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Mean ergodicity for compact operators

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Abstract. A mean ergodic theorem is proved for a compact operator on a Banach space without assuming mean-boundedness. Some related results are also presented.

1. Introduction. In the literature, mean ergodic theorems for linear operators usually deal with operators which are power bounded (see, e.g., [YK] and [Z]). However, already in 1945, Hille [H] gave an example of an operator T on $X = L_1[0,1]$ which is *mean ergodic* (i.e., the sequence of averages $(n^{-1}\sum_{j=1}^{n} T^j x)_{n=1}^{\infty}$ converges strongly for every $x \in X$) but not power-bounded. By the Banach–Steinhaus theorem, a necessary condition for mean ergodicity is *mean-boundedness*, i.e., $\sup_n n^{-1}\|\sum_{j=1}^{n} T^j\| < \infty$ (which is C-mean-boundedness in [E]). Also the strong (resp. weak) convergence of $(n^{-1}\sum_{j=1}^{n} T^j x)_{n=1}^{\infty}$ clearly implies that $(n^{-1}T^n x)_{n=1}^{\infty} \to 0$ strongly (resp. weakly). In the treatment of mean ergodic theory in the book of Dunford and Schwartz [DS], the operator T is assumed to be mean-bounded (Theorem VIII.5.1, p. 661), or the sequence $(n^{-1}T^n)_{n=1}^{\infty}$ is assumed to converge to zero weakly (Theorem VIII.8.3, p. 711). In 1985, Émilion [E] gave an example of a positive operator on L_p $(1 < p < \infty)$ which is mean ergodic and not power-bounded; he also showed by an example (due to I. Assani) that mean-boundedness of a compact operator T does not imply $(n^{-1}\|T^n\|)_{n=1}^{\infty} \to 0$. More recently, Derriennic [D] constructed a mean ergodic operator T on a Hilbert space such that $\|T^n\| \ge n$ for every positive integer n; moreover, T^* is weakly mean ergodic (i.e., the averages converge weakly for every point of the Hilbert space) but not mean ergodic. Moreover, Yoshimoto [Y1, Y2] obtained, under the assumption that $(n^{-w}\|T^n\|)_{n=1}^{\infty} \to 0$ (resp.

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$(n^{-w}T^n)_{n=1}^\infty \to 0$ in the strong operator topology), the equivalence between the convergence of $C_n^{(\alpha)}[T]$ in the uniform operator norm (resp., in the strong operator topology) and that of the so-called Dirichlet methods (which generalize the Abel method), where $w = \min(1, \alpha)$, and $C_n^{(1)}[T] = n^{-1}\sum_{j=0}^{n-1} T^j$.

In this paper, under a fairly weak condition (cf. Proposition 2.1(4) below), we shall first obtain a general mean ergodic theorem for operators T which are not necessarily mean-bounded nor satisfy $(n^{-1}T^n)_{n=1}^\infty \to 0$ on X uniformly, operator strongly, or operator weakly (cf. also Proposition 2.2). We next obtain a mean ergodic theorem for compact operators on a Banach space, which need not be mean-bounded nor satisfy $(n^{-1}T^n)_{n=1}^\infty \to 0$ on X uniformly, operator strongly or operator weakly (cf. Theorem 2.3 and its corollaries below). Finally, in Theorem 2.10, we present a relation between our condition and power-boundedness.

2. Ergodic theorems. If $(X, \|\cdot\|)$ is a normed space, we denote by $\mathcal{B}(X)$ the space of all bounded linear operators on X. If $A \in \mathcal{B}(X)$, then $x \in X$ is called a *fixed point* of A if $Ax = x$, and $\sigma(A)$ denotes the spectrum of A.

