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A perturbation characterization of compactness of self-adjoint operators

by

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Abstract. A characterization of compactness of a given self-adjoint bounded operator A on a separable infinite-dimensional Hilbert space is established in terms of the spectrum of perturbations. An example is presented to show that without separability, the perturbation condition, which is always necessary, is not sufficient. For non-separable spaces, another condition on the self-adjoint operator A, which is necessary and sufficient for the perturbation, is given.

1. Introduction and preliminaries. Let H be a Hilbert space. We shall denote by $\mathcal{B}(H)$ the space of all bounded linear operators on H. If $A \in \mathcal{B}(H)$, then $\sigma(A)$ and $\sigma_{\mathrm{ess}}(A)$ denote the spectrum and essential spectrum [KA, p. 243] of A respectively.

In this paper, we first establish a characterization of compactness of a given self-adjoint operator on a separable infinite-dimensional Hilbert space in terms of the spectrum of perturbations (see Theorem 2.1 below). The point of departure is the result of Lemma 1 in [CLT, p. 73]: For each $n \times n$ non-zero real symmetric matrix A, there exists an $n \times n$ real symmetric matrix B such that B and $B + A$ have no eigenvalue in common. An example is presented to show that without separability, the perturbation condition, which is always necessary, is not sufficient. Our results show that the dimension of the Hilbert space plays a vital role here: for a non-separable Hilbert space case, the suitable generalization of compact (self-adjoint) operators, in this regard, is (self-adjoint) operators without any non-zero element in σ_{s} (cf. Definition 2.2 below). Altogether we have the following:

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THEOREM. *Let H be a (real or complex) finite-dimensional (respectively, separable infinite-dimensional, non-separable) Hilbert space, $A = A^* \in \mathcal{B}(H)$. In order that for all $B = B^* \in \mathcal{B}(H)$, $\sigma(B) \cap \sigma(B+A) \neq \emptyset$, it is necessary and sufficient that $\sigma(A)$ (respectively $\sigma_{\mathrm{ess}}(A), \sigma_{\mathrm{s}}(A)$) does not have a non-zero element.*

This differs from similar perturbation problems considered in [BPS] and [DPR], where $B \in \mathcal{B}(H)$ is a nilpotent, respectively general operator. The characterizations obtained are also different: ours are in terms of spectral sets while [BPS] and [DPR] are in terms of maximal two-sided ideals.

2. Perturbation theorems

THEOREM 2.1. *Let H be a separable infinite-dimensional (real or complex) Hilbert space and $A \in \mathcal{B}(H)$ be self-adjoint. Then A is compact if and only if for every self-adjoint $B \in \mathcal{B}(H)$, $\sigma(B) \cap \sigma(B+A) \neq \emptyset$.*

Proof. It is well known that for a compact $A \in \mathcal{B}(H)$ we have $\emptyset \neq \sigma_{\mathrm{ess}}(B) = \sigma_{\mathrm{ess}}(B+A)$ for any $B \in \mathcal{B}(H)$ (see, for example, [GGK, Vol. 1, p. 191 & p. 205]; the necessity of the present theorem follows immediately (note that the separability of H is not involved here).

To prove the sufficiency, we will assume that the self-adjoint $A \in \mathcal{B}(H)$ is non-compact, and we will construct a self-adjoint $B \in \mathcal{B}(H)$ such that $\sigma(B) \cap \sigma(B+A) = \emptyset$. Assume without loss of generality that $\|A\| < 1$.

Since the self-adjoint operator A is non-compact, there exists a non-zero real number p in $\sigma_{\mathrm{ess}}(A)$ (cf. [RU, Theorem 12.30, pp. 312–313]). Without loss of generality assume $p > 0$ (otherwise work with $-A$ instead of A). Note that $p < 1$, since $\|A\| < 1$.

