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Shilov boundary for holomorphic functions on some classical Banach spaces

by

María D. Acosta (Granada) and
Mary Lilian Lourenço (São Paulo)

Abstract. Let $\mathcal{A}_\infty(B_X)$ be the Banach space of all bounded and continuous functions on the closed unit ball B_X of a complex Banach space X and holomorphic on the open unit ball, with sup norm, and let $\mathcal{A}_{\rm u}(B_X)$ be the subspace of $\mathcal{A}_\infty(B_X)$ of those functions which are uniformly continuous on B_X. A subset $B \subset B_X$ is a boundary for $\mathcal{A}_\infty(B_X)$ if $\|f\| = \sup_{x\in B} |f(x)|$ for every $f \in \mathcal{A}_\infty(B_X)$. We prove that for $X = d(w,1)$ (the Lorentz sequence space) and $X = C_1(H)$, the trace class operators, there is a minimal closed boundary for $\mathcal{A}_\infty(B_X)$. On the other hand, for $X = \mathcal{S}$, the Schreier space, and $X = K(\ell_p, \ell_q)$ $(1 \le p \le q < \infty)$, there is no minimal closed boundary for the corresponding spaces of holomorphic functions.

1. Introduction. A result of Shilov asserts that if $\mathfrak{A}$ is a unital separating algebra of $\mathcal{C}(K)$ (K a compact Hausdorff topological space), then there is a smallest closed subset $S \subset K$ such that every function of $\mathfrak{A}$ attains its norm at some point of S [6, Theorem I.4.2]. Bishop [4] proved that if K is metrizable, then, in fact, there is a minimal subset of K satisfying the above condition for every separating algebra of $\mathcal{C}(K)$. That subset is the set of peak points for $\mathfrak{A}$ (see definition below).

Globevnik introduced the corresponding concepts for a subalgebra of $\mathcal{C}_{\rm b}(\Omega)$, the space of bounded continuous functions on a topological space Ω not necessarily compact [9]. In fact, he considered the case $\Omega = B_X$, where X is a Banach space. If $\mathfrak{A}$ is a subspace of $\mathcal{C}_{\rm b}(\Omega)$, we will say that a subset $B \subset \Omega$ is a *boundary* for $\mathfrak{A}$ if

$$\|f\| = \sup_{b\in B} |f(b)|, \qquad \forall f \in \mathfrak{A}.$$

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If there is a closed boundary B that is contained in all closed boundaries for $\mathfrak{A}$, we will say that B is the *Shilov boundary* of $\mathfrak{A}$.

If X is a complex Banach space, we will denote by $\mathcal{A}_{\mathrm{u}}(B_X)$ the space of uniformly continuous functions on the closed unit ball of X which are holomorphic on the open unit ball. Globevnik [9] described the boundaries of $\mathcal{A}_{\mathrm{u}}(B_{c_0})$. As a consequence of the description, he showed that this algebra has no Shilov boundary. Aron, Choi, Lourenço and Paques [3] gave examples of boundaries for $\mathcal{A}_{\mathrm{u}}(B_{\ell_\infty})$ and proved that there is no Shilov boundary for this algebra. They also showed that the unit sphere of ℓ_1 is the Shilov boundary for $\mathcal{A}_{\mathrm{u}}(B_{\ell_1})$.

Moraes and Romero [14] gave a characterization of the boundaries of $\mathcal{A}_{\mathrm{u}}(B_{d_*(w,1)})$, where $d_*(w,1)$ is the canonical predual of the Lorentz sequence space $d(w,1)$ when $w=(1/n)$. Later Acosta, Moraes and Romero [2] generalized that characterization proving it for any space $d_*(w,1)$ and obtained another one in terms of the strong peak sets of the unit ball. In this case, there is no Shilov boundary. Choi, García, Kim and Maestre [5] proved that there is no Shilov boundary for $\mathcal{A}_{\mathrm{u}}(B_{C(K)})$, when K is infinite and scattered. Acosta showed the same result for every infinite K and also proved that for this space the set of extreme points of the unit ball of $C(K)$ is a boundary for $\mathcal{A}_{\mathrm{u}}(B_{C(K)})$ (see [1]).

Before going on it is convenient to recall some definitions. Let $\mathcal{A}$ be a function space on a metric space Ω. An element $y \in \Omega$ is called a *peak point* for $\mathcal{A}$ if there is some $f \in \mathcal{A}$ such that $f(y)=1$ and $|f(x)|<1$ for all $x \in \Omega \setminus \{y\}$. In this case we say that f *peaks* at y. An element $y \in \Omega$ is called a *strong peak point* for $\mathcal{A}$ if there is some $f \in \mathcal{A}$ satisfying $f(y)=1$ and such that given any $\varepsilon>0$ there is some $\delta>0$ such that $\operatorname{dist}(x,y)>\varepsilon$ implies that $|f(x)|<1-\delta$. It is clear that every closed boundary for $\mathcal{A}$ contains all the strong peak points.