We begin with the following result:

PROPOSITION 2.1. *Let $(X, \|\cdot\|)$ be a (real or complex) normed space, $A \in \mathcal{B}(X)$ and $x \in X$. Denote by I the identity operator on X, and $\overline{(I-A)X}$ the norm closure of $(I-A)X$ in X.*

(1) *If for some subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$, $n_k^{-1}\sum_{j=1}^{n_k} A^j x \to 0$ weakly as $k \to \infty$, then $x \in \overline{(I-A)X}$.*

(2) *If $n^{-1}A^n x \to 0$ weakly as $n \to \infty$ and $n_k^{-1}\sum_{j=1}^{n_k} A^j x \to \overline{x}$ weakly as $k \to \infty$ for some subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$, then $A\overline{x} = \overline{x}$, and $x - \overline{x} \in \overline{(I-A)X}$.*

(3) *If $x = (I-A)y$ with $n^{-1}A^n y \to 0$ weakly (resp. strongly) as $n \to \infty$, then $n^{-1}\sum_{j=1}^{n} A^j x \to 0$ weakly (resp. strongly) as $n \to \infty$.*

(4) *Suppose $n^{-1}A^n x \to 0$ weakly as $n \to \infty$ and for some subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$, $n_k^{-1}\sum_{j=1}^{n_k} A^j x \to \overline{x}$ weakly as $k \to \infty$. If*

$$(*) \qquad x - \overline{x} = (I-A)y \quad \text{with} \quad n^{-1}\|A^n y\| \to 0 \text{ as } n \to \infty,$$

then $\|n^{-1}\sum_{j=1}^{n} A^j x - \overline{x}\| \to 0$ as $n \to \infty$.

Proof. (1) Note that

$$(I-A)\left(I + \frac{n_k - 1}{n_k} A + \frac{n_k - 2}{n_k} A^2 + \ldots + \frac{1}{n_k} A^{n_k - 1}\right)x$$
$$= \left[I - \frac{1}{n_k}(A + A^2 + \ldots + A^{n_k})\right]x \to x \quad \text{weakly as } k \to \infty.$$

The desired conclusion follows.

(2) By our assumption, setting $x_n = n^{-1}\sum_{j=1}^n A^j x$, we have

$$
\begin{aligned}
Ax_{n_k} &= \frac{1}{n_k}\sum_{j=2}^{n_k+1} A^k x = \frac{1}{n_k}\Big[\Big(\sum_{j=1}^{n_k} A^k x\Big) + A^{n_k+1}x - Ax\Big] \\
&= x_{n_k} + \frac{1}{n_k}(A^{n_k+1}x) - \frac{1}{n_k}Ax \to \overline{x} \quad \text{weakly as } k\to\infty.
\end{aligned}
$$

Since A is also weakly continuous, $Ax_{n_k} \to A\overline{x}$ weakly as $k\to\infty$; thus we conclude that $A\overline{x} = \overline{x}$. Therefore

$$
\frac{1}{n_k}\sum_{j=1}^{n_k} A^j(x-\overline{x}) = \frac{1}{n_k}\sum_{j=1}^{n_k} A^j x - \overline{x} \to 0 \quad \text{weakly as } k\to\infty.
$$

By (1), $x-\overline{x} \in \overline{(I-A)X}$.

(3) Since

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} A^j x &= \frac{1}{n}\sum_{j=1}^{n} A^j(I-A)y = \frac{1}{n}\sum_{j=1}^{n} A^j y - \frac{1}{n}\sum_{j=1}^{n} A^{j+1} y \\
&= \frac{1}{n}Ay - \frac{1}{n}A^{n+1}y,
\end{aligned}
$$

the desired conclusions hold.

(4) By (2), $A\overline{x}=\overline{x}$. Thus by (3), we have

$$
\frac{1}{n}\sum_{j=1}^{n} A^j x - \overline{x} = \frac{1}{n}\sum_{j=1}^{n} A^j(x-\overline{x}) \to 0 \quad \text{strongly as } n\to\infty. \ \blacksquare
$$

PROPOSITION 2.2. *Let $(X,\|\cdot\|)$ be a (real or complex) Banach space, $A\in\mathcal{B}(X)$ and $x\in X$. Suppose $I-A$ is one-to-one and has closed range, $\|A^n x\|/n \to 0$ as $n\to\infty$ and $n_k^{-1}\sum_{j=1}^{n_k} A^j x \to \overline{x}$ weakly as $k\to\infty$ for a subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$. Then there exists $y\in X$ satisfying the condition $(*)$ of Proposition 2.1 above, and $\|n^{-1}\sum_{j=1}^n A^j x - \overline{x}\| \to 0$ as $n\to\infty$.*

