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On self-commutators of Toeplitz operators with rational symbols

by

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Abstract. We prove that the self-commutator of a Toeplitz operator with unbounded analytic rational symbol has a dense domain in both the Bergman space and the Hardy space of the unit disc. This is a basic step towards establishing whether the self-commutator has a compact or trace-class extension.

1. Introduction. Let $\mathcal{H}$ be a complex, separable Hilbert space. For a linear operator $T : \mathcal{H} \to \mathcal{H}$, $\mathcal{D}(T)$ and $\ker T$ denote the domain and kernel of T, respectively; that is, $\mathcal{D}(T) = \{h \in \mathcal{H} : Th \in \mathcal{H}\}$ and $\ker T = \{h \in \mathcal{D}(T) : Th = 0\}$. T is called *densely defined* if $\overline{\mathcal{D}(T)} = \mathcal{H}$, where the closure is taken with respect to the norm in $\mathcal{H}$. In fact, T has a unique adjoint T^* if and only if T is densely defined (see [1] for more details). For a densely defined operator T, the *self-commutator* of T is defined by $[T^*, T] = T^*T - TT^*$. Throughout this paper $\mathcal{H}$ stands for either the *Bergman space* $L^2_{\rm a}$ or the *Hardy space* H^2 of the open unit disc $\mathbb{D}$. Specifically, $L^2_{\rm a}$ is the space of all analytic functions on $\mathbb{D}$ for which

$$\|f\|_{L^2_{\rm a}} := \Big(\int_{\mathbb{D}} |f(z)|^2\, dA(z)\Big)^{1/2} < \infty,$$

where dA denotes the normalized Lebesgue area measure restricted to $\mathbb{D}$; and H^2 is the space of all analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H^2} := \sup_{0<r<1} \Big(\int_{\mathbb{T}} |f(r\zeta)|^2\, dm(\zeta)\Big)^{1/2} < \infty,$$

where dm denotes the normalized Lebesgue arc length measure restricted to

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the unit circle $\mathbb{T}$. The algebra of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞.

Using the almost everywhere existence of the non-tangential limit of each function in H^2, one can identify H^2 as a closed subspace of $L^2(\mathbb{T})$ consisting of functions (or equivalence classes of functions) with vanishing negative Fourier coefficients. In this connection, the norm of H^2 can be alternatively defined as

$$\|f\|_{H^2} := \Big(\int_{\mathbb{T}} |f(\zeta)|^2 \, dm(\zeta) \Big)^{1/2} < \infty,$$

where f denotes the boundary function. We will use this property of the Hardy space H^2 throughout without further references. We also assume some basics from the theory of Hardy and Bergman spaces (see [4] and [5] for more details).

We will consider the (unbounded) *Toeplitz operator* T_φ with symbol φ on $\mathcal{H}$; that is, if φ is a measurable function on $\mathbb{D}$ and $\mathcal{D}(T_\varphi) = \{f \in L^2_{\rm a} : \varphi f \in L^2(\mathbb{D})\}$, then $T_\varphi : \mathcal{D}(T_\varphi) \to \mathcal{H}$ is defined by $T_\varphi f = P\varphi f$, where (if $\mathcal{H} = L^2_{\rm a}$) $P = P_{\rm B}$ is the Bergman orthogonal projection of $L^2(\mathbb{D})$ onto $L^2_{\rm a}$ and (if $\mathcal{H} = H^2$) $P = P_{\rm H}$ is the Hardy orthogonal projection of $L^2(\mathbb{T})$ onto H^2. Observing that T_φ belongs to the larger class of unbounded subnormal operators, one can easily prove the next lemma (see [2] for details and a proof).

LEMMA 1. *If T_φ is a Toeplitz operator with symbol φ on $\mathcal{H}$, then*

(a) $\mathcal{D}(T_\varphi) \subseteq \mathcal{D}(T^*_\varphi)$.
(b) $T^*_\varphi f = T_{\overline{\varphi}} f$ *for all* f *in* $\mathcal{D}(T_\varphi)$.

