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Bounded evaluation operators from H^p into ℓ^q

by

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Abstract. Given $0 < p, q < \infty$ and any sequence $\mathbf{z} = \{z_n\}$ in the unit disc $\mathbb{D}$, we define an operator from functions on $\mathbb{D}$ to sequences by $T_{\mathbf{z},p}(f) = \{(1 - |z_n|^2)^{1/p} f(z_n)\}$. Necessary and sufficient conditions on $\{z_n\}$ are given such that $T_{\mathbf{z},p}$ maps the Hardy space H^p boundedly into the sequence space ℓ^q. A corresponding result for Bergman spaces is also stated.

1. Introduction. For $0 < p < \infty$ let ℓ^p denote the classical sequence space and H^p denote the classical Hardy space of the unit disc, $\mathbb{D}$. It is well known that, for all $f \in H^p$ and $z \in \mathbb{D}$,

$$|f(z)| \le \|f\|_{H^p}(1 - |z|^2)^{-1/p} \tag{1}$$

(see e.g. [4, p. 36]), and that this gives a sharp rate of growth for H^p functions. Given any sequence $\mathbf{z} = \{z_n\}$ in $\mathbb{D}$ we define the operator $T_{\mathbf{z},p}$ by

$$T_{\mathbf{z},p}(f) = \{(1 - |z_n|^2)^{1/p} f(z_n)\} \quad \text{for } f \text{ a function on } \mathbb{D}. \tag{2}$$

The operator plays a key role in interpolation theory, indeed, $\mathbf{z}$ is said to be an *interpolating sequence* for H^p if $T_{\mathbf{z},p}$ maps H^p onto ℓ^p. Note that (1) trivially implies that $\|T_{\mathbf{z},p}(f)\|_{\ell^\infty} \le \|f\|_{H^p}$ for all $f \in H^p$. It is also straightforward to show that for an infinite sequence $\mathbf{z}$, $T_{\mathbf{z},p}$ maps H^p into c_0, the space of sequences which tend to zero, if and only if $|z_n| \to 1$ as $n \to \infty$.

The aim of this paper is as follows: given $0 < p, q < \infty$, describe all sequences $\mathbf{z}$ such that there exists a constant C such that

$$\|T_{\mathbf{z},p}(f)\|_{\ell^q} \le C\|f\|_{H^p} \quad \text{for all } f \in H^p. \tag{3}$$

Given $z, w \in \mathbb{D}$, let ϕ_w denote the corresponding Möbius transform and $\mathrm{d}(z, w)$ the *pseudohyperbolic distance*, i.e.

$$\phi_w(z) = \frac{z - w}{1 - \overline{w}z} \quad \text{and} \quad \mathrm{d}(z, w) = |\phi_w(z)|.$$

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A sequence of points $\{z_n\}$ in $\mathbb{D}$ is said to be *uniformly discrete* if

$$\inf_{n \neq m} \mathrm{d}(z_n, z_m) > \delta > 0 \quad \text{for some } \delta,$$

and *uniformly separated* if

$$\inf_{n} \prod_{m \neq n} \mathrm{d}(z_n, z_m) > \delta > 0 \quad \text{for some } \delta.$$

Perhaps surprisingly, the characterisation of sequences $\mathbf{z}$ such that (3) holds forms a trichotomy depending only upon whether p is less than, equal to or greater than q:

THEOREM 1. *Given $0 < p, q < \infty$ and a sequence $\{z_n\}$ in $\mathbb{D}$, the following are equivalent*:

(1) *There exists a constant C such that*

$$\sum_n (1 - |z_n|^2)^{q/p} |f(z_n)|^q \leq C \|f\|_{H^p}^q \quad \textit{for all } f \in H^p.$$

(2) (a) *$p < q$ and $\{z_n\}$ is a finite union of uniformly discrete sequences*;
(b) *$p = q$ and $\{z_n\}$ is a finite union of uniformly separated sequences*;
(c) *$p > q$ and $\{z_n\}$ is a finite sequence.*

The conclusion of Theorem 1 when $p = q$ is well known; see [5], [8] and [9]. It is closely related to the fact that $T_{\mathbf{z},p}$ maps H^p onto ℓ^p if and only if the infinite sequence $\{z_n\}$ is uniformly separated; this was proved by Carleson [2] when $p = \infty$, Shapiro and Shields [12] when $1 \leq p < \infty$ and Kabaĭla [7] when $0 < p < 1$; see e.g. [4, Chapter 9].

We shall therefore concentrate on the cases when $p \neq q$, where the characterisations given do not appear to be stated in the literature.