In this paper we prove that there is no Shilov boundary for $\mathcal{A}_{\mathrm{u}}(B_X)$ when X is the Schreier space or the space $K(\ell_p,\ell_q)$ $(1 \le p \le q < \infty)$. For the spaces $X=C_1(H)$, the trace class operators on a complex Hilbert space H, or $X=d(w,1)$, the Shilov boundary for $\mathcal{A}_{\mathrm{u}}(B_X)$ exists. In fact, all the points in the unit sphere of $d(w,1)$ are strong peak points for $\mathcal{A}_{\mathrm{u}}(B_{d(w,1)})$, and so in this case the Shilov boundary is the unit sphere. For ℓ_1 the same result holds. That fact was proved in [3] for the finitely supported sequences in the unit sphere. If K is infinite, we also prove that there are no strong peak points for $\mathcal{A}_{\mathrm{u}}(B_{C(K)})$. The set of peak points for $\mathcal{A}_{\mathrm{u}}(B_{C(K)})$ is the set of extreme points of $B_{C(K)}$ if K is separable.

Throughout this paper, all the Banach spaces considered are complex. For a Banach space X, B_X and S_X will be the closed unit ball and the unit sphere of X, respectively. We will denote by $\mathcal{A}_\infty(B_X)$ the Banach space of all bounded and continuous functions on B_X which are holomorphic on the open unit ball, and by $\mathcal{A}_{\mathrm{u}}(B_X)$ the space of all functions in $\mathcal{A}_\infty(B_X)$ which are uniformly continuous.

2. Existence of the Shilov boundary on the Lorentz sequence space. Given a decreasing sequence w of positive real numbers satisfying $w_1 = 1$ and $w \in c_0 \setminus \ell_1$, the complex Lorentz sequence space $d(w,1)$ is given by

$$d(w,1) = \Big\{x : \mathbb{N} \to \mathbb{C} : \sup\Big\{\sum_{n=1}^{\infty} |x(\sigma(n))| w_n : \sigma : \mathbb{N} \to \mathbb{N} \text{ injective}\Big\} < \infty\Big\}.$$

The norm is given by

$$\|x\| = \sup\Big\{\sum_{n=1}^{\infty} w_n |x(\sigma(n))| : \sigma : \mathbb{N} \to \mathbb{N} \text{ injective}\Big\} \qquad (x \in d(w,1)).$$

It is well known and easy to verify that the above supremum is attained for the decreasing rearrangement of x. The usual vector basis (e_n) is a monotone Schauder basis (see [12]).

A canonical predual $d_*(w,1)$ of $d(w,1)$ is given by

$$d_*(w,1) = \left\{x \in c_0 : \lim_n \frac{\sum_{k=1}^n x^*(k)}{W_n} = 0\right\}$$

where $W_n = \sum_{k=1}^n w_k$ and x^* is the decreasing rearrangement of x. This space is a Banach space endowed with the norm

$$\|x\| = \sup_n \left\{\frac{\sum_{k=1}^n x^*(k)}{W_n}\right\}$$

(see [16] and [7]). $d_*(w,1)$ has a Schauder basis whose sequence of biorthogonal functionals is, in fact, the canonical basis of $d(w,1)$.

We begin by presenting some useful lemmas.

Lemma 2.1. *If (z_n) is a bounded sequence of complex numbers such that the sequence $(1 + |z_n| - |1 + z_n|)$ converges to zero, then so does $(|z_n| - z_n)$.*

Proof. We consider the following identity for a complex number z:

$$\begin{aligned}(1 + |z| - |1 + z|)^2 &= 1 + |z|^2 + 2|z| + |1 + z|^2 - 2(1 + |z|)\,|1 + z| \\ &= 2(\operatorname{Re} z - |z|) + 2(1 + |z|)(1 + |z| - |1 + z|).\end{aligned}$$

If we apply the above identity to the sequence (z_n) and use the assumption, we find that the sequence $(|z_n| - \operatorname{Re} z_n)$ converges to zero.

Now if we consider the expression

$$\begin{aligned}(|z| - \operatorname{Re} z)^2 &= 2(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 - 2|z| \operatorname{Re} z \\ &= (\operatorname{Im} z)^2 + 2(\operatorname{Re} z - |z|) \operatorname{Re} z,\end{aligned}$$

and we apply the identity to the sequence (z_n), we deduce that $\operatorname{Im} z_n \to 0$. Hence

$$|z_n| - z_n = |z_n| - \operatorname{Re} z_n - i \operatorname{Im} z_n \to 0. \blacksquare$$

LEMMA 2.2 ([3, Lemma 9]). *Let* $0 < a < 1$. *The real-valued function given by*

$$g_a(x) = \left(1 + \frac{x}{1-a}\right)\left(1 + \frac{1-x}{a}\right) \quad (x \in \mathbb{R})$$

attains its maximum at $x = a$ *and*

$$g_a(x) < g_a(a) = \frac{1}{a(1-a)}, \quad \forall x \in \mathbb{R} \setminus \{a\}.$$

LEMMA 2.3. *The set of peak points in* S_X *for* $\mathcal{A}_\infty(B_X)$ *is invariant under surjective linear isometries on* X. *The same holds for the set of strong peak points in* S_X.