In [7], the authors prove that the self-commutator of the Bergman–Toeplitz operator T_φ, where the symbol φ is a conformal mapping of the unit disc onto a region of bounded area, has a trace class extension to $L^2_{\rm a}$. In view of the generalization of the Berger–Shaw theorem obtained in [2], the proof given in [7] is based on the fact that $\mathcal{D}([T^*_\varphi, T_\varphi])$ is a dense subset of $L^2_{\rm a}$. Furthermore, using a rather technical argument, the authors were also able to establish the density of $\mathcal{D}([T^*_\varphi, T_\varphi])$ in $L^2_{\rm a}$ under the assumption that φ is a rational symbol of the form $\varphi(z) = (1-z)^{-1}$ or $\varphi(z) = (1-z)^{-2}$ (see [7]). In this paper, we prove the general case: if φ is an analytic rational symbol with poles on the unit circle, then the self-commutator of the Toeplitz operator T_φ on $\mathcal{H}$ is densely defined. As a result, this paper provides the first necessary step in investigating the open problem of whether $[T^*_\varphi, T_\varphi]$ has a compact (or trace-class) extension to $\mathcal{H}$.

2. Density Theorem. The main result of this paper is the following Density Theorem.

THEOREM A. *If φ is an analytic rational symbol in $\mathbb{D}$ with poles on the unit circle $\mathbb{T}$, then the self-commutator of the Toeplitz operator T_φ is densely defined with respect to both the Bergman space L^2_{a} and Hardy space H^2.*

Before giving the proof of Theorem A, we need the following important property of the adjoint operator T^*_φ. Note also that $\mathcal{P}$ denotes the linear space of analytic polynomials in variable z.

LEMMA 2. *If $\varphi = f/g$ where $f, g \in H^\infty$ and g is an outer function, then T^*_φ leaves the space of analytic polynomials invariant; that is, $T^*_\varphi \mathcal{P} \subseteq \mathcal{P}$.*

Proof. The condition "g is an outer function" guarantees the density of $\mathcal{D}(T_\varphi)$ in $\mathcal{H}$ so that T^*_φ is well defined. To give a proof, we note that $g[\mathcal{H}] = \{gh : h \in \mathcal{H}\}$ is clearly contained in $\mathcal{D}(T_\varphi)$. Now, in view of Beurling's theorem (see [4]), $g[H^2]$ is dense in H^2. The proof of the Bergman space case follows from the fact that H^2 is a dense subset of L^2_{a}.

To prove the claim, we first assume that $\varphi \in H^\infty$ where φ's Taylor expansion is given by $\varphi(z) = \sum_{n=0}^\infty a_n z^n$. Fix $n \ge 0$. The linearity of the projection operator P and Lemma 1(b) imply

$$T^*_\varphi z^n = P\Big[\Big(\sum_k \overline{a}_k \overline{z}^k\Big) z^n\Big] = \sum_k \overline{a}_k P(\overline{z}^k z^n).$$

A straightforward calculation (see [7] for details) shows that $P(\overline{z}^k z^n)$ vanishes for all $k > n$ and $P(\overline{z}^k z^n) = C_{kn} z^{n-k}$ for $k \le n$, where C_{kn} are constants depending only on k and n. This proves the bounded case.

Next assume that $1/\varphi$ is bounded. One can easily verify that $T^*_\varphi T^*_{1/\varphi} = I$, where I denotes the identity operator on $\mathcal{H}$. Thus we are done since $T^*_{1/\varphi}\mathcal{P} \subseteq \mathcal{P}$ by the bounded case. Finally, the general case ($\varphi = f/g$) follows from the above special cases together with the observation that $T^*_{1/g} T^*_f \subseteq T^*_\varphi$. ■

Proof of Theorem A. Suppose $\varphi = h/r$ where $h, r \in \mathcal{P}$. Let $\xi_1, \ldots, \xi_n \in \mathbb{T}$ denote r's distinct zeros of orders $\alpha_1, \ldots, \alpha_n$, respectively. Throughout the rest of the proof, $n \ge 1$ is fixed. Define

$$\mathcal{Q} = r[\mathcal{P}] = \{rp : p \in \mathcal{P}\} \quad \text{where} \quad r(z) = \prod_{i=1}^n (z - \xi_i)^{\alpha_i}.$$

First, we prove that $\mathcal{Q}$ is dense in $\mathcal{H}$. For $\xi \in \mathbb{T}$, the fact that $\overline{\xi}(\xi - z)$ has a positive real part on $\mathbb{D}$ implies that $F(z) = \xi - z$ is an outer function in H^2 (see [4]). Since the product of outer functions is again an outer function, the above fact implies that r is an outer function in H^2. Now a similar argument to the one given in the proof of Lemma 2 proves the density of $\mathcal{Q}$ in $\mathcal{H}$.