Proof. By Proposition 2.1(2), $A\overline{x}=\overline{x}$ and $x-\overline{x}\in\overline{(I-A)X}$. As $\|A^n x\|/n\to 0$ as $n\to\infty$, we have $\|A^n(x-\overline{x})\|/n \to 0$ as $n\to\infty$. Since $I-A$ has closed range and is one-to-one, there is a (unique) $y\in X$ such that $(I-A)y = x-\overline{x}$. Since $(I-A)X$ is (closed in X, hence) a Banach space, by the open mapping theorem $(I-A)^{-1} : (I-A)X\to X$ is bounded. Thus

$$
\begin{aligned}
\frac{\|A^n y\|}{n} &= \frac{\|(I-A)^{-1}A^n(x-\overline{x})\|}{n} \\
&\le \|(I-A)^{-1}\|\,\frac{\|A^n(x-\overline{x})\|}{n} \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

By Proposition 2.1(4), $\|n^{-1}\sum_{j=1}^n A^j x - \overline{x}\| \to 0$ as $n\to\infty$. $\blacksquare$

We now present our main result:

THEOREM 2.3. *Let $(X, \|\cdot\|)$ be a (real or complex) Banach space, $A \in \mathcal{B}(X)$ be a compact operator and $x \in X$ be such that $\|A^n x\|/n \to 0$ as $n \to \infty$. If the sequence $(n^{-1}\sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$ is bounded, then it converges strongly to a fixed point of A.*

Proof. As A is compact and the sequence $(n^{-1}\sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$ is bounded, every subsequence of the sequence $(n^{-1}\sum_{j=2}^{n+1} A^j x)_{n=1}^{\infty}$ has a convergent subsequence. Because $\|A^n x\|/n \to 0$ and

$$\frac{1}{n}\sum_{j=1}^{n} A^j x = \frac{1}{n}\sum_{j=2}^{n+1} A^j x + \frac{1}{n}(Ax - A^{n+1}x),$$

every subsequence of the sequence $(n^{-1}\sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$ also has a convergent subsequence.

CASE 1: *A has no non-zero fixed point.* Let $(n_k)_{k=1}^{\infty}$ be any subsequence of $(n)_{n=1}^{\infty}$ and $\overline{x} \in X$ such that $n_k^{-1}\sum_{j=1}^{n_k} A^j x \to \overline{x}$ as $k \to \infty$. By Proposition 2.1(2), $A\overline{x} = \overline{x}$. Since A has no non-zero fixed point, we must have $\overline{x} = 0$. It follows that $n^{-1}\sum_{j=1}^{n} A^j x \to 0$ as $n \to \infty$.

CASE 2: *A has non-zero fixed points.*

SUBCASE 1. Suppose X is a complex Banach space. Let $\sigma_2 = \sigma(A) \setminus \{1\}$. Then there is a Riesz decomposition of $X = X_1 \oplus X_2$, where X_1 and X_2 are closed A-invariant subspaces of X, X_1 is finite-dimensional, $\sigma(A_1) = \{1\}$ and $\sigma(A_2) = \sigma_2$, where $A_j = A|_{X_j}$ for $j = 1, 2$. Clearly each A_j is compact on X_j, and the projection E_j on X_j corresponding to the decomposition satisfies $E_j A = AE_j = A_j E_j$. Let $x = x_1 + x_2$, where $x_j \in X_j$ for $j = 1, 2$. Then

$$\frac{\|A_k^n x_k\|}{n} = \frac{\|A_k^n E_k x\|}{n} = \frac{\|E_k A^n x\|}{n} \le \|E_k\| \frac{\|A^n x\|}{n} \to 0,$$
$$\left\|\frac{1}{n}\sum_{j=1}^{n} A_k^j x_k\right\| = \left\|\frac{1}{n}\sum_{j=1}^{n} E_k A^j x\right\| \le \|E_k\| \cdot \left\|\frac{1}{n}\sum_{j=1}^{n} A^j x\right\|.$$

By Case 1, we have $n^{-1}\sum_{j=1}^{n} A_2^j x_2 \to 0$ as $n \to \infty$.