By the maximum modulus theorem, every peak point for a subspace of $\mathcal{A}_\infty(B_X)$ belongs to S_X. As a consequence, so does every strong peak point. The following result shows the converse for the subspace of all polynomials on $d(w,1)$.

THEOREM 2.4. *The set of strong peak points for the space of polynomials of degree less than or equal to* 2 *on* $d(w,1)$ *contains the unit sphere of* $d(w,1)$.

Proof. Let $y_0 \in S_{d(w,1)}$. By Lemma 2.3 we can assume that $\operatorname{supp} y_0$ is an interval of positive integers containing $\{1\}$ and

$$(1) \qquad y_0(j) \in \mathbb{R}^+, \quad \forall j \in \operatorname{supp} y_0, \qquad y_0(n) \geq y_0(n+1), \quad \forall n \in \mathbb{N}.$$

We will prove that y_0 is a strong peak point for $\mathcal{A}_{\mathrm{u}}(d(w,1))$.

If the support of y_0 contains just one element, then $y_0 = e_1$ and it is sufficient to consider the first-degree polynomial given by

$$f(x) = 1 + x(1) \quad (x \in d(w,1)).$$

Clearly $\|f\| = 2 = f(y_0)$. By using the fact that in $S_{d(w,1)}$ the weak and $\sigma(d(w,1), d_*(w,1))$ convergences coincide ([16, Proposition 2.2] and [10, Corollary III.2.15]) and that every point of the unit sphere is a point of weak-norm continuity of the unit ball [13, Proposition 4], it is easily checked that f strongly peaks in the unit ball at y_0.

Now assume that $J := \operatorname{supp} y_0$ has at least two elements. Since $\|y_0\| = 1$, by (1), we know that $\sum_{i \in J} w_i y_0(i) = 1$ and so $0 < w_i y_0(i) < 1$ for every $i \in J$.

For every $k \in J$ we define

$$f_k(x) = \frac{1}{M_k}\left(1 + \frac{w_k x(k)}{1 - w_k y_0(k)}\right)\left(1 + \frac{1}{w_k y_0(k)} \sum_{j \in J \setminus \{k\}} w_j x(j)\right) \quad (x \in d(w,1)),$$

where

$$M_k = \frac{1}{w_k y_0(k)(1 - w_k y_0(k))}.$$

Then f_k is clearly a non-homogeneous polynomial on $d(w,1)$ of degree 2 and $f_k(y_0) = 1$. We will check that $\|f_k\| = 1$.

If $x \in B_{d(w,1)}$, then

$$\begin{aligned}
(2) \qquad |f_k(x)| &= \frac{1}{M_k}\left|1 + \frac{w_k x(k)}{1 - w_k y_0(k)}\right| \left|1 + \frac{1}{w_k y_0(k)} \sum_{j\in J\setminus\{k\}} w_j x(j)\right| \\
&\le \frac{1}{M_k}\left(1 + \frac{w_k |x(k)|}{1 - w_k y_0(k)}\right)\left(1 + \frac{1}{w_k y_0(k)} \sum_{j\in J\setminus\{k\}} |w_j x(j)|\right) \\
&\le \frac{1}{M_k}\left(1 + \frac{w_k |x(k)|}{1 - w_k y_0(k)}\right)\left(1 + \frac{1 - w_k|x(k)|}{w_k y_0(k)}\right) \quad (\text{since } x \in B_X) \\
&\le \frac{1}{M_k}\left(1 + \frac{w_k y_0(k)}{1 - w_k y_0(k)}\right)\left(1 + \frac{1 - w_k y_0(k)}{w_k y_0(k)}\right) \quad (\text{by Lemma 2.2}) \\
&= 1.
\end{aligned}$$

Hence $\|f_k\| = 1$.

Our intention is to show that y_0 is a strong peak point for the space of second-degree polynomials. To this end, we will prove that

$$(3) \qquad x_n \in B_{d(w,1)},\ \forall n, \qquad |f_k(x_n)| \underset{n}{\to} 1 \ \Rightarrow\ x_n(k) \underset{n}{\to} y_0(k).$$

For every fixed k, we write

$$u_n = \frac{w_k x_n(k)}{1 - w_k y_0(k)}, \qquad v_n = \sum_{\substack{j\in J\\ j\neq k}} \frac{w_j x_n(j)}{w_k y_0(k)}.$$

We rewrite the inequality (2) in terms of the above sequences:

$$|f_k(x_n)| = \frac{1}{M_k}\,|1 + u_n|\,|1 + v_n| \le \frac{1}{M_k}\,(1 + |u_n|)(1 + |v_n|) \le 1.$$