2. The case $p < q$. Our main tool is the following generalisation of Carleson's measure theorem due to Duren [3]. Given $\theta_0 \in [0, 2\pi)$ and $0 < h < 1$, let

$$S(\theta_0, h) = \{re^{i\theta} : 1 - h \leq r < 1,\ \theta_0 \leq \theta \leq \theta_0 + h\}$$

be the corresponding Carleson square.

THEOREM 2. *Given a finite positive Borel measure μ on $\mathbb{D}$ and $0 < p \leq q < \infty$, there exists a constant C such that*

$$\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq C \|f\|_{H^p}^q \quad \textit{for all } f \in H^p$$

if and only if there exists a constant $\widetilde{C}$ such that $\mu(S(\theta_0, h)) \leq \widetilde{C} h^{q/p}$ for all Carleson squares $S(\theta_0, h)$.

We can now prove Theorem 1 in the case that $p < q$.

THEOREM 3. *Given a sequence $\{z_n\}$ in $\mathbb{D}$, the following are equivalent*:

(1) *For all $0 < p < q < \infty$, there exists a constant C such that*

$$\sum_n (1-|z_n|^2)^{q/p}|f(z_n)|^q \le C\|f\|_{H^p}^q \qquad \text{for all } f \in H^p.$$

(2) *$\{z_n\}$ is a finite union of uniformly discrete sequences.*

(3) *For some $r > 1$,*

$$\sup_{z\in\mathbb{D}} \sum_n (1-|\phi_{z_n}(z)|^2)^r < \infty.$$

(4) *For all $r > 1$,*

$$\sup_{z\in\mathbb{D}} \sum_n (1-|\phi_{z_n}(z)|^2)^r < \infty.$$

Proof. (1)$\Rightarrow$(4). Given any $0 < p < \infty$, let $q = rp$. For all $z \in \mathbb{D}$, let

$$f_z(w) = \frac{(1-|z|^2)^{1/p}}{(1-\overline{z}w)^{2/p}},$$

so $\|f_z\|_{H^p} \equiv 1$. Consequently,

$$\sup_{z\in\mathbb{D}} \sum_n \left(\frac{(1-|z|^2)(1-|z_n|^2)}{|1-\overline{z}z_n|^2}\right)^r \le C.$$

The result now follows from the identity

$$1-|\phi_{z_n}(z)|^2 = \frac{(1-|z|^2)(1-|z_n|^2)}{|1-\overline{z}z_n|^2}.$$

(4)$\Rightarrow$(3) is trivial so we show that (3)$\Rightarrow$(2), following a method from [5] and [9]. For any point $z \in \mathbb{D}$, let $N(z)$ denote the number of points of $\{z_n\}$ contained in $\Delta(z,1/2) := \{w \in \mathbb{D} : \mathrm{d}(w,z) < 1/2\}$. Then there exist $K > 0$ such that, for all $z \in \mathbb{D}$,

$$K \ge \sum_n (1-|\phi_{z_n}(z)|^2)^r \ge \sum_{z_n\in\Delta(z,1/2)} (1-|\phi_{z_n}(z)|^2)^r \ge (3/4)^r N(z),$$

so $N(z) \le K(4/3)^r$. Since there exists an integer N such that $N(z) \le N$ for all $z \in \mathbb{D}$, it follows that $\{z_n\}$ can be split into the union of at most N uniformly discrete sequences (see e.g. [6, p. 69]).

(2)$\Rightarrow$(1). We may as well suppose that $\{z_n\}$ is uniformly discrete. Then, letting

$$Q(\theta_0,h) = \{re^{i\theta} : 1-h \le r < 1-h/2,\ \theta_0 \le \theta \le \theta_0 + h\}$$

be the top half of the Carleson square $S(\theta_0,h)$, it is easily shown that there exists an integer M such that every set $Q(\theta_0,h)$ contains at most M points of the sequence $\{z_n\}$. So, letting μ be the discrete measure

$$\mu = \sum_n (1-|z_n|^2)^{q/p}\delta_{z_n},$$

we have for any $S(\theta_0, h)$,

$$\begin{aligned}\mu(S(\theta_0,h)) &= \sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}\mu(Q(\theta_0+2^{-k}jh, 2^{-k}h))\\ &\le \sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1} M(1-(1-2^{-k-1}h)^2)^{q/p}\\ &\le M\sum_{k=0}^{\infty}2^k(2^{-k}h)^{q/p} = Mh^{q/p}\sum_{k=0}^{\infty}2^{k(1-q/p)} = Ch^{q/p},\end{aligned}$$

for some C as $q > p$. Now (1) follows from Theorem 2. ∎

The surprising arithmetic fact that (3) implies (4) in Theorem 3 generalises [9, Theorem 4].

