

(cf. [25] and our Example 5, with $p = 2$). Then

$$\|x\|_p \leq \|x\|_{\lambda,p} \leq \lambda\|x\|_p \quad \text{for all } x \in \ell_p,$$

and, according to Corollary 4,

$$\begin{aligned} J(X_{\lambda,p}) &\leq \min\{2, \lambda J(\ell_p)\} = \min\{2, \lambda 2^{1-1/p}\}, \\ C_{\text{NJ}}(X_{\lambda,p}) &\leq \min\{2, \lambda^2 C_{\text{NJ}}(\ell_p)\} = \min\{2, \lambda^2 2^{1-2/p}\}, \\ N(X_{\lambda,p}) &\leq \max\{1, N(\ell_p)/\lambda\} = \max\{1, 2^{1/p}/\lambda\}. \end{aligned}$$

Taking $x = (a/\lambda, a/\lambda, 0, 0, \dots)$ and $y = (a/\lambda, -a/\lambda, 0, 0, \dots)$ with $a = \min(1, \lambda 2^{-1/p})$ we even obtain equalities for the first two constants:

$$J(X_{\lambda,p}) = \min\{2, \lambda 2^{1-1/p}\}, \quad C_{\text{NJ}}(X_{\lambda,p}) = \min\{2, \lambda^2 2^{1-2/p}\}.$$

If we take, as in [1, p. 146] where the case of $p = 2$ was considered, the set A as the closure of $\text{conv}(\{e_n : n \in \mathbb{N}\})$, then $\text{diam}(A) = \max\{2^{1/p}, \lambda\}$ and $r(A) = \lambda$ and so

$$N(X_{\lambda,p}) = \max\{1, 2^{1/p}/\lambda\}.$$

(b) For $1 \leq p \leq 2$ and $\lambda \geq 1$ let $Y_{\lambda,p}$ be the space $L_p[0, 1]$ with the norm

$$\|x\|_{\lambda,p} = \max\{\|x\|_p, \lambda\|x\|_1\}.$$

As in (a) we obtain

$$J(Y_{\lambda,p}) = \min\{2, \lambda 2^{1/p}\}, \quad C_{\text{NJ}}(Y_{\lambda,p}) = \min\{2, \lambda^2 2^{2/p-1}\}$$

(equalities are realized for $x = a\chi_{[0,1/2]}$ and $y = a\chi_{[1/2,1]}$ with $a = \min\{2^{1/p}, 2/\lambda\}$) and

$$N(Y_{\lambda,p}) \geq \max\{1, 2^{1-1/p}/\lambda\}.$$

Taking as K the set formed by the Rademacher functions r_1, \dots, r_m we have $\|r_k\|_{\lambda,p} = 1$ and $\|r_k - r_j\|_{\lambda,p} = \max\{2^{1-1/p}, \lambda\}$ for all $k \neq j$ and also (cf. [1, p. 129]) $r(K) \geq (1 - 1/m)\lambda$, which gives

$$\text{diam}(K)/r(K) \leq \frac{\max\{2^{1-1/p}, \lambda\}}{(1 - 1/m)\lambda} \rightarrow \max\{1, 2^{1-1/p}/\lambda\} \quad \text{as } m \rightarrow \infty.$$

Therefore $N(Y_{\lambda,p}) = \max\{1, 2^{1-1/p}/\lambda\}$.

EXAMPLE 7. For $\lambda > 0$ let Z_λ be \mathbb{R}^2 with the norm

$$|x|_\lambda = (\|x\|_2^2 + \lambda\|x\|_\infty^2)^{1/2}.$$

Then

$$\|x\|_2 \leq |x|_\lambda \leq (1 + \lambda)^{1/2}\|x\|_2 \quad \text{for all } x \in \mathbb{R}^2,$$

and the estimates we get by using Corollary 4: $J(Z_\lambda) \leq \min\{2, (2 + 2\lambda)^{1/2}\}$, $C_{\text{NJ}}(Z_\lambda) \leq \min\{2, 1 + \lambda\}$, $N(Z_\lambda) \geq \max\{1, [(2/(1 + \lambda))^{1/2}]\}$, are rather poor, because calculations show that

$$J(Z_\lambda) = 2[(\lambda + 1)/(\lambda + 2)]^{1/2} \quad \text{and} \quad C_{\text{NJ}}(Z_\lambda) = 2(\lambda + 1)/(\lambda + 2).$$