If we assume that $|f_k(x_n)| \to 1$ as $n \to \infty$, then the sequence $(1 + v_n)$ has no subsequence converging to zero. From the above inequality we deduce that

$$|1 + u_n| - 1 - |u_n| \to 0.$$

Since k is fixed, Lemma 2.1 implies that $(|u_n| - u_n)$ converges to zero, that is, $|x_n(k)| - x_n(k) \to 0$ as $n \to \infty$. Also by Lemma 2.2, we know that

$$w_k |x_n(k)| \to w_k y_0(k) \qquad \text{as } n \to \infty.$$

Hence we deduce that $x_n(k) \to y_0(k)$ as $n \to \infty$.

Now we choose a sequence (α_n) in ℓ_1 such that $\operatorname{supp}\alpha = J$, $\alpha_n > 0$ for all $n \in J$ and $\sum_{n\in J} \alpha_n = 1$. Define

$$f(x) = \sum_{k\in J} \alpha_k f_k(x) \quad (x \in B_{d(w,1)}).$$

Then f is a polynomial of degree at most 2 in $d(w,1)$ and $\|f\| \le 1 = f(y_0)$.

We now prove that this function strongly peaks in the unit ball of $d(w,1)$ at y_0. So assume that $|f(x_n)| \to 1$ for some sequence (x_n) in the unit ball. Then clearly $f_k(x_n) \to 1$ as $n \to \infty$ for every $k \in J$.

Since $y_0 \in S_{d(w,1)}$, by condition (3), we know that (x_n) converges pointwise to y_0. All the elements involved in the argument are in the unit ball of $d(w,1)$ and so (x_n) converges to y_0 in the $\sigma(d(w,1), d_*(w,1))$-topology. Since $d_*(w,1)$ is an M-ideal in its dual (see [16, Proposition 2.2] or [10, Examples III.1.4c]), in the unit ball of $d(w,1)$, the weak and weak* topologies coincide on the unit sphere, in view of [10, Corollary III.2.15]. By applying this to the element y_0, which is the w^*-limit of (x_n), we see that in fact (x_n) converges weakly to y_0. Since all the points of the unit sphere of $d(w,1)$ are points of weak-norm continuity [13, Proposition 4], we conclude that (x_n) converges in norm to y_0 and y_0 is a strong peak point, as we wanted to show. ∎

Corollary 2.5. *The Shilov boundary for the space of second-degree polynomials on $d(w,1)$ is $S_{d(w,1)}$. Hence $S_{d(w,1)}$ is also the Shilov boundary for $\mathcal{A}_{\mathrm{u}}(B_{d(w,1)})$ and $\mathcal{A}_\infty(B_{d(w,1)})$.*

It is known that all the finitely supported elements in S_{ℓ_1} are strong peak points for the space of second-degree polynomials on ℓ_1 [3, Theorem 10]. We now extend that result.

Theorem 2.6. *S_{ℓ_1} is the set of strong peak points for the space of second-degree polynomials on ℓ_1.*

Proof. If $y_0 \in S_{\ell_1}$, then, by Lemma 2.3, we can assume that $y_0(n) \ge 0$ for every n. If $|\operatorname{supp} y_0| = 1$ and $\{n\} = \operatorname{supp} y_0$, the function $x \mapsto 1 + x(n)$ strongly peaks in the unit ball of ℓ_1 at y_0. Otherwise, if $J := \operatorname{supp} y_0$ satisfies $|J| \ge 2$, then the second-degree polynomial given by

$$f_k(x) := \frac{1}{y_0(k)(1-y_0(k))}\left(1 + \frac{x(k)}{1-y_0(k)}\right)\left(1 + \frac{\sum_{i\ne k} x(i)}{y_0(k)}\right) \quad (x \in \ell_1)$$

satisfies $f_k(y_0) = 1$. In view of Lemma 2.2, also $\|f_k\| = 1$ and now we can follow the argument in the proof of Theorem 2.4. ∎

3. Boundaries for the Schreier space and $C(K)$. A subset $E = \{n_1 < \cdots < n_k\}$ of the natural numbers $\mathbb{N}$ is said to be *admissible* if $k \le n_1$. The *Schreier space* $\mathcal{S}$ is the completion of the space c_{00} of all scalar sequences of finite support with respect to the norm $\|x\| = \sup \sum_{j\in E} |x_j|$, where the supremum is taken over all admissible sets E of natural numbers.

The following theorem shows in particular that the intersection of all boundaries for $\mathcal{A}_\infty(B_\mathcal{S})$ is empty.