We shall now show that $A_1 x_1 = x_1$, hence $n^{-1}\sum_{j=1}^{n} A_1^k x_1 = x_1$ for all $n \ge 1$. This will show that $(n^{-1}\sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$ converges to $x_1 + 0 \in X_1 + X_2$, which is a fixed point of A, thus completing the proof.

Indeed, it suffices to show that for any $m \times m$ cell K (where $m \ge 2$) in the Jordan form of A_1,

$$K = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix},$$

and any $y = [y_1, y_2, \ldots, y_m]^t \in \mathbb{C}^m$ with $\|K^n y\|/n \to 0$, we have $y_2 = y_3 = \ldots = y_m = 0$, hence $Ky = y$. To this end, for each $p \geq m$, let $K^p y = [k_1^{(p)}, k_2^{(p)}, \ldots, k_m^{(p)}]^t$. Then for each $j = 1, \ldots, m$,

$$k_j^{(p)} = y_j + \binom{p}{1} y_{j+1} + \ldots + \binom{p}{m-j} y_m.$$

Since

$$\frac{1}{p}\, k_{m-1}^{(p)} = \frac{1}{p}\, y_{m-1} + \frac{1}{p}\binom{p}{1} y_m \to 0 \quad \text{as } p \to \infty,$$

we must have $y_m = 0$. If $y_m = \ldots = y_j = 0$ for $j \geq 3$, then since

$$\begin{aligned} \frac{1}{p}\, k_{j-2}^{(p)} &= \frac{1}{p}\, y_{j-2} + \frac{1}{p}\binom{p}{1} y_{j-1} + \ldots + \frac{1}{p}\binom{p}{m-j+2} y_m \\ &= \frac{1}{p}\, y_{j-2} + \frac{1}{p}\binom{p}{1} y_{j-1} \to 0 \quad \text{as } p \to \infty, \end{aligned}$$

we must have $y_{j-1} = 0$. Thus by induction, $y_2 = y_3 = \ldots = y_m = 0$, and we are done in Subcase 1.

SUBCASE 2. Suppose $(X, \|\cdot\|)$ is a real Banach space. Let $X_{\mathbb{C}}$ be the complexification of X and let $A_{\mathbb{C}}$ be the complexification of A (see e.g. [PS] or [ERT, pp. 118–119]). Then $n^{-1}\|A_{\mathbb{C}}^n(x,0)\| = n^{-1}\|A^n x\| \to 0$ as $n \to \infty$ and the sequence $(n^{-1}\sum_{j=1}^n A_{\mathbb{C}}^j(x,0))_{n=1}^\infty = (n^{-1}\sum_{j=1}^n A^j x, 0)_{n=1}^\infty$ is bounded. By Subcase 1, the sequence $(n^{-1}\sum_{j=1}^n A_{\mathbb{C}}^j(x,0))_{n=1}^\infty$ converges to a fixed point $(\overline{x}, 0)$ of $A_{\mathbb{C}}$. It follows that $(n^{-1}\sum_{j=1}^n A^j x)_{n=1}^\infty$ converges to $\overline{x}$ which is a fixed point of A. ∎

It is clear that the conditions in Proposition 2.1(4) are satisfied if x, X, A are as given in Theorem 2.3. We note also that as briefly mentioned previously, in [E] there is given an example of a real 2×2 matrix A which, regarded as an operator on $X = \mathbb{R}^2$, satisfies $\sup_n n^{-1}\sum_{j=1}^n \|A^j\| < \infty$, but for some $x \in X$, the sequence $(n^{-1}\|A^n x\|)_{n=1}^\infty$ does not tend to 0. The following theorem is an easy but interesting consequence of Theorem 2.3; for some related results, the reader is referred to [BGM].