3. The case $p > q$. Using (1), it is easily shown that (3) holds when $\{z_n\}$ is a finite sequence.

PROPOSITION 4. *Let $0 < q < p < \infty$ and $\{z_n\}$ be a sequence in $\mathbb{D}$. Suppose that there exists a constant C such that*

$$\sum_n (1-|z_n|^2)^{q/p}|f(z_n)|^q \le C\|f\|_{H^p}^q \quad \text{for all } f \in H^p. \tag{4}$$

Then $\{z_n\}$ is a finite sequence.

Proof. Suppose that (4) holds for an infinite sequence $\{z_n\}$. Then, for all $f \in H^p$,

$$\sum_n (1-|z_n|^2)|f(z_n)|^p \le \Big(\sum_n (1-|z_n|^2)^{q/p}|f(z_n)|^q\Big)^{p/q} \le C^{p/q}\|f\|_{H^p}^p.$$

So, by Theorem 1, $\{z_n\}$ is a finite union of uniformly separated sequences. By removing superfluous terms if necessary, we may suppose that $\{z_n\}$ is an infinite uniformly separated sequence. Then the map $T_{\mathbf{z},p} : H^p \to \ell^p$ as defined in (2) is onto (see the comments after Theorem 1). By Banach's open mapping theorem there exists a constant N such that for all $\{\alpha_n\} \in \ell^p$, there exists $f \in H^p$ with $T_{\mathbf{z},p}f = \{\alpha_n\}$ and $\|f\|_{H^p} \le N\|\{\alpha_n\}\|_{\ell^p}$ (see e.g. [4, p. 149]). So, in view of (4), $\|\{\alpha_n\}\|_{\ell^q} \le C^{1/q}\|f\|_{H^p} \le C^{1/q}N\|\{\alpha_n\}\|_{\ell^p}$ for all $\{\alpha_n\} \in \ell^p$, which gives a contradiction. ∎

4. Remarks and acknowledgements. The inequality (3) has a dual formulation. For $1 < p, q < \infty$, let $p' = p/(p-1)$ and $q' = q/(q-1)$. Then we may identify the dual space of ℓ^q with $\ell^{q'}$ and the dual space of H^p with $H^{p'}$ (under the pairing induced by the inner product in H^2; see e.g. [4, p. 113]). Given $z \in \mathbb{D}$, let k_z denote the corresponding Cauchy kernel, so

$k_z(w) = 1/(1-\overline{z}w)$. The following reproducing property holds: for $f \in H^p$, $f(z) = \langle f, k_z \rangle$. By considering the adjoint of $T_{\mathbf{z},p}$, it is easily shown that for p, q as above, (3) holds if and only if there exists a constant $\widetilde{C}$ such that

$$\Big\|\sum_n \alpha_n (1-|z_n|^2)^{1/p} k_{z_n}\Big\|_{H^{p'}} \le \widetilde{C} \|\{\alpha_n\}\|_{\ell^{q'}} \quad \text{for all } \{\alpha_n\} \in \ell^{q'}.$$

Using this equivalent formulation, an application for Theorem 1 in the classification of Schatten class Hankel operators has been found in [10].

We can also consider an analogous problem for Bergman spaces. For $0 < p < \infty$ let A^p denote the classical Bergman space of the unit disc. It is well known that $|f(z)| \le \|f\|_{A^p}(1-|z|^2)^{-2/p}$ for all $f \in A^p$ and $z \in \mathbb{D}$ (see e.g. [6, p. 36]. Given any sequence $\mathbf{z} = \{z_n\}$ in $\mathbb{D}$ we define the operator $R_{\mathbf{z},p}$ by $R_{\mathbf{z},p}(f) = \{(1-|z_n|^2)^{2/p} f(z_n)\}$.

THEOREM 5. *Given $0 < p, q < \infty$ and a sequence $\{z_n\}$ in $\mathbb{D}$, the following are equivalent*:

(1) *There exists a constant C such that*

$$\sum_n (1-|z_n|^2)^{2q/p} |f(z_n)|^q \le C \|f\|_{A^p}^q \quad \textit{for all } f \in A^p.$$

(2) (a) *$p \le q$ and $\{z_n\}$ is a finite union of uniformly discrete sequences*;
(b) *$p > q$ and $\{z_n\}$ is a finite sequence.*

The conclusion when $p = q$ may be found in [13]; see also [6, p. 70]. It is closely related to Amar's result that, if $\{z_n\}$ is uniformly discrete, then $\{z_n\}$ is the finite union of sequences $\{z_n^{(i)}\}$ such that each $R_{\mathbf{z}^{(i)},p}$ maps A^p onto ℓ^p (see [1, Theorem 2.1.1], also [11]). The proofs when $p \ne q$ are similar to the Hardy space cases but simpler, and so are omitted.

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