THEOREM 3.1. *Let $\mathcal{S}$ be the Schreier space and B be a boundary for $\mathcal{A}_\infty(B_\mathcal{S})$. If $x_0 \in B$ and $0 < r < 1$, then $B \setminus (x_0 + rB_\mathcal{S})$ is a boundary for $\mathcal{A}_\infty(B_\mathcal{S})$. As a consequence, there is no Shilov boundary for $\mathcal{A}_\infty(B_\mathcal{S})$.*

Proof. Assume that $h \in \mathcal{A}_\infty(B_\mathcal{S})$. For every $0 < \varepsilon < (1-r)/2$, there is $y_0 \in c_{00}$ such that $\|y_0\| < 1$ and

$$|h(y_0)| > \|h\| - \varepsilon.$$

We write $k = \max \operatorname{supp} y_0$ and denote by (P_m) the sequence of canonical projections associated to the usual basis of $\mathcal{S}$. Choose a positive integer n such that $n > k/(1 - \|y_0\|)$ and $\|(I - P_n)(x_0)\| < \varepsilon$. We will check that $y_0 + \lambda y \in B_\mathcal{S}$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $y = \sum_{j=n+1}^{2n} (1/n) e_j$.

Let $A = E \cup F$ be an admissible set such that $E \subset \{1, \ldots, k\}$ and $\min F > k$. If $E \neq \emptyset$, then $|E| + |F| \le k$ and

$$\sum_{i\in E\cup F} |y_0 + \lambda y(i)| \le \sum_{i\in E} |y_0(i)| + \sum_{i\in F} |y(i)| \le \|y_0\| + \frac{k}{n} \le 1\ .$$

If $E = \emptyset$, then $\sum_{i\in F} |(y_0 + \lambda y)(i)| = \sum_{i\in F} |y(i)| \le 1$. So $\|y_0 + \lambda y\| \le 1$.

By the maximum modulus theorem, there is $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| = 1$ such that

$$|h(y_0 + \lambda_0 y)| \ge |h(y_0)| > \|h\| - \varepsilon.$$

Fix $\lambda_1 \in \mathbb{C}$ satisfying $|\lambda_1| = 1$ and

$$|h(y_0 + \lambda_0 y) + \lambda_1| = |h(y_0 + \lambda_0 y)| + 1.$$

Since $\|y\| = 1$ and $P_n(y) = 0$, there is $y^* \in S_\mathcal{S}$ such that $y^*(\lambda_0 y) = 1$, $y^*(e_j) = 0$ for all $j \le n$ and so $y^*(y_0) = 0$. Now, we define a holomorphic function g by

$$g(x) := h(x) + \lambda_1 y^*(x) \qquad (x \in B_\mathcal{S}).$$

Clearly $g \in \mathcal{A}_\infty(B_\mathcal{S})$ and

$$\begin{aligned}\|h\| - \varepsilon + 1 < |h(y_0)| + 1 &\le |h(y_0 + \lambda_0 y)| + y^*(\lambda_0 y)\\ &= |g(y_0 + \lambda_0 y)| \le \|g\| \le \|h\| + 1.\end{aligned}$$

Since B is a boundary there is $z_0 \in B$ such that

$$|g(z_0)| > \|h\| - \varepsilon + 1.$$

On the other hand,

$$|g(z_0)| \le |h(z_0)| + |y^*(z_0)| \le \|h\| + |y^*(z_0)| \le \|h\| + 1.$$

This implies $|y^*(z_0)| > 1 - \varepsilon$. Hence

$$\|(I - P_n)(z_0)\| \geq |y^*(z_0)| > 1 - \varepsilon.$$

Consequently,

$$\begin{aligned}\|z_0 - x_0\| &\geq \|(I - P_n)(z_0 - x_0)\| \\ &\geq \|(I - P_n)(z_0)\| - \|(I - P_n)x_0\| \geq 1 - 2\varepsilon > r.\end{aligned}$$

Also $|h(z_0)| + 1 \geq \|h\| + 1 - \varepsilon$ and hence $|h(z_0)| > \|h\| - \varepsilon$. Therefore $z_0 \in B \backslash (x_0 + rB_{\mathcal{S}})$ and this set is a boundary for $\mathcal{A}_\infty(B_{\mathcal{S}})$. As a consequence, the Shilov boundary of this space does not exist. ∎

We recall that a point $x \in B_X$ is a $\mathbb{C}$-*extreme point* of the unit ball if

$$(y \in X,\ \|x + \lambda y\| \leq 1,\ \forall \lambda \in \mathbb{C},\ |\lambda| = 1) \ \Rightarrow \ y = 0.$$

THEOREM 3.2. *If K is any infinite compact Hausdorff topological space, then there are no strong peak points for $\mathcal{A}_\infty(B_{\mathcal{C}(K)})$. If K is separable, then all the extreme points in $B_{\mathcal{C}(K)}$ are peak points for the space of first-degree polynomials on $\mathcal{C}(K)$.*