From the definition of $\mathcal{Q}$ and φ, it follows that $\mathcal{Q} \subseteq \mathcal{D}(T_\varphi)$ and $T_\varphi \mathcal{Q} \subseteq \mathcal{P}$. Consequently, in view of Lemma 2, $T_\varphi \mathcal{Q} \subseteq \mathcal{D}(T_\varphi^*)$; that is, $\mathcal{Q} \subseteq \mathcal{D}(T_\varphi^* T_\varphi)$. The proof is then complete if one can show that $\mathcal{Q}$ is also contained in $\mathcal{D}(T_\varphi T_\varphi^*)$. It turns out, however, that $\mathcal{Q}$ is too large for our purposes. In fact, we will show the existence of a dense subset of $\mathcal{Q}$ which is contained in $\mathcal{D}(T_\varphi T_\varphi^*)$.

For fixed $1 \le k \le n$ and $0 \le l \le \alpha_k - 1$, define the linear functional $L_{kl} : \mathcal{Q} \to \mathbb{C}$ by

$$L_{kl} : q \mapsto (T_\varphi^* q)^{(l)}(\xi_k) \quad \text{for } q \in \mathcal{Q}.$$

In the above definition, and the rest of the proof, $(\cdot)^{(i)}$ denotes $\frac{d^i}{dz^i}(\cdot)$. Next, we put $\mathcal{L} = \bigcap_{k,l} \ker L_{kl}$. Since $\mathcal{L} \subseteq \mathcal{Q}$, we have $\mathcal{L} \subseteq \mathcal{D}(T_\varphi^*)$. Moreover, the definition of L_{kl} directly implies that $T_\varphi^* \mathcal{L} \subseteq \mathcal{D}(T_\varphi)$. Thus $\mathcal{L} \subseteq \mathcal{D}(T_\varphi T_\varphi^*)$ and we are done if it can be shown that $\mathcal{L}$ is a dense subset of $\mathcal{Q}$.

Let $s(z) = \prod_{i=1}^n (z - \xi_i)$ and put $R(z) = r(z) \cdot s(z)$. For $p \in \mathcal{P}$, define

$$q(z) = (T_R^* p)(z) - \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} (T_R^* p)^{(j)}(\xi_i) t_{ij}(z), \tag{2.1}$$

where t_{ij} are polynomials satisfying $t_{ij}^{(l)}(\xi_k) = \delta_{ik}\delta_{jl}$ for $1 \le i \le n$, $0 \le j \le \alpha_i - 1$, $1 \le k \le n$, $0 \le l \le \alpha_k - 1$, and δ_{ij} stands for Kronecker's delta (see for example [3] for details). It follows easily that $q \in \mathcal{Q}$. Now, in view of the definition for L_{kl} and (2.1), we get

$$L_{kl}(q) = (T_S^* p)^{(l)}(\xi_k) - \sum_{i=1}^{n} \sum_{j=0}^{\alpha_i - 1} (T_R^* p)^{(j)}(\xi_i) (T_\varphi^* t_{ij})^{(l)}(\xi_k), \tag{2.2}$$

where $p \in \mathcal{P}$ and $S = sh$. To proceed further, we need two auxiliary results which are stated in Claims 1 and 2.

Claim 1. *Fix $1 \le i \le n$ and $0 \le j \le \alpha_i - 1$. If $p \in \mathcal{P}$, then there is a constant $C_{ij} > 0$ (independent of p) such that $|(T_R^* p)^{(j)}(\xi_i)| \le C_{ij} \|p\|_{\mathcal{H}}$.*

Proof. We only prove the Bergman case when $\mathcal{H} = L_{\mathrm{a}}^2$ and omit the similar proof of the Hardy case. Since $R \in H^\infty$, Lemma 2 implies $T_R^* \mathcal{P} \subseteq \mathcal{P}$. Thus

$$(T_R^* p)^{(j)}(\xi_i) = \lim_{r \to 1^-} (T_R^* p)^{(j)}(r\xi_i). \tag{2.3}$$

Recall that the Bergman kernel $k(z,w) = \overline{k_z(w)} = (1 - \overline{w}z)^{-2}$ has the reproducing property $f(z) = \langle f, k_z \rangle = \int_{\mathbb{D}} f(w) k_z(w)\, d\mathcal{A}(w)$ for all $f \in L_{\mathrm{a}}^2$ and $z \in \mathbb{D}$ (see [5] for more details). In particular, for $z = r\xi_i \in \mathbb{D}$ $(0 < r < 1)$ we have