THEOREM 2.4. *Let $(X, \|\cdot\|)$ be a (real or complex) Banach space and $A \in \mathcal{B}(X)$ be a compact operator. Let $x \in X$ be such that a subsequence of $(A^n x)_{n=1}^\infty$ is bounded. Then $(n^{-1}\sum_{j=1}^n A^j x)_{n=1}^\infty$ converges to a fixed point of A.*

Proof. By Theorem 4 in [ERT, pp. 117–118], the whole sequence $(A^n x)_{n=1}^{\infty}$ is bounded. The desired conclusion then follows readily from our Theorem 2.3. ∎

In particular, we have the following result which is Theorem 2.1 of [TT]:

COROLLARY 2.5. *Let A be an $m \times m$ complex (respectively, real) matrix and x be an $m \times 1$ complex (respectively, real) vector. If $(A^n x)_{n=1}^{\infty}$ has a bounded subsequence, then $(n^{-1} \sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$ converges to a fixed vector of A.*

We emphasize that the compact operator A in Theorem 2.3 (respectively, in Theorem 2.4, and the $m \times m$ matrix A in Corollary 2.5) is not assumed to be mean-bounded. Indeed, we shall provide in the following a simple example of a compact operator A satisfying the conditions in Theorem 2.3, Theorem 2.4 and Corollary 2.5 respectively, but which is not mean-bounded.

EXAMPLE 2.6. Let $X = \mathbb{R}^3$ or $\mathbb{C}^3$ and

$$A = \begin{bmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{bmatrix}$$

where $|b| < 1$, $|c| = 1$, $|d| > 1$. Then A is a compact operator on X which is not power-bounded and not mean-bounded so that Theorem 1 in [YK] is not applicable. Let $x = [r, s, u]^t$. Then the sequence $\|A^n x\|/n \to 0$ as $n \to \infty$ if and only if $u = 0$, if and only if $(A^n x)_{n=1}^{\infty}$ has a bounded subsequence; moreover, in that case, the sequence $(n^{-1} \sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$ (is bounded and) converges to $\overline{x}$, where

$$\overline{x} = \begin{cases} [0, s, 0]^t & \text{if } c = 1, \\ 0 & \text{if } c \neq 1, \end{cases}$$

and $\overline{x}$ is a fixed point of A. Note that in the present example, the condition $u = 0$ is even necessary for the boundedness of the sequence $(n^{-1} \sum_{j=1}^{n} A^j x)_{n=1}^{\infty}$.

We now consider the conditions (a) A is power-bounded (i.e., $\sup_{n \ge 1} \|A^n\| < \infty$), and (b) $\|A^n\|/n \to 0$ as $n \to \infty$. In general, (b) is strictly weaker than (a) (see, e.g., [S]). However, in [MZ, Theorem 3], it is shown that for a Riesz operator A on a complex Banach space, (a) and (b) are equivalent. In Theorem 2.10 below we present a slightly more general result for a not necessarily Riesz operator. It also generalizes the result of Sz.-Nagy [N] from a compact operator on a complex Hilbert space to an operator more general than a Riesz operator on a real or complex Hilbert space. For related results for more restrictive classes of operators, we refer the reader to [Ze].

We will need (parts of) three lemmas which are of some independent interest. In the first lemma, we consider a real Banach space $(X, \|\cdot\|)$, and $A \in \mathcal{B}(X)$. Let $(X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}})$, and $A_{\mathbb{C}}$ be its complexification. If $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$ i.e., if X is a real Hilbert space, then we let $(X_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ be the (Hilbert space) complexification of X; and $\|\cdot\|_{\mathbb{C}}$ is induced by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ (see, e.g., [PS] or [ERT, pp. 118–119]).

LEMMA 2.7. *We use the above notations.*

(1) *Let $(X, \|\cdot\|)$ be a real Banach space, and let $A \in \mathcal{B}(X)$. Then A is power-bounded if and only if its complexification $A_{\mathbb{C}}$ is power-bounded. Moreover, $\|A^n\|/n \to 0$ as $n \to \infty$ if and only if $\|A_{\mathbb{C}}^n\|/n \to 0$ as $n \to \infty$.*

(2) *Let X be a real Hilbert space, and $A \in \mathcal{B}(X)$. Then A is similar to a contraction on X if and only if its complexification $A_{\mathbb{C}}$ is similar to a contraction on $X_{\mathbb{C}}$.*

Proof. (1) Since there is a positive constant d such that for every positive integer n, $\|A^n\| \le \|A_{\mathbb{C}}^n\| \le d\|A^n\|$, the assertions are obviously true.