Proof. It is known that every peak point is a $\mathbb{C}$-extreme point [8, Theorem 4]. So we will prove that $\mathbb{C}$-extreme points of $B_{\mathcal{C}(K)}$ are not strong peak points. Assume that $x_0 \in S_{\mathcal{C}(K)}$ is an extreme point of the unit ball. Since K is infinite, there is a sequence $(x_n) \subset \mathcal{C}(K)$ satisfying

$$0 \leq x_n \leq 1,\ \|x_n\| = 1,\ \forall n, \qquad \operatorname{supp} x_n \cap \operatorname{supp} x_m = \emptyset,\ \forall n \neq m.$$

Assume that $h \in B_{\mathcal{A}_\infty(B_{\mathcal{C}(K)})}$ with $h(x_0) = 1$. Since (x_n) is equivalent to the c_0-basis, it converges weakly to zero. Then the sequence $(x_0(1-x_n))$ is in the unit ball of $\mathcal{C}(K)$ and converges weakly to x_0. Since $\mathcal{C}(K)$ has the Dunford–Pettis property, it also has the polynomial Dunford–Pettis property [15], and so the argument in the proof of [1, Proposition 4.1] shows that

$$h(x_0(1 - x_n)) \to 1.$$

Since x_n are non-negative elements in the unit sphere, for every n there is $t_n \in K$ such that $x_n(t_n) = 1$ and so

$$\|x_0(1 - x_n) - x_0\| \geq \|x_0 x_n\| \geq |x_0(t_n)x_n(t_n)| = 1.$$

Hence x_0 is not a strong peak point for $\mathcal{A}_\infty(B_{\mathcal{C}(K)})$.

If K is separable and $\{t_n : n \in \mathbb{N}\}$ is a dense set in K, we will prove that the function u such that $u(K) = \{1\}$ is a peak point for the space of first-degree polynomials. In view of Lemma 2.3, this proves the stated assertion.

Define

$$f(x) := \sum_{n=1}^{\infty} \alpha_n (1 + x(t_n)) \qquad (x \in \mathcal{C}(K)),$$

where $(\alpha_n) \subset S_{\ell_1}$ with $\alpha_n > 0$ for every n. Then f is clearly a first-degree polynomial on $\mathcal{C}(K)$ and $f(u) = \|f\| = 2$. If $x \in B_{\mathcal{C}(K)}$ and $|f(x)| = 2$, then $|1 + x(t_n)| = 2$ for every n and so $x(t_n) = 1$ for all n, that is, $x = u$. ■

Since ℓ_∞ has a countable subset of functionals that separate points and attain the norm at the same element of the unit ball, we can also obtain:

COROLLARY 3.3 ([3]). *All the extreme points in B_{ℓ_∞} are peak points for the space of first-degree polynomials on ℓ_∞.*

4. Shilov boundary on the trace class operators. Let H be a complex Hilbert space. An operator $T : H \to H$ is called a *trace class operator* if there are orthonormal sequences (e_n) and (f_n) in H such that $T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle f_n$ for every $x \in H$ and the sequence (λ_n) is in ℓ_1. In that case, the norm of T is given by $\|T\| = \sum_{n=1}^{\infty} |\lambda_n|$. We denote by $C_1(H)$ the Banach space of all trace class operators on H.

THEOREM 4.1. *If H is a complex Hilbert space, then the Shilov boundaries for $\mathcal{A}_{\mathrm{u}}(C_1(H))$ and $\mathcal{A}_\infty(C_1(H))$ both exist and coincide.*

Proof. Assume that $\{e_i : i \in I\}$ is an orthonormal basis of H and $F \subset I$ is any subset. Then the operator Π_F given by

$$\Pi_F(T) := P_F T P_F \quad (T \in C_1(H)),$$

where $P_F(x) = \sum_{i\in F} x(i)e_i$ $(x \in H)$, is a norm one projection on $C_1(H)$. Since $\mathrm{Lin}\{e_i \otimes e_j : i, j \in I\}$ is dense in $C_1(H)$, for every $h \in \mathcal{A}_\infty(B_{C_1(H)})$ we have

$$\|h\| = \sup_{\substack{F\subset I\\ F \text{ finite}}} \|h \circ \Pi_F\|.$$

For every complex finite-dimensional space Y, the subset of peak points of B_Y is a boundary for $\mathcal{A}_{\mathrm{u}}(B_Y)$ [4, Theorem 1]. We will prove that for every finite subset $F \subset I$, every peak point of the unit ball of $\Pi_F(C_1(H))$ for the space of bounded and continuous functions on the unit ball of $\Pi_F(C_1(H))$ which are holomorphic on the open unit ball, is a strong peak point for $\mathcal{A}_{\mathrm{u}}(B_{C_1(H)})$.