$$(2.4)\qquad (T_R^* p)^{(j)}(r\xi_i) = \overline{\xi}_i^{\,j} \frac{d^j}{dr^j} \langle T_R^* p, k_{r\xi_i}\rangle = \overline{\xi}_i^{\,j} \frac{d^j}{dr^j} \langle p, T_R k_{r\xi_i}\rangle$$
$$= \overline{\xi}_i^{\,j} \frac{d^j}{dr^j} \int_{\mathbb{D}} p(w)\, \overline{r_{\xi_i}(w)}\, \overline{(w-\xi_i)}^{\alpha_i+1} \frac{1}{(1-\overline{w}r\xi_i)^2}\, d\mathcal{A}(w)$$
$$= (j+1)!\, \overline{\xi}_i^{\,2(j+1)} \int_{\mathbb{D}} p(w)\, \overline{r_{\xi_i}(w)}\, \overline{w}^j \overline{\left(\frac{w-\xi_i}{rw-\xi_i}\right)}^{\,j+2} \overline{(w-\xi_i)}^{\alpha_i-j-1}\, d\mathcal{A}(w),$$

where $r_{\xi_i}(z) = R(z)/(z-\xi_i)^{\alpha_i+1}$. By hypothesis, $\alpha_i - j - 1 \geq 0$; hence, in view of (2.3) and (2.4), the result follows from the dominated convergence theorem together with an application of Hölder's inequality. ■

Claim 2. *Fix $1 \leq k \leq n$, $0 \leq l \leq \alpha_k - 1$ and let $N = n + \deg h$. If p is a polynomial of the form $p(z) = a_N z^N + \cdots + a_{N+M} z^{N+M}$ $(M \geq 0)$, then there are constants C_{ikl} (independent of p) such that*

$$(T_S^* p)^{(l)}(\xi_k) = \begin{cases} \sum_{i=0}^{l} C_{ikl}\, p^{(i)}(\xi_k) & \text{if } \mathcal{H} = H^2, \\ \sum_{i=0}^{l+1} C_{ikl}\, P^{(i)}(\xi_k) & \text{if } \mathcal{H} = L_{\mathrm{a}}^2, \end{cases}$$

where $P(z) = \int_0^z p(w)\, dw$.

Proof. We will again only consider the Bergman case $\mathcal{H} = L_{\mathrm{a}}^2$ and omit the similar proof of the Hardy case. Recall that $S(z) = h(z) \cdot \prod_{i=1}^n (z-\xi_i) := \sum_{j=0}^N s_j z^j$. Moreover, as already noticed in the proof of Lemma 2,

$$P_{\mathrm{B}}(z^i\, \overline{z}^j) = \begin{cases} 0 & \text{for } j > i, \\ \dfrac{i+1-j}{i+1}\, z^{i-j} & \text{for } j \leq i. \end{cases}$$

Now for fixed $z \in \mathbb{D}$, in light of Lemma 1(b), one obtains

$$(T_S^* p)(z) = P_{\mathrm{B}}\Big(\sum_{i=N}^{N+M} a_i \sum_{j=0}^{N} \overline{s}_j z^i \overline{z}^j\Big) = \sum_{i=N}^{N+M} a_i \sum_{j=0}^{N} \overline{s}_j \frac{i+1-j}{i+1} z^{i-j}$$
$$= \frac{d}{dz} \sum_{i=N}^{N+M} a_i \frac{1}{i+1} z^{i+1} \sum_{j=0}^{N} \overline{s}_j z^{-j}$$
$$= \frac{d}{dz}\Big[\sum_{i=N}^{N+M} a_i \int_0^z w^i\, dw \cdot \overline{S(1/\overline{z})}\Big]$$
$$= \frac{d}{dz}\Big[\int_0^z p(w)\, dw \cdot \overline{S(1/\overline{z})}\Big].$$

Differentiation of the above equality l times with respect to z yields

$$(2.5)\qquad (T_S^*p)^{(l)}(z)=\frac{d^{l+1}}{dz^{l+1}}[P(z)\cdot\overline{S(1/\overline{z})}]=\sum_{i=0}^{l+1}\binom{l+1}{i}P^{(i)}(z)\,\overline{S(1/\overline{z})}^{(l+1-i)}.$$

Thus the claim follows from the evaluation of (2.5) at $z=\xi_k$. ■

Now suppose that $\mathcal{L}$ is not dense in $\mathcal{Q}$. Then, by the Hahn–Banach theorem, there is a non-zero bounded linear functional L on $\mathcal{H}$ such that $\mathcal{L}\subseteq \ker L$. Hence there are constants $\lambda_{kl}\in\mathbb{C}$ ($1\le k\le n$ and $0\le l\le \alpha_k-1$) such that $L(q)=\sum_{k,l}\lambda_{kl}\,L_{kl}(q)$ for all $q\in\mathcal{Q}$ (see for example [1]). For $p\in\mathcal{P}$, let q be defined as in (2.1). It follows from (2.2) that

$$L(q)=\sum_{k,l}\lambda_{kl}(T_S^*p)^{(l)}(\xi_k)-\sum_{i,j,k,l}\lambda_{kl}(T_R^*p)^{(j)}(\xi_i)(T_\varphi^*t_{ij})^{(l)}(\xi_k),$$

where $1\le i\le n$, $0\le j\le\alpha_i-1$, $1\le k\le n$, and $0\le l\le\alpha_k-1$.