If A is invertible, then $B = 0$ does the job. So assume $0 \in \sigma(A)$. Let $q = p/m$ for some sufficiently large positive integer m so that $p + q < 1$ and $2q < p$. By considering the spectral representation $A = \int_{[-1,1]} \lambda\, dP_\lambda$ and defining the operators

$$R = \int\limits_{[-q,q]} \lambda\, dP_\lambda, \qquad S = \int\limits_{[p-q,p+q]} \lambda\, dP_\lambda,$$

we can represent A on $H = H_1 \oplus H_2 \oplus H_3$ as

$$A = \begin{bmatrix} T & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & R \end{bmatrix},$$

where $\|R\| \leq q$, $\|S - p\| \leq q$ (or equivalently, $p - q \leq S \leq p + q$) and T is some invertible operator. Furthermore, H_2 is infinite-dimensional, because $p \in \sigma_{\mathrm{ess}}(A)$ (cf. [KA, Remark X.1.11, p. 520]). The subspace H_3 on which R lives could have finite or infinite dimension; T could be absent.

As $\dim H_2 \geq \dim H_3$ (recall H is separable and infinite-dimensional), there is an isometry J from H_3 into H_2 (so that J^*J is the identity operator on H_3). Let

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & J \\ 0 & J^* & 0 \end{bmatrix}.$$

Then B^2 is a projection and thus $\sigma(B) \subset \{0, 1, -1\}$. We shall show that the spectrum of the self-adjoint operator

$$A + B = \begin{bmatrix} T & 0 & 0 \\ 0 & S & J \\ 0 & J^* & R \end{bmatrix}$$

does not contain 0, 1, or -1. Since T is invertible and $\|T\| < 1$, $\sigma(T)$ cannot contain any of these points; we will prove this for the spectrum of $\begin{bmatrix} S & J \\ J^* & R \end{bmatrix}$, whose union with $\sigma(T)$ constitutes the spectrum of $A + B$. We shall make use of the fact that for normal operators, every point in the spectrum is in the approximate point spectrum.

(i) $0 \notin \sigma\left(\begin{bmatrix} S & J \\ J^* & R \end{bmatrix}\right)$.

Assume $0 \in \sigma\left(\begin{bmatrix} S & J \\ J^* & R \end{bmatrix}\right)$. Then 0 is in the approximate point spectrum, i.e., there are sequences $(x_n)_{n=1}^{\infty}$ in H_2 and $(y_n)_{n=1}^{\infty}$ in H_3 such that $\|x_n\|^2 + \|y_n\|^2 = 1$ and

$$\begin{bmatrix} S & J \\ J^* & R \end{bmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to 0 \quad \text{as } n \to \infty,$$

or

$$(+) \qquad \begin{cases} Sx_n + Jy_n \to 0, \\ J^*x_n + Ry_n \to 0, \quad \text{as } n \to \infty. \end{cases}$$

Thus $x_n + S^{-1}Jy_n \to 0$ and $J^*x_n + J^*S^{-1}Jy_n \to 0$ as $n \to \infty$. Hence $J^*S^{-1}Jy_n - Ry_n \to 0$ and $(S^{-1}Jy_n \,|\, Jy_n) - (Ry_n \,|\, y_n) \to 0$ as $n \to \infty$. But $(p+q)^{-1} \leq S^{-1}$ (recall $S \leq p + q$). Thus for each $n \geq 1$,

$$(S^{-1}Jy_n \,|\, Jy_n) \geq (p+q)^{-1}\|Jy_n\|^2 = (p+q)^{-1}\|y_n\|^2.$$

For each $n \geq 1$, as $(Ry_n \,|\, y_n) \leq q\|y_n\|^2$, we have

$$(S^{-1}Jy_n \,|\, Jy_n) - (Ry_n \,|\, y_n) \geq \left(\frac{1}{p+q} - q\right)\|y_n\|^2.$$

Since the left hand side tends to 0 as $n \to \infty$ (and since $q = p/m < 1 < 1/(p+q)$), we get $y_n \to 0$ as $n \to \infty$. Thus $\|x_n\| \to 1$ as $n \to \infty$ and the first equation of (+) gives $Sx_n \to 0$ as $n \to \infty$, which is a contradiction because S is invertible.