EXAMPLE 8 (Day–James ℓ_p - ℓ_q spaces and Bynum $b_{p,q}$ -spaces). For $1 \leq p, q \leq \infty$ denote by ℓ_p - ℓ_q the Day–James space, i.e., \mathbb{R}^2 with the norm defined by

$$|x|_{p,q} = \begin{cases} \|x\|_p & \text{if } x_1 x_2 \geq 0, \\ \|x\|_q & \text{if } x_1 x_2 \leq 0. \end{cases}$$

James [19, p. 561] considered the ℓ_p - $\ell_{p'}$ space as an example of a Banach space which is isometric to its dual but which is not given by a Hilbert norm when $p \neq 2$. Day [10] considered even more general spaces, namely, if $(X, \|\cdot\|)$ is a two-dimensional Banach space and $(X^*, \|\cdot\|^*)$ its dual, then

the X - X^* space is the space X with the norm defined by

$$|x|_{X, X^*} = \begin{cases} \|x\| & \text{if } x_1 x_2 \geq 0, \\ \|x\|^* & \text{if } x_1 x_2 \leq 0. \end{cases}$$

The infinite-dimensional version of the Day–James ℓ_p - ℓ_q space is the space $b_{p,q}$ which was investigated by Bynum [4] (usually this space is denoted by $\ell_{p,q}$, but this notation suggests the standard Lorentz sequence spaces, therefore we use the notation $b_{p,q}$ instead). For $1 \leq p, q \leq \infty$ denote by $b_{p,q}$ the ℓ_p space with the norm

$$|x|_{p,q} = \begin{cases} (\|x^+\|_p^q + \|x^-\|_p^q)^{1/q} & \text{when } 1 \leq q < \infty, \\ \max\{\|x^+\|_p, \|x^-\|_p\} & \text{when } q = \infty, \end{cases}$$

where x^+ and x^- are positive and negative parts of $x \in \ell_p$, i.e., $(x^+)_n = \max\{x_n, 0\}$ and $(x^-)_n = \max\{-x_n, 0\}$. Note that if $p \geq q \geq 1$, then, by the convexity of the function $f(u) = u^{p/q}$, we have

$$\|x\|_p \leq |x|_{p,q} \leq 2^{1/q-1/p} \|x\|_p.$$

In view of Corollary 4, this yields that if either $X = \ell_p$ - ℓ_q or $X = b_{p,q}$, then $J(X) \leq \min\{2, 2^{1/q-1/p} J(\ell_p)\}$, $C_{\text{NJ}}(X) \leq \min\{2, 2^{2/q-2/p} C_{\text{NJ}}(\ell_p)\}$ and $N(X) \geq \max\{1, 2^{1/p-1/q} N(\ell_p)\}$. From these rough estimates we cannot get the following information (cf. our Examples 2 and 4):

$$J(\ell_2\text{-}\ell_1) = \sqrt{8/3}, \quad C_{\text{NJ}}(\ell_2\text{-}\ell_1) \geq 3/2$$

and

$$J(\ell_\infty\text{-}\ell_1) = 3/2, \quad C_{\text{NJ}}(\ell_\infty\text{-}\ell_1) \geq 5/4.$$

PROBLEM 1 (cf. [12]). Let $1 < p < \infty$, $1 \leq q < \infty$. Compute $N(b_{p,q})$.

PROBLEM 2. Compute $J(X)$ and $C_{\text{NJ}}(X)$ for $X = \ell_p$ - ℓ_q or $X = b_{p,q}$.

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