(2) Suppose A is similar to a contraction on the real Hilbert space X, and let S be an invertible operator in $\mathcal{B}(X)$ such that $\|SAS^{-1}\| \le 1$. Let $T = S \times S$. Then $T^{-1} = S^{-1} \times S^{-1}$ in $\mathcal{B}(Y)$, and $\|TA_{\mathbb{C}}T^{-1}\| \le 1$, so $A_{\mathbb{C}}$ is similar to a contraction.

Conversely, suppose $A_{\mathbb{C}}$ is similar to a contraction on $X_{\mathbb{C}}$. We shall show that A is similar to a contraction on X. Indeed, let W be an invertible operator in $\mathcal{B}(X_{\mathbb{C}})$ such that $\|WA_{\mathbb{C}}W^{-1}\| \le 1$. By the Riesz representation theorem and spectral theorem, there exists a positive operator $P \in \mathcal{B}(X)$ such that $\langle Py, Px \rangle = \mathrm{Re}\langle W(y,0), W(x,0) \rangle_{\mathbb{C}}$; here, $\mathrm{Re}\, z$ denotes the real part of the complex number z. Then P is bijective, hence invertible in $\mathcal{B}(X)$. Now for each $x \in X$,

$$\begin{aligned}\|PAP^{-1}x\| &= \|W(AP^{-1}x, 0)\|_{\mathbb{C}} = \|WA_{\mathbb{C}}W^{-1}W(P^{-1}x, 0)\|_{\mathbb{C}} \\ &\le \|W(P^{-1}x, 0)\|_{\mathbb{C}} = \|P(P^{-1}x)\| = \|x\|;\end{aligned}$$

thus $\|PAP^{-1}\| \le 1$ and A is similar to a contraction on X. ∎

LEMMA 2.8. *Let $(X, \|\cdot\|)$ be a* (*real or complex*) *Banach space, let $A \in \mathcal{B}(X)$, and let X_j, $j = 1, 2$, be A-invariant closed subspaces of X such that $X = X_1 + X_2$. Let A_j denote the restriction of A to X_j, $j = 1, 2$. Then A is power-bounded if and only if A_j, $j = 1, 2$, are power-bounded. Moreover, $\lim_{n\to\infty} \|A^n\|/n = 0$ if and only if $\lim_{n\to\infty} \|A_j^n\|/n = 0$ for $j = 1, 2$.*

Proof. Since A_j^n is the restriction of A^n to X_j, $\|A_j^n\| \le \|A^n\|$ and the necessity of both assertions are obviously true. For the sufficiency, suppose first X is a complex Banach space. Note that by [R, Theorem 5.20, p. 130], there exists a positive constant r such that for each $x \in X$, there are $x_j \in X_j$, $j = 1, 2$, satisfying $x = x_1 + x_2$ and $\|x_1\| + \|x_2\| \le r\|x\|$. Hence for each positive integer n,

$$\|A^n x\| \le \|A_1^n x_1\| + \|A_2^n x_2\| \le (\|A_1^n\| + \|A_2^n\|)(\|x_1\| + \|x_2\|) \\ \le (\|A_1^n\| + \|A_2^n\|) r \|x\|,$$

and the sufficiency of both assertions in the complex Banach space case follows readily.

Suppose now that X is a real Banach space and each A_j is power-bounded. For notational simplicity, let $Y = X_{\mathbb{C}}$ and $B = A_{\mathbb{C}}$ be their complexifications. Define $Y_j = X_j \times X_j$ for $j = 1, 2$. Then each Y_j is a closed B-invariant subspace of Y, and $Y = X_{\mathbb{C}} = Y_1 + Y_2$. Let B_j be the restriction of B to Y_j, $j = 1, 2$. Then $B_j = A_j \times A_j = (A_j)_{\mathbb{C}}$. By Lemma 2.7, each B_j is power-bounded. By the preceding paragraph, B is power-bounded. By Lemma 2.7 again, A is power-bounded. Similarly the sufficiency of the other assertion is proved. ∎

LEMMA 2.9. *Let* $(X, \|\cdot\|)$ *be a* (*real or complex*) *Hilbert space*, $A \in \mathcal{B}(X)$, *and* X_j, $j = 1, 2$, *be* A*-invariant closed subspaces of* X *such that* $X = X_1 + X_2$. *Let* A_j *denote the restriction of* A *to* X_j, $j = 1, 2$. *Then* A *is similar to a contraction on* X *if and only if each* A_j *is similar to a contraction on* X_j, $j = 1, 2$.