Let $T_0 \in S_{C_1(H)} \cap \Pi_F(C_1(H))$ be a peak point. Then there is a continuous function g on the unit ball of $\Pi_F(C_1(H))$, which is holomorphic on the open unit ball and satisfies

$$g(T_0) = \|g\| = 1 \quad \text{and} \quad |g(T)| < 1,\ \forall T \in (B_{C_1(H)} \cap \Pi_F(C_1(H))) \setminus \{T_0\}.$$

Now we extend g to $B_{C_1(H)}$ by

$$\widetilde{g}(T) = g(\Pi_F(T)) \quad (T \in B_{C_1(H)}).$$

Clearly $\widetilde{g} \in \mathcal{A}_{\mathrm{u}}(B_{C_1(H)})$, $\|\widetilde{g}\| \le \|g\| = 1$ and $\widetilde{g}(T_0) = 1$. Assume that $(T_n) \subset B_{C_1(H)}$ with $|\widetilde{g}(T_n)| \to 1$, that is, $|g(\Pi_F(T_n))| \to 1$. Since $\Pi_F(C_1(H))$

is a finite-dimensional space and T_0 is a peak point, we have $\Pi_F(T_n) \to T_0$. Since $\|T_0\| = 1$, it follows that $\|\Pi_F(T_n)\| \to 1$. By using [11, Proposition 2.2], we have

$$\|P_F T_n P_F\|^2 + \|P_F T_n(I-P_F)\|^2 + \|(I-P_F)T_n P_F\|^2 + \|(I-P_F)T_n(I-P_F)\|^2 \leq \|T_n\|^2 \leq 1,$$

and so $\|\Pi_F(T_n) - T_n\| = \|P_F T_n P_F - T_n\| \to 0$. Since we know that $(\Pi_F(T_n))$ converges to T_0, so does (T_n), and T_0 is a strong peak point, as we wanted to show. Since the strong peak points are contained in any closed boundary and in this case the set of strong peak points is a boundary for $\mathcal{A}_{\mathrm{u}}(B_{C_1(H)})$, the Shilov boundary for this space is the closure of the set of strong peak points of $\mathcal{A}_{\mathrm{u}}(B_{C_1(H)})$. The same argument works for $\mathcal{A}_\infty(B_{C_1(H)})$. ∎

5. Boundaries for $K(\ell_p, \ell_q)$. We now study the properties of the boundaries for $\mathcal{A}_\infty(B_X)$, where X is the space of all compact operators on ℓ_p for $1 \leq p < \infty$.

THEOREM 5.1. *If $1 \leq p \leq q < \infty$, then there is no Shilov boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. In fact, if B is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$, $0 < r < 1$ and $S_0 \in B$, then $B \backslash (S_0 + rB_{K(\ell_p,\ell_q)})$ is also a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. There are closed boundaries A, B for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ such that* $\operatorname{dist}(A,B) \geq 1$. *The same assertions hold for $\mathcal{A}_{\mathrm{u}}(B_{K(\ell_p,\ell_q)})$.*

Proof. We denote by (P_n) and (Q_n) the sequences of canonical projections associated to the usual bases of ℓ_p and ℓ_q, respectively.

Assume that $B \subset B_{K(\ell_p,\ell_q)}$ is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$, $0 < r < 1$ and $S_0 \in B$. If $h \in \mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ and $0 < \varepsilon < (1-r)/3$, then there are $N \in \mathbb{N}$ and $F \in B_{K(\ell_p,\ell_q)}$ which satisfy $Q_N F P_N = F$ and

$$|h(F)| > \|h\| - \varepsilon.$$

Since S_0 is a compact operator, there exists $n > N$ with

$$\|(I - Q_n)S_0(I - P_n)\| < \varepsilon.$$

Choose $R \in S_{K(\ell_p,\ell_q)}$ such that

$$(I - Q_n)R(I - P_n) = R,$$

and $x_0 \in S_{\ell_p}$ satisfying $P_n x_0 = 0$ and $\|R(x_0)\| = 1$. Then there exists $y^* \in S_{\ell_q^*}$ with $Q_n^*(y^*) = 0$ and $y^*(R(x_0)) = 1$. Notice that $\|F + \lambda R\| \leq 1$ for every complex number λ with $|\lambda| = 1$. By the maximum modulus theorem, there is $\lambda_0 \in \mathbb{C}$ such that $|\lambda_0| = 1$ and

$$|h(F)| \leq |h(F + \lambda_0 R)| \leq \sup_{|\lambda|=1} |h(F + \lambda R)|.$$

If $\lambda_1 \in \mathbb{C}$ is a modulus one scalar satisfying

$$|h(F + \lambda_0 R) + \lambda_1 y^*(\lambda_0 R(x_0))| = |h(F + \lambda_0 R)| + 1,$$

we define a holomorphic function g by

$$g(T) := h(T) + \lambda_1 y^*(Tx_0) \quad (T \in B_{K(\ell_p,\ell_q)}).$$

Clearly $g \in \mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ and

$$\begin{aligned}\|g\| &\geq |g(F + \lambda_0 R)| = |h(F + \lambda_0 R) + \lambda_1 y^*(\lambda_0 R x_0)| \\ &= |h(F + \lambda_0 R)| + 1 \geq |h(F)| + |y^*(Rx_0)| > \|h\| - \varepsilon + 1.\end{aligned}$$

