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Almost everywhere convergence of the inverse Jacobi transform and endpoint results for a disc multiplier

by

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Abstract. The maximal operator S_* for the spherical summation operator (or *disc multiplier*) S_R associated with the Jacobi transform through the defining relation $\widehat{S_R f}(\lambda) = 1_{\{|\lambda|\le R\}}\widehat{f}(t)$ for a function f on $\mathbb{R}$ is shown to be bounded from $L^p(\mathbb{R}_+, d\mu)$ into $L^p(\mathbb{R}, d\mu) + L^2(\mathbb{R}, d\mu)$ for $\frac{4\alpha+4}{2\alpha+3} < p \le 2$. Moreover S_* is bounded from $L^{p_0,1}(\mathbb{R}_+, d\mu)$ into $L^{p_0,\infty}(\mathbb{R}, d\mu) + L^2(\mathbb{R}, d\mu)$. In particular $\{S_R f(t)\}_{R>0}$ converges almost everywhere towards f, for $f \in L^p(\mathbb{R}_+, d\mu)$, whenever $\frac{4\alpha+4}{2\alpha+3} < p \le 2$.

1. Introduction. The importance of the disc multiplier in Euclidean harmonic analysis—defined as the operator S_R satisfying the relation $\widehat{S_R f}(\xi) = 1_{\|\xi\|\le R}\widehat{f}(\xi)$—was firmly established by Fefferman's groundbreaking result in [7] that S_R is not bounded on $L^p(\mathbb{R}^n)$, $n \ge 2$, unless $p = 2$. The operator has since then played a role in other areas of mathematics. It usually appears whenever one studies convergence properties of eigenfunction expansions for differential operators on manifolds, and it also appears as an extreme endpoint case of Bochner–Riesz means. An interesting aspect, however, is that the operator behaves much better when restricted to radial L^p-functions. Indeed, according to [11], the operator is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ for $2n/(n+1) < p < 2n/(n-1)$. This result has later been improved in several directions, and we shall recall them one by one in the main text.

A natural analogue of the disc multiplier in the framework of spherical analysis on Riemannian symmetric spaces of rank one was introduced by Meaney and Prestini in the mid-90's and the study was completed in the paper [18] with almost sharp statements about the mapping properties of the maximal operator associated with the disc multiplier. In the present paper we follow in their footsteps and generalize their results to Jacobi analysis, and we establish the missing endpoint results in the setting of

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Jacobi analysis. In particular we complement the paper [1]. This implies almost everywhere convergence of $\{S_R f(x)\}_{R>0}$ for $f \in L^p(d\mu)$ for a certain range of p, most directly related to [23] in the Euclidean case, whereas the extension to Hankel transforms was considered in [4].

There are other ways to obtain almost everywhere convergence of $\{S_R f(x)\}_{R>1}$. In [2], the authors obtain equiconvergence results for $\{S_R f\}$ in the slightly more general framework of (noncompact) Chébli–Trimèche hypergroups. The results of the present paper should generalize to their setting without much effort. Our endpoint results are stronger, however, as we are able to determine the endpoint behavior of the maximal operator at the level of Lorentz spaces. Moreover, and this is a fundamental advantage of working with maximal operators, we will use the results of the present paper as part of a complex interpolation argument in a companion paper to obtain convergence results for Bochner–Riesz means in Jacobi analysis below the critical order of integrability. In order for this to work we need norm estimates in the first place.

Finally we wish to point out that a "flat" version of our results on the disc multipliers was recently obtained in [5]. By "flat" we refer to the modern habit of regarding Dunkl analysis on $\mathbb{R}$ as a "zero curvature limit" of harmonic analysis in rank one root systems, in the sense of Cherednik, Heckman and Opdam. The proofs of [5] are more or less straightforward adaptations of techniques from [23] and [21], since the size of balls, measured in terms of the relevant measures in Dunkl theory, does not grow exponentially fast, in contrast to what happens for noncompact Riemannian symmetric spaces. It is well-understood that the "curved" situation—be it analysis on symmetric spaces or slightly more generally, Jacobi analysis—is complicated by balls having exponential volume growth.

We employ the same techniques as in [18], carried out in the more general setting of Jacobi analysis. Most proofs are therefore structurally identical to those in [18], which we wish to acknowledge at this point. There are several technical difficulties, however, like the precise asymptotic expansion for the **c**-function in Lemma 2.1. Also of importance is that we are able to incorporate the paper [22] by Prestini. The careful analysis, in turn, allows us to establish new endpoint results, thereby showing to exactly what extent one can generalize the spherical analysis on symmetric spaces of rank one. Since we never use the actual formula for the measure $d\mu(t)$, but rather just its behavior for $t \sim 1$ and $t \gg 1$, and since the key ingredients for the proofs—asymptotic estimates for φ_λ and the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$—are also available for Chébli–Trimèche hypergroups (see Theorem 1.2, Section 1.3, Theorems 2.1 and 2.2 in [2]) the exact same calculations can be carried out in the context of such hypergroups.

2. Jacobi analysis. Let $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$. The *hypergeometric function* ${}_2F_1(a,b;c,z)$ is defined by

$$ {}_2F_1(a,b;c,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \qquad |z| < 1; $$

the function $z \mapsto {}_2F_1(a,b;c,z)$ is the unique solution of the differential equation

$$ z(1-z)u''(z) + (c-(a+b+1)z)u'(z) - abu(z) = 0 $$

which is regular at 0 and equals 1 there. The *Jacobi functions* for parameters (α,β) are defined by

$$ \varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\big(\tfrac{1}{2}(\alpha+\beta+1-i\lambda), \tfrac{1}{2}(\alpha+\beta+1+i\lambda); \alpha+1, -\sinh^2 t\big). $$

It is thereby clear that $\lambda \mapsto \varphi_\lambda(t)$ is analytic for all $t \geq 0$. Moreover, for $\Im\lambda \geq 0$, there exists a unique solution ϕ_λ to the same equation satisfying $\phi_\lambda(t) = e^{(i\lambda-\rho)t}(1+o(1))$ as $t \to \infty$, and $\lambda \mapsto \phi_\lambda(t)$ is therefore also analytic for $t \geq 0$.

In what follows we assume that $\alpha \neq -1,-2,\ldots$, $\alpha > \beta > -1/2$, and $|\beta| < \alpha+1$. Let $\rho = \alpha+\beta+1$. The usual Lebesgue space on $\mathbb{R}_+$ will simply be denoted L^p, whereas by $L^p(d\mu)$ we understand the weighted Lebesgue space, with $d\mu(t) = d\mu_{\alpha,\beta}(t) = \Delta(t)\,dt$, where

$$ \Delta(t) = \Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}, \qquad t > 0. $$

We adopt the notational convention of writing $\mu(A)$ for the weighted measure of a measurable subset A of $\mathbb{R}$, that is, $\mu(A) = \|1_A\|_{L^1(d\mu)}$. It is of paramount importance to stress that the behavior of $\Delta(t)$ depends on the "size" of t. More precisely,

$$ |\Delta(t)| \leq \begin{cases} t^{2\alpha+1} & \text{for } t \lesssim 1, \\ e^{2\rho t} & \text{for } t \gg 1. \end{cases} $$

In analogy with the case of symmetric spaces, one proceeds to show the existence of a function $\mathbf{c} = \mathbf{c}_{\alpha,\beta}$ for which $\varphi_\lambda(t) = \mathbf{c}(\lambda)e^{(i\lambda-\rho)t}\phi_\lambda(t) + \mathbf{c}(-\lambda)e^{(-i\lambda-\rho)t}\phi_{-\lambda}(t)$. Since we adhere to the conventions and normalization used in [9], the $\mathbf{c}$-function is given by

$$ \mathbf{c}(\lambda) = \frac{2^\rho \Gamma(i\lambda)\Gamma\big(\frac{1}{2}(1+i\lambda)\big)}{\Gamma\big(\frac{1}{2}(\rho+i\lambda)\big)\Gamma\big(\frac{1}{2}(\rho+i\lambda)-\beta\big)}. $$

Observe that for $\alpha,\beta \neq -1,-2,\ldots$, $\mathbf{c}(-\lambda)^{-1}$ has finitely many poles for $\Im\lambda < 0$ and none if $\Im\lambda \geq 0$ and $\Re\rho > 0$. It follows from Stirling's formula that for every $r > 0$ there exists a positive constant c_r such that

(2.1) $\quad |\mathbf{c}(-\lambda)|^{-1} \leq c_r(1+|\lambda|)^{\alpha+1/2}$ if $\Im\lambda \geq 0$ and $\mathbf{c}(-\lambda') \neq 0$ for $|\lambda'-\lambda| \leq r$.

The following statement on the precise asymptotic expansion of the density $|\mathbf{c}(\lambda)|^{-2}$ will play an important role later in the paper. We have included a detailed proof since the result cannot be deduced directly from [25] or [18]; α and β need not correspond to integer-valued root multiplicities, so the expression for $\mathbf{c}(\lambda)$ does not really simplify, unlike for rank one symmetric spaces.

LEMMA 2.1. *Assume $\alpha > \beta > -1/2$.*

(i) *For every integer M there exist constants c_i, $i = 0, \ldots, M-1$, (depending on α, β, and M) such that*

$$|\mathbf{c}(\lambda)|^{-2} \sim c_0 |\lambda|^{2\alpha+1} \Big\{ 1 + \sum_{j=1}^{M-1} c_j \lambda^{-j} + O(\lambda^{-M}) \Big\} \qquad \text{as } |\lambda| \to \infty.$$

(ii) *Let $\mathbf{d}(\lambda) = |\mathbf{c}(\lambda)|^{-2}$, $\lambda \geq 0$, and $k \in \mathbb{N}_0$. There exists a constant $c_k = c_{k,\alpha,\beta}$ such that*

$$\Big| \frac{d^k}{d\lambda^k} \mathbf{d}(\lambda) \Big| \leq c_k (1 + |\lambda|)^{2\alpha+1-k}.$$

(iii) $\mathbf{c}'(\lambda) \sim \mathbf{c}(\lambda) O(\lambda^{-1})$ *and* $\mathbf{c}''(\lambda) \sim \mathbf{c}(\lambda) O(\lambda^{-2})$.

This improves on the usual asymptotic statement that $|\mathbf{c}(\lambda)|^{-2} \sim |\lambda|^{2\alpha+1}$ as $|\lambda| \to \infty$ and we will need this improvement at a later stage. This was already observed in [18].

Proof. (i) Following the technique in [20, Subsection 2.2.1] we introduce the auxiliary function

$$Q(\lambda) = \Big(\prod_{r=1}^{q} \Gamma(1 - b_r + \beta_r \lambda) \Big) \Big(\prod_{r=1}^{p} \Gamma(1 - a_r + \alpha_r \lambda) \Big)^{-1},$$

where we of course have in mind the particular parameters

$$\text{(2.2)} \qquad \begin{cases} p = 2,\ q = 4, \quad b_1 = b_2 = 1 - \rho/2, \quad b_3 = b_4 = 1 + \beta - \rho/2, \\ a_1 = a_2 = 1, \quad \beta_1 = \beta_2 = \beta_3 = \beta_4 = i/2, \quad \alpha_1 = \alpha_2 = i, \end{cases}$$

so that $|Q(\lambda)| = |\mathbf{c}(\lambda)|^{-2}$ by the duplication formula for the Γ-function. Recall that by Stirling's formula,

$$\log \Gamma(z) = \big(z - \tfrac{1}{2}\big) \log z - z + \tfrac{1}{2} \log(2\pi) + \Omega(z),$$

where

$$\Omega(z) \sim \sum_{r=1}^{\infty} \frac{B_{2r}}{2r(2r-1) z^{2r-1}}$$

for suitable numbers B_{2n} (the Bernoulli numbers). Moreover,

$$\Omega(z) = \sum_{r=1}^{n-1} \frac{B_{2r}}{2r(2r-1)z^{2r-1}} + R_n(z)$$

for every positive integer n, where—upon writing $z = xe^{i\theta}$—the remainder term $R_n(z)$ may be estimated according to

$$|R_n(z)| \le \frac{|B_{2n}|}{2n(2n-1)} \frac{\left(\sec\frac{\theta}{2}\right)^{2n}}{|z|^{2n-1}} \quad \text{for } |\arg z| < \pi \tag{2.3}$$

(see [20, equation (2.1.6)]). Presently z will be of the form $z = \alpha_r\lambda + 1 - a_r$ with $\alpha_r \ge 0, a_r \in \mathbb{C}$, and $\lambda \in \mathbb{R}_+$, so that $\arg z$ remains constant as $\lambda \to \infty$.

For fixed $M \in \mathbb{N}$ and large $|\lambda|$ we thus have

$$\begin{aligned}
\log Q(\lambda) &= \sum_{r=1}^{q} \log \Gamma(1 - b_r + \beta_r\lambda) - \sum_{r=1}^{p} \log \Gamma(1 - a_r + \alpha_r\lambda) \\
&= \sum_{r=1}^{q} \big\{ \big(\tfrac12 - b_r + \beta_r\lambda\big) \log(1 - b_r + \beta_r\lambda) - (1 - b_r + \beta_r\lambda) \\
&\quad + \tfrac12 \log(2\pi) + \Omega(1 - b_r + \beta_r\lambda)\big\} \\
&\quad - \sum_{r=1}^{p} \big\{ \big(\tfrac12 - a_r + \alpha_r\lambda\big) \log(1 - a_r + \alpha_r\lambda) - (1 - a_r + \alpha_r\lambda) \\
&\quad + \tfrac12 \log(2\pi) + \Omega(1 - a_r + \alpha_r\lambda)\big\} \\
&\lessapprox \sum_{r=1}^{q} \big(\tfrac12 - b_r + \beta_r\lambda\big) \log(\beta_r\lambda) - \sum_{r=1}^{p} \big(\tfrac12 - a_r + \alpha_r\lambda\big) \log(\alpha_r\lambda) \\
&\quad + \tfrac12 (q-p)(\log(2\pi) - 2) - s\kappa - \theta \\
&\quad + \sum_{r=1}^{q} \Omega(1 - b_r + \beta_r\lambda) - \sum_{r=1}^{p} \Omega(1 - a_r + \alpha_r\lambda) \\
&\sim \sum_{r=1}^{q} \big(\tfrac12 - b_r + \beta_r\lambda\big)(\log\beta_r + \log\lambda) \\
&\quad - \sum_{r=1}^{p} \big(\tfrac12 - a_r + \alpha_r\lambda\big)(\log\alpha_r + \log\lambda) \\
&\quad + \tfrac12 (q-p)(\log(2\pi) - 2) - s\kappa - \theta + \sum_{j=1}^{M-1} c_j\lambda^{-j} + O(\lambda^{-M}) \\
&= \big[\tfrac12 (q-p) + \theta + \lambda\kappa\big] \log\lambda - \lambda(\log h + \kappa) \\
&\quad + \log \widetilde{c}_0 - \theta + \tfrac12 (q-p)(\log(2\pi) - 2) + \sum_{j=1}^{M-1} c_j\lambda^{-j} + O(\lambda^{-M})
\end{aligned}$$

where

$$h := \Big(\prod_{r=1}^{p} \alpha_r^{\alpha_r}\Big)\Big(\prod_{r=1}^{q} \beta_r^{-\beta_r}\Big), \quad \widetilde{c}_0 := \Big(\prod_{r=1}^{q} \beta_r^{1/2-b_r}\Big)\Big(\prod_{r=1}^{p} \alpha_r^{a_r-1/2}\Big),$$
$$\theta := \sum_{r=1}^{p} a_r - \sum_{r=1}^{q} b_r, \qquad \kappa := \sum_{r=1}^{q} \beta_r - \sum_{r=1}^{q} \alpha_r.$$

With the parameters defined as in (2.2), one sees that $\kappa = 0$, $\theta = 1 + 1 - (1 - \rho/2) \cdot 2 - (1 - \rho/2 + \beta) \cdot 2 = 2\alpha$, and $(q - p)/2 = 1$, whence

$$Q(\lambda) \sim c_0 \lambda^{(q-p)/2+\theta+s\kappa} e^{-s\kappa}\Big\{1 + \sum_{j=1}^{M-1} c_j \lambda^{-j} + O(\lambda^{-M})\Big\}$$
$$= c_0 \lambda^{2\alpha+1}\Big\{1 + \sum_{j=1}^{M-1} c_j \lambda^{-j} + O(\lambda^{-M})\Big\} \quad \text{as } |\lambda| \to \infty.$$

(ii)&(iii) We have $\mathbf{d}'(\lambda) = -2\mathbf{d}(\lambda)\mathbf{c}'(\lambda)/\mathbf{c}(\lambda)$, so it suffices to show that $\mathbf{c}'(\lambda)/\mathbf{c}(\lambda) \simeq O(1/\lambda)$. This may be seen as in the proof of [19, Lemma 8] as follows: Since

$$\frac{\mathbf{c}'(\lambda)}{\mathbf{c}(\lambda)} = ic_{\alpha,\beta}\Big\{\psi(i\lambda) - \psi(\alpha - \beta + i\lambda) + \frac{1}{2}\psi\Big(\frac{\alpha - \beta + i\lambda}{2}\Big) - \frac{1}{2}\psi\Big(\frac{\rho + i\lambda}{2}\Big)\Big\},$$

where

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)},$$

γ being the Euler constant, it follows that

$$\frac{\mathbf{c}'(\lambda)}{\mathbf{c}(\lambda)} = c_{\alpha,\beta}\Big\{-\frac{1}{i\lambda} + \frac{1}{\rho + i\lambda} + \sum_{n=1}^{\infty} \frac{z_1 - z_3}{(z_1 + n)(z_3 + n)} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{z_2 - z_4}{(z_2 + n)(z_4 + n)}\Big\}$$

with $z_1 = i\lambda$, $z_2 = \frac{1}{2}(\alpha - \beta + i\lambda)$, $z_3 = \alpha - \beta + i\lambda$, and $z_4 = \frac{1}{2}(\rho + i\lambda)$. Observe that

$$\Big|-\frac{1}{i\lambda} + \frac{1}{\rho + i\lambda}\Big| = \Big|\frac{-\rho}{i\lambda(\rho + i\lambda)}\Big| \le \frac{1}{|\lambda|}\,\frac{|\rho|^2 + |\rho|\,|\lambda|}{|\rho|^2 + |\lambda|^2} \le \frac{c}{|\lambda|}.$$

The assertion for $k = 1$ now follows from the estimate

$$\frac{|\alpha - \beta|}{2}\sum_{n=1}^{\infty} \frac{1}{|z_1 + n|\,|z_3 + n|} + \frac{|\beta + 1/2|}{2}\sum_{n=1}^{\infty} \frac{1}{|z_2 + n|\,|z_4 + n|}$$
$$\le c_{\alpha,\beta}\int_1^{\infty} \frac{1}{x^2 + \lambda^2}\,dx \le \frac{c_{\alpha,\beta}}{|\lambda|}.$$

The required estimate for $\mathbf{d}''(\lambda)$ is obtained analogously: First observe that

$$\mathbf{d}''(\lambda) = -2\mathbf{d}'(\lambda)\frac{\mathbf{c}'(\lambda)}{\mathbf{c}(\lambda)} - 2\mathbf{d}(\lambda)\left(\frac{\mathbf{c}''(\lambda)}{\mathbf{c}(\lambda)} + \left(\frac{\mathbf{c}'(\lambda)}{\mathbf{c}(\lambda)}\right)^2\right).$$

In order to establish the assertion for $k=2$ it suffices to prove that $\mathbf{c}''(\lambda)/\mathbf{c}(\lambda) \simeq O(1/\lambda^2)$. This can also be established as in the proof of [19, Lemma 8]; indeed,

$$\frac{\mathbf{c}''(\lambda)}{\mathbf{c}(\lambda)} = c_{\alpha,\beta}\big(\psi'(z_1) - \psi'(z_2) + \tfrac{1}{4}\psi'(z_3) - \tfrac{1}{4}\psi'(z_4)\big) + c_{\alpha,\beta}\frac{\mathbf{c}'(\lambda)}{\mathbf{c}(\lambda)}\big(\psi(z_1) - \psi(z_2) + \tfrac{1}{2}\psi(z_3) - \tfrac{1}{2}\psi(z_4)\big)$$

where $\psi'(z) = z^{-2} + \sum_{n=1}^{\infty}(z+n)^{-2}$, evaluated at the four points z_i. Heuristically, it is now easy to prove that the left hand side is $O(\lambda^{-2})$. The second half of the right hand side behaves as $\frac{1}{\lambda}\frac{1}{\lambda}$ according to what we have already established in the case of $k = 1$, so it suffices to show that

$$\left|\sum_{n=1}^{\infty}\frac{1}{(z_i + n)^2} - \frac{1}{(z_{i+2} + n)^2}\right| \le \frac{c}{|\lambda|^2} \qquad \text{for } i = 1, 2.$$

If, say, $i = 1$, the required estimate follows like this:

$$\left|\sum_{n=1}^{\infty}\frac{1}{(z_1 + n)^2} - \frac{1}{(z_3 + n)^2}\right| \le \sum_{n=1}^{\infty}\left|\frac{(\alpha-\beta)(\alpha-\beta+2i\lambda+2n)}{(i\lambda+n)^2(\alpha-\beta+i\lambda+n)^2}\right| \le c_{\alpha,\beta}\int_1^{\infty}\frac{1}{x^3+\lambda^3}\,dx \le \frac{c'_{\alpha,\beta}}{|\lambda|^2}.$$

One proves by induction that $\mathbf{c}^{(k)}(\lambda)/\mathbf{c}(\lambda) = O(\lambda^{-k})$ for $k = 0, 1, \ldots$, and one would then formally have to carry out another proof by induction that the estimate for $\mathbf{d}^{(k)}(\lambda)$ has the right order in $|\lambda|$. We leave the tedious details to the energetic reader. ∎

REMARK 2.2. In principle one should be able to obtain the asymptotic expansion for $|\mathbf{c}(\lambda)|^{-2}$ from the expansion of $|\mathbf{c}(\lambda)|^2$ in [2, Theorem 2.2] by long division of the asymptotic series. Such computations are indeed justified (cf. [6, Section 1.5]). We have opted for a self-contained proof, however, that is inspired by [20, Section 2.2]. We found it worthwhile to use the explicit formula for the $\mathbf{c}$-function since we still need similar estimates for various derivatives of $\mathbf{c}(\lambda)$ and $|\mathbf{c}(\lambda)|^{\pm 2}$. The asymptotic expansion for $|\mathbf{c}(\lambda)|^{-2}$ will therefore be more explicit than what could be obtained from [2].

EXAMPLE 2.3 (Specialization to rank one symmetric spaces). For special values of α and β, determined by the root system of a rank one Riemannian symmetric space, the functions φ_λ are the usual spherical functions of

Harish-Chandra, and the Jacobi transform is the spherical transform. To be more precise assume G/K is a rank one Riemannian symmetric space of noncompact type, with positive roots α and 2α. Furthermore let p denote the multiplicity of α, and q the multiplicity of 2α (we allow q to be zero). With $\alpha := \frac{1}{2}(p+q-1)$ and $\beta := \frac{1}{2}(q-1)$ both real, and $p = 2(\alpha-\beta)$ and $q = 2\beta+1$, the function $\varphi_\lambda^{(\alpha,\beta)}$ is precisely the usual elementary spherical function φ_λ as considered by Harish-Chandra, $\rho = \alpha+\beta+1 = \frac{1}{2}(p+2q)$ as it should be, and $\dim(G/K) = p+q+1 = 2(\alpha+1)$. According to Lemma 2.1 we write $|\mathbf{c}(\lambda)|^{-2} = P(\lambda) + E(\lambda)$, where

$$|E(\lambda)| = |P(\lambda)| \cdot \begin{cases} 0 & \text{whenever } q = 0,\ p = 2k, \\ |1-\coth(\pi\lambda/2)| & \text{whenever } q = 2l+1,\ p = 4k+2, \\ |1-\tanh(\pi\lambda/2)| & \text{otherwise} \end{cases}$$

(cf. the proof of Lemma 4.2 in [25]). Since $|\mathbf{c}(\lambda)|^{-2} \sim \lambda^{n-1}$ as $\lambda \to \infty$, we can at least say that $\deg P(\lambda) = n-1$.

A similar choice of parameters α, β reveals that even spherical analysis on Damek–Ricci spaces is subsumed under the present setup. This was already exploited in [1] in order to extend results from spherical analysis on rank one symmetric spaces to the framework of Damek–Ricci spaces.

Let $d\nu(\lambda) = d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-1/2}|\mathbf{c}(\lambda)|^{-2}\,d\lambda$ and denote by $L^p(d\nu)$ the associated weighted Lebesgue space on $\mathbb{R}_+$; note that $\mathbf{c}(\lambda)\mathbf{c}(-\lambda) = \mathbf{c}(\lambda)\overline{\mathbf{c}(\lambda)} = |\mathbf{c}(\lambda)|^2$ whenever $\alpha, \beta, \lambda \in \mathbb{R}$. The Jacobi transform, initially defined for, say, a function $f \in C_c^\infty(\mathbb{R}_+)$ by

$$\widehat{f}(\lambda) = \frac{\sqrt{\pi}}{\Gamma(\alpha+1)} \int_0^\infty f(t)\varphi_\lambda(t)\,d\mu(t)$$

extends to a unitary isomorphism from $L^2(d\mu)$ onto $L^2(d\nu)$, and the inversion formula is the statement that

$$f(t) = \int_0^\infty \widehat{f}(\lambda)\varphi_\lambda(t)\,d\nu(\lambda)$$

holds in the L^2-sense (cf. [16, formula (4.5)]. The limiting case $\alpha = \beta = -1/2$ is the Fourier-cosine transform, which we will not study. One easily verifies that $\widehat{\mathcal{L}f}(\lambda) = -(\lambda^2+\rho^2)\widehat{f}(\lambda)$.

3. The disc multiplier: statement of results. Our starting point in defining the disc multiplier is the inversion formula for the Jacobi transform, that is,

$$f(t) = \int_0^\infty \widehat{f}(\lambda)\varphi_\lambda(t)\,d\nu(\lambda),$$

where $d\nu(\lambda) = |\mathbf{c}(\lambda)|^{-2}\, d\lambda$. Let $S_R f(t) = \int_0^R \widehat{f}(\lambda)\varphi_\lambda(t)\, d\nu(\lambda)$ and notice that for well-behaved functions f (say, in $C_c^\infty(\mathbb{R}_+)$), $S_R f$ may be written as an integral operator

$$S_R f(t) = \int_0^R \Big\{ \int_0^\infty f(r)\varphi_\lambda(r)\, d\mu(r) \Big\} \varphi_\lambda(t)\, d\nu(\lambda) = \int_0^\infty K_R(t,r) f(r)\, d\mu(r)$$

where $K_R(t,r) = \int_0^R \varphi_\lambda(t)\varphi_\lambda(r)\, d\nu(\lambda