Proof. (1) Suppose X is a complex Hilbert space. By Paulsen's result [P, Corollary 3.5], the lemma is equivalent to the assertion that A is completely polynomially bounded if and only if each A_j, $j = 1, 2$, is completely polynomially bounded. To show the latter assertion, note that for every square matrix $[p_{lk}]$ of (complex) polynomials (of one variable), $[p_{lk}(A_j)]$ is a restriction of $[p_{lk}(A)]$, so the necessity is clear. For the sufficiency, let c_j, $j = 1, 2$, be constants such that for every square matrix $[p_{lk}]$ of polynomials, $\|[p_{lk}(A_j)]\| \le c_j \|[p_{lk}]\|_\infty$, $j = 1, 2$.

Consider $[p_{lk}(A)]_{1\le l,k\le n}$ as an operator on the direct sum $\widetilde{X} = \sum_{k=1}^n \oplus X$ of n copies of X, and let $\widetilde{x} = [x^{(k)}] \in \widetilde{X}$ be arbitrary. As in Lemma 2.8 above, there is a positive constant r (independent of n and $\widetilde{x}$) and for each $k = 1, \ldots, n$, there are $x_j^{(k)} \in X_j$, $j = 1, 2$, satisfying $x^{(k)} = x_1^{(k)} + x_2^{(k)}$ and $\|x_1^{(k)}\| + \|x_2^{(k)}\| \le r\|x^{(k)}\|$. Hence

$$\begin{aligned} \|[x_1^{(k)}]\| + \|[x_2^{(k)}]\| &\le \Big(\sum_{k=1}^n \|x_1^{(k)}\|^2\Big)^{1/2} + \Big(\sum_{k=1}^n \|x_2^{(k)}\|^2\Big)^{1/2} \\ &\le 2^{1/2}\Big[\sum_{k=1}^n (\|x_1^{(k)}\|^2 + \|x_2^{(k)}\|^2)\Big]^{1/2} \\ &\le 2^{1/2}\Big[\sum_{k=1}^n r^2\|x^{(k)}\|^2\Big]^{1/2} \le c\|\widetilde{x}\|, \end{aligned}$$

where $c = 2^{1/2}r$. Now we have

$$\begin{aligned}\|[p_{lk}(A)]\widetilde{x}\| &= \Big\|\Big[\sum_{k=1}^{n} p_{lk}(A)x^{(k)}\Big]\Big\| = \Big\|\Big[\sum_{k=1}^{n}\sum_{j=1}^{2} p_{lk}(A_j)x_j^{(k)}\Big]\Big\| \\ &= \Big\|\sum_{j=1}^{2}\Big[\sum_{k=1}^{n} p_{lk}(A_j)x_j^{(k)}\Big]\Big\| \\ &\le \Big\|\Big[\sum_{k=1}^{n} p_{lk}(A_1)x_1^{(k)}\Big]\Big\| + \Big\|\Big[\sum_{k=1}^{n} p_{lk}(A_2)x_2^{(k)}\Big]\Big\| \\ &= \|[p_{lk}(A_1)]\ [x_1^{(k)}]\| + \|[p_{lk}(A_2)]\ [x_2^{(k)}]\| \\ &\le \|[p_{lk}(A_1)]\|\,\|[x_1^{(k)}]\| + \|[p_{lk}(A_2)]\|\,\|[x_2^{(k)}]\| \\ &\le \max(c_1,c_2)\|[p_{lk}]\|_\infty(\|[x_1^{(k)}]\| + \|[x_2^{(k)}]\|) \\ &\le c\max(c_1,c_2)\|[p_{lk}]\|_\infty\|\widetilde{x}\|,\end{aligned}$$

so that $\|[p_{lk}(A)]\| \le c\,\max(c_1,c_2)\|[p_{lk}]\|_\infty$. Thus A is completely polynomially bounded. So the lemma is proved in the complex Hilbert space case.