(ii) $1 \notin \sigma\left(\begin{bmatrix} S & J \\ J^* & R \end{bmatrix}\right)$.

Assume not. Then there are sequences $(x_n)_{n=1}^\infty$ in H_2 and $(y_n)_{n=1}^\infty$ in H_3 such that $\|x_n\|^2 + \|y_n\|^2 = 1$ and

$$(++) \qquad \begin{cases} (S-1)x_n + Jy_n \to 0, \\ J^*x_n + (R-1)y_n \to 0, \quad \text{as } n \to \infty. \end{cases}$$

Then $x_n - (1-S)^{-1}Jy_n \to 0$ as $n \to \infty$. Multiplying by J^* and comparing to the second equation in (++) we get

$$J^*(1-S)^{-1}Jy_n - (1-R)y_n \to 0 \quad \text{as } n \to \infty$$

and thus $((1-S)^{-1}Jy_n \mid Jy_n) - ((1-R)y_n \mid y_n) \to 0$ as $n \to \infty$.

Since $1-p-q \le 1-S \le 1-p+q$, we also have

$$(1-p+q)^{-1} \le (1-S)^{-1}.$$

Also $1-q \le 1-R \le 1+q$. Thus, as in (i), for each $n \ge 1$,

$$\begin{aligned} &((1-S)^{-1}Jy_n \mid Jy_n) - ((1-R)y_n \mid y_n) \\ &\qquad \ge \frac{1}{1-p+q}\|y_n\|^2 - (1+q)\|y_n\|^2 = \left(\frac{1}{1-p+q} - 1 - q\right)\|y_n\|^2, \end{aligned}$$

which implies $y_n \to 0$ as $n \to \infty$. The first equation of (++) now gives $(S-1)x_n \to 0$ as $n \to \infty$; since $\|x_n\| \to 1$ as $n \to \infty$ (and $1-S$ is invertible), this is a contradiction.

(iii) $-1 \notin \sigma\left(\begin{bmatrix} S & J \\ J^* & R \end{bmatrix}\right)$.

Assume not. As above, there are sequences $(x_n)_{n=1}^\infty$ in H_2 and $(y_n)_{n=1}^\infty$ in H_3 such that $\|x_n\|^2 + \|y_n\|^2 = 1$ and

$$\begin{cases} (S+1)x_n + Jy_n \to 0, \\ J^*x_n + (R+1)y_n \to 0, \quad \text{as } n \to \infty. \end{cases}$$

The proof is similar to (ii): we obtain

$$J^*(1+S)^{-1}Jy_n - (1+R)y_n \to 0 \quad \text{as } n \to \infty;$$

using the relations $1-q \le 1+R$ and $(1+S)^{-1} \le (1+p-q)^{-1}$ we conclude that for each $n \ge 1$,

$$[1-q-(1+p-q)^{-1}]\|y_n\|^2 \le ((1+R)y_n \mid y_n) - ((1+S)^{-1}Jy_n \mid Jy_n)$$

so that $y_n \to 0$ and $\|x_n\| \to 1$ as $n \to \infty$, contradicting $(1+S)x_n \to 0$ as $n \to \infty$. ■

We remark here that our Theorem 2.1 above is closely related to Corollary B in [DPR] which is stated as follows: "Suppose H is a separable infinite-dimensional complex Hilbert space and $A \in \mathcal{B}(H)$. Then A is compact if and only if for all $B \in \mathcal{B}(H), \sigma(B) \cap \sigma(B+A) \neq \emptyset$." (Note that in this corollary, unlike our Theorem 2.1, A and B are not necessarily self-adjoint.) Our proof of Theorem 2.1 is quite different from that of Corollary B in [DPR] which is obtained from Theorem B in [DPR] and a result of Calkin [C]. Moreover, there does not seem to be an easy or direct way to obtain our Theorem 2.1 from the said Corollary B (or Theorem B) in [DPR]. For perturbations by nilpotent operators, we refer the reader to [BPS].