Since B is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$, there is $S \in B$ such that $|g(S)| > \|g\| - \varepsilon$. Hence

$$\|h\| - 2\varepsilon + 1 \leq \|g\| - \varepsilon < |g(S)| \leq |h(S)| + |y^*(Sx_0)|, \tag{4}$$

and so

$$|y^*(Sx_0)| \geq 1 - 2\varepsilon.$$

By the choice of x_0 and y^*,

$$\|(I - Q_n)S(I - P_n)\| \geq |y^*(I - Q_n)S(I - P_n)x_0| = |y^*(Sx_0)| \geq 1 - 2\varepsilon.$$

Finally, we deduce that

$$\begin{aligned}\|S - S_0\| &\geq \|(I - Q_n)(S - S_0)(I - P_n)\| \\ &\geq \|(I - Q_n)S(I - P_n)\| - \|(I - Q_n)S_0(I - P_n)\| \geq 1 - 3\varepsilon > r.\end{aligned}$$

From inequality (4), we also obtain

$$|h(S)| \geq \|h\| - 2\varepsilon.$$

We have just proved that $B \backslash (S_0 + rB_{K(\ell_p,\ell_q)})$ is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. As a consequence, the Shilov boundary of this space does not exist.

Now we give a procedure to construct boundaries for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. Since $\operatorname{Lin}\{x \otimes y : x \in (\ell_p)^*,\ y \in \ell_q,\ \operatorname{supp} x, \operatorname{supp} y \text{ are finite}\}$ is dense in $K(\ell_p,\ell_q)$, for every $h \in \mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ we have

$$\|h\| = \sup\{\|h_F\| : F \subset \mathbb{N} \text{ finite}\},$$

where $h_F(T) := h(Q_F T P_F)$ for $T \in K(\ell_p,\ell_q)$ and P_F, Q_F are the projections given by

$$P_F(x) = \sum_{n \in F} x(n) e_n \quad (x \in \ell_p), \qquad Q_F(x) = \sum_{n \in F} x(n) e_n \quad (x \in \ell_q).$$

Note also that $\|h_F\| \leq \|h_G\|$ for $F \subset G$.

Assume that (F_n) is an increasing sequence of finite subsets of $\mathbb{N}$ such that $G_n := F_{n+1} \setminus F_n$ is non-empty and $\bigcup_n F_n = \mathbb{N}$. We consider the subsets A_n whose elements are those operators $T \in B_{K(\ell_p,\ell_q)}$ that admit a decomposition $T = R + S$ satisfying

$$\|R\| = \|S\| = 1, \quad R = Q_{F_n} R P_{F_n}, \quad Q_{F_n} S P_{F_n} = 0, \quad Q_{G_n} S P_{G_n} = S.$$

Note that A_n is closed for every n.

We now check that $B = \bigcup_n A_n$ is a closed boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. Given $h \in \mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$ and $\varepsilon > 0$, there is some finite subset $F \subset \mathbb{N}$ such that $\|h_F\| > \|h\| - \varepsilon$. If $F \subset F_m$, then also $\|h_{F_m}\| \geq \|h\| - \varepsilon$. Hence there is an operator $R \in S_{K(\ell_p,\ell_q)}$ such that $Q_{F_m} R P_{F_m} = R$ where h_{F_m} attains its norm and so

$$|h(R)| \geq \|h\| - \varepsilon.$$

If $S \in S_{K(\ell_p,\ell_q)}$ satisfies $Q_{F_m} S P_{F_m} = 0$ and $Q_{G_m} S P_{G_m} = S$, then the operator $R + \lambda S$ is in the unit ball of $K(\ell_p, \ell_q)$, for every complex number λ in the unit disk. The maximum modulus theorem applied to the function $\lambda \mapsto h(R + \lambda S)$ defined on the complex unit disk shows that there is a complex number λ_0 with $|\lambda_0| = 1$ and such that

$$|h(R + \lambda_0 S)| \geq |h(R)| \geq \|h\| - \varepsilon.$$

Since $R + \lambda_0 S \in A_m$, B is a boundary for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$.

Note that for two positive integers $n < m$, if $T_n \in A_n$ and $T_m \in A_m$, then

$$\|T_m - T_n\| \geq \|Q_{G_m}(T_m - T_n) P_{G_m}\| = \|Q_{G_m} T_m P_{G_m}\| = 1. \tag{5}$$

Since every A_n is closed, the above inequality shows that B is also closed.

By the same argument, $\bigcup_n A_{2n}$ and $\bigcup_n A_{2n-1}$ are also closed boundaries for $\mathcal{A}_\infty(B_{K(\ell_p,\ell_q)})$. In view of (5), the distance between them is at least 1. ∎

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Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: dacosta@ugr.es

Departamento de Matemática e Estatística
Universidade de São Paulo
CP 66281 CEP
05311-970 São Paulo, Brazil
E-mail: mllouren@ime.usp.br

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