(2) Suppose X is a real Hilbert space. Then as in Lemma 2.8, we consider the complexifications. With the notation therein and by Lemma 2.7, A (respectively A_j) is similar to a contraction if and only if so is B (respectively B_j). By (1) above, the desired conclusion follows readily. ■

THEOREM 2.10. *Let $(X, \|\cdot\|)$ be a (real or complex) Banach space and let $A \in \mathcal{B}(X)$. Let X_1, X_2 be closed A-invariant subspaces of X such that X_1 is finite-dimensional, $X = X_1 + X_2$, and the spectral radius $r_\sigma(A_2) = \lim_{n\to\infty}\|A_2^n\|^{1/n}$ is less than 1, where A_j denotes the restriction of A to X_j. Suppose $\|A^n\|/n \to 0$ as $n \to \infty$. Then A is power-bounded. If X is a Hilbert space, then A is similar to a contraction on X.*

Proof. (1) Suppose X is a complex Banach space. By Lemma 2.8, $\lim_{n\to\infty}\|A_j^n\|/n = 0$ for $j = 1, 2$. Since A_1 is compact, A_1 is power-bounded by [MZ, Theorem 3]. Since $r_\sigma(A_2) < 1$ and $\|A_2^n\| \to 0$ as $n \to \infty$, A_2 is power-bounded. By Lemma 2.8, A is power-bounded.

(2) Let X be a real Banach space. As in Lemma 2.8, we consider the complexifications. Using the notations therein and by Lemma 2.7, each Y_j is a B-invariant closed subspace of Y, Y_1 is finite-dimensional, $Y = Y_1 + Y_2$, $\lim_{n\to\infty}\|B^n\|/n = 0$, and $r_\sigma(B_2) = \lim_{n\to\infty}\|B_2^n\|^{1/n} = \lim_{n\to\infty}\|A_2^n\|^{1/n} < 1$. By (1) above, B is power-bounded. By Lemma 2.7, A is power-bounded.

(3) Let X be a complex Hilbert space. Since $r_\sigma(A_2) < 1$, by Rota's result [RO], A_2 is similar to a (proper) contraction on X_2. On the other hand, since $\|A_1^n\|/n \to 0$ as $n \to \infty$, $r_\sigma(A_1) \le 1$. If $r_\sigma(A_1) < 1$, then again by Rota's result [RO], A_1 is similar to a (proper) contraction on X_1. If $r_\sigma(A_1) = 1$,

then the condition $\lim_{n\to\infty} \|A_1^n\|/n = 0$ together with Jordan canonical form (since X_1 is finite-dimensional) implies that A_1 is diagonalizable, and A_1 is similar to a contraction on X_1. Therefore each A_j is similar to a contraction on X_j for $j = 1, 2$. By Lemma 2.9, A is similar to a contraction on X.

(4) Finally, let X be a real Hilbert space. As in (2) above, we have $Y = X_{\mathbb{C}} = Y_1 + Y_2$ (all complex Hilbert spaces) with Y_1 finite-dimensional, $B = A_{\mathbb{C}} = B_1 + B_2$, $r_\sigma(B_2) < 1$. Thus by (3) above, $A_{\mathbb{C}}$ is similar to a contraction on $X_{\mathbb{C}}$. By Lemma 2.9, A is similar to a contraction on X. ∎

We note that in Lemma 2.8, Lemma 2.9, and Theorem 2.10, the sum $X = X_1 + X_2$ need not be a direct sum; in particular, when X is a Hilbert space, the sum $X = X_1 + X_2$ need not be an orthogonal (or direct) sum.

Theorem 2.10 implies readily the following result in which the case of a compact operator on a complex Hilbert space was proved by Sz.-Nagy in [N]:

COROLLARY 2.11. *Let A be a power-bounded compact operator* (*respectively*, *a Riesz operator*) *on a real or complex* (*respectively*, *complex*) *Hilbert space H. Then A is similar to a contraction on H.*

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