DEFINITION 2.2. Let H be a (real or complex) Hilbert space of infinite dimension, and $A \in \mathcal{B}(H)$ be self-adjoint with spectral representation $A = \int_{\mathbb{R}} \lambda \, dP_\lambda$. For any Borel subset Ω of $\mathbb{R}$, define

$$H_\Omega := \Big\{ \Big(\int_\Omega dP_\lambda \Big) x : x \in H \Big\}.$$

Also define

$$\sigma_{\mathrm{s}}(A) := \{ t \in \mathbb{R} : \text{for every } \delta > 0,\ \dim H_{(t-\delta, t+\delta)} = \dim H \}.$$

Clearly $\sigma_{\mathrm{s}}(A) \subset \sigma(A)$. [Indeed, $\sigma_{\mathrm{s}}(A) \subset \sigma_{\mathrm{ess}}(A)$, cf. [KA, Remark X.1.11, p. 520]).

LEMMA 2.3. *Let H be a* (*real or complex*) *Hilbert space of infinite dimension*, *and let $A \in \mathcal{B}(H)$ be self-adjoint. Then $\sigma_{\mathrm{s}}(A) \neq \emptyset$.*

Proof. Suppose $\sigma_{\mathrm{s}}(A) = \emptyset$. Then each point t of $\sigma(A)$ has an open neighborhood $(t-\delta, t+\delta)$ such that $H_{(t-\delta, t+\delta)}$ has smaller dimension than H. By compactness, $\sigma(A)$ is covered by a finite number of these neighborhoods, implying that the dimension of $H_{\sigma(A)}$ is less than that of H, but $H_{\sigma(A)} = H$. This contradiction proves the lemma. ∎

The following result shows that Theorem 2.1 is false if the given Hilbert space H is not separable:

PROPOSITION 2.4. *There exist a non-separable* (*real or complex*) *Hilbert space H and a non-compact self-adjoint operator $A \in \mathcal{B}(H)$ such that for every self-adjoint operator $B \in \mathcal{B}(H)$, $\sigma(B) \cap \sigma(B+A) \neq \emptyset$.*

Proof. We will present two proofs; the second proof illustrates the situation when $\sigma_{\mathrm{s}}(A) = \{0\}$.

For the first proof, let $H = H_1 \oplus H_2$, where H_1 is a separable infinite-dimensional Hilbert space and H_2 is a non-separable Hilbert space, and let $A = I \oplus O$, where $I \in B(H_1)$ is the identity operator and $O \in B(H_2)$ is the zero operator. Then A is self-adjoint and not compact. Let $B \in \mathcal{B}(H)$ be self-adjoint. To see that $\sigma(B) \cap \sigma(B+A) \neq \emptyset$, let $\widehat{H}_1 = \overline{\mathrm{Sp}}(\bigcup_{n=0}^\infty B^n H_1)$ (where $\overline{\mathrm{Sp}}(K)$ denotes the closure of the span of the subset K of H) and let H_2' be the orthogonal complement of $\widehat{H}_1$ in H. Then $H = \widehat{H}_1 \oplus H_2'$, and with respect to this new direct sum decomposition of H,

$$A = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad \text{where} \quad E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $\sigma(B) = \sigma(B_1) \cup \sigma(B_2)$, $\sigma(B+A) = \sigma(B_1+E) \cup \sigma(B_2)$; hence $\sigma(B) \cap \sigma(B+A) \supset \sigma(B_2)$, which is non-empty. Therefore $\sigma(B) \cap \sigma(B+A) \neq \emptyset$.

For the second proof, let $H = \sum_{k=1}^{\infty} \oplus H_k$ where H_k is a Hilbert space with $\dim H_k = \aleph_{k-1}$, $k = 1, 2, \ldots$, and let $A = \sum_{k=1}^{\infty} \oplus k^{-1} I_k$ where $I_k \in B(H_k)$ is the identity operator. Then $\dim H = \aleph_\omega$, A is self-adjoint, not compact, $\sigma(A) \subset [0,1]$, and the spectral representation $A = \int_{[0,1]} \lambda\, dP_\lambda$ is given by

$$P_\lambda = \begin{cases} 0 & \text{if } \lambda = 0, \\ \Pr\Big(\displaystyle\sum_{k=n+1}^{\infty} \oplus H_k\Big) & \text{if } 1/(n+1) \le \lambda < 1/n,\ n \ge 1, \\ I & \text{if } \lambda = 1, \end{cases}$$

where $\Pr(\sum_{k=n}^{\infty} \oplus H_k)$ denotes the (orthogonal) projection of H onto the closed subspace $\sum_{k=n}^{\infty} \oplus H_k$, and I denotes the identity operator in $\mathcal{B}(H)$.

Let $B \in \mathcal{B}(H)$ be self-adjoint. By Lemma 2.3 above, it suffices to show that $\sigma_{\mathrm{s}}(B) \subset \sigma(B+A)$. To this end, let $t \in \sigma_{\mathrm{s}}(B)$ and $B = \int_R \lambda\, dQ_\lambda$ be the spectral representation of B. Given $\varepsilon > 0$, we shall show the existence of a unit vector x such that $\|(A+B-t)x\| < \varepsilon$. Consider the ranges K_1 and K_2 of the two projections

$$\int_{(-\varepsilon/2,\varepsilon/2)} dP_\lambda \quad \text{and} \quad \int_{(t-\varepsilon/2,t+\varepsilon/2)} dQ_\lambda$$

respectively. Observe that $\dim K_2 = \dim H$, but the dimension of the orthogonal complement of K_1 is strictly less than $\dim H$ by construction of A. Thus a cardinality argument shows that $K_1 \cap K_2 \neq \{0\}$, and in fact, $\dim(K_1 \cap K_2) = \dim K_2$. Pick a unit vector x in this intersection. Then

$$\|(A+B-t)x\| \le \|Ax\| + \|(B-t)x\| \le \frac{\varepsilon}{2}\|x\| + \frac{\varepsilon}{2}\|x\|.$$

This proves $t \in \sigma(A+B)$; in fact, it is clear from the argument that $t \in \sigma_{\mathrm{s}}(A+B)$. ∎

Remark 2.5. (i) With slight modification (using the compactness of $[-\|A\|, \|A\|]$), the above proof of Proposition 2.4 works for any self-adjoint $A \in \mathcal{B}(H)$ with $\sigma_{\mathrm{s}}(A) = \{0\}$: Let H be an infinite-dimensional Hilbert space, and let $B = B^* \in \mathcal{B}(H)$. Then $\sigma_{\mathrm{s}}(B) \subset \sigma(B+A)$ for any $A = A^* \in \mathcal{B}(H)$ with $\sigma_{\mathrm{s}}(A) = \{0\}$.

(ii) The proof of the sufficiency part of Theorem 2.1 above works for any self-adjoint $A \in \mathcal{B}(H)$ with a non-zero $t \in \sigma_{\mathrm{s}}(A)$. Thus we have: Let H be an infinite-dimensional Hilbert space, and $A \in \mathcal{B}(H)$ be self-adjoint. Then $\sigma_{\mathrm{s}}(A) = \{0\}$ if and only if for every self-adjoint $B \in \mathcal{B}(H)$, $\sigma(B) \cap \sigma(B+A) \neq \emptyset$. In other words, $\sigma_{\mathrm{s}}(A)$ has a non-zero element if and only if there exists $B = B^* \in \mathcal{B}(H)$ such that $\sigma(B) \cap \sigma(B+A) = \emptyset$.

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