Using the result of Claim 1 and the fact that $|(T_\varphi^*t_{ij})^{(l)}(\xi_k)|$ are independent of p, we get

$$(2.6)\qquad |L(q)|\ge\Big|\sum_{k,l}\lambda_{kl}(T_s^*p)^{(l)}(\xi_k)\Big|-\sum_{i,j,k,l}C_{ijkl}\|p\|_{\mathcal{H}},$$

where C_{ijkl} denote non-negative constants which do not depend on p.

Since $L\ne 0$, there is at least a pair of integers $1\le s\le n$ and $0\le t\le\alpha_s-1$ for which $\lambda_{st}\ne 0$. Fix $m\ge 1$ and suppose there exists a polynomial p_m which satisfies the following conditions:

(a_1) p_m is of the form $p_m(z)=a_Nz^N+\cdots+a_{N+M}z^{N+M}$ where $N=n+\deg h$;

(a_2) $p_m^{(t)}(\xi_s)=m$;

(a_3) $p_m^{(l)}(\xi_k)=0$ for $1\le k\ne s\le n$ and $0\le l\le\alpha_k-1$; if $\mathcal{H}=L^2_{\mathrm{a}}$, then p must satisfy the additional conditions $\int_0^{\xi_k}p(w)\,dw=0$ for $1\le k\le n$;

(a_4) $\|p_m\|_{\mathcal{H}}\le 1$.

Let q_m denote the corresponding polynomial for p_m in accordance with (2.1). Use Claim 2 together with the properties (a_1)–(a_4) to deduce from (2.6) that

$$|L(q_m)|\ge m\cdot|C_{st}|-C$$

where C_{st} is a constant independent of m and $C=\sum_{ijkl}C_{ijkl}$. But then $|L(q_m)|$ can be made arbitrarily large, as $m\to\infty$, which contradicts the boundedness of L. Thus $\mathcal{L}$ must be dense in $\mathcal{Q}$ (see [1]) and the proof of the theorem is complete if the existence of such a polynomial p_m can be proved.

We only give a proof of the more involved case of the Bergman space $\mathcal{H}=L^2_{\mathrm{a}}$. (The proof of the Hardy space case $\mathcal{H}=H^2$ is similar and easier.)

Let H_m denote the Hermite interpolating polynomial (see for example [8]) which satisfies the conditions:

(b$_1$) H_m is of the form $H_m(z) = a_N z^N + \cdots + a_{N+M} z^{N+M}$ where $N = n + \deg h$;
(b$_2$) $H_m^{(t)}(\xi_s) = m$;
(b$_3$) $H_m^{(l)}(\xi_k) = 0$ for $1 \le k \ne s \le n$ and $0 \le l \le \alpha_k$.

Put $\beta = \frac{1}{2}\min\{\|H_m\|_\infty^{-1}, \|H'_m\|_\infty^{-1}\}$. It is not hard to see that one can choose a polynomial K_m such that

(c$_1$) $K_m(\xi_s) = 1, K'_m(\xi_s) = 0, \ldots, K_m^{(\alpha_s - t)}(\xi_s) = 0$,
(c$_2$) $\|K_m\|_{L^2_a} \le \|K'_m\|_{L^2_a} \le \beta$.

Indeed, let $K_m(z) = 1 - C^{-1}(z - \xi_s)^M$, where $M \ge \alpha_s - t$ and the constant C is chosen appropriately to satisfy (c$_2$). Now, if we let $P_m = H_m K_m$ and set $p_m(z) = P'_m(z)$, then (b$_1$)–(b$_3$) and (c$_1$) imply that p satisfies (a$_1$)–(a$_3$). Moreover, it follows from (c$_2$) that

$$\|p_m\|_{L^2_a} = \|P'_m\|_{L^2_a} \le \|H'_m\|_\infty \|K_m\|_{L^2_a} + \|H_m\|_\infty \|K'_m\|_{L^2_a} \le 1.$$

Thus p_m also satisfies (a$_4$), as was required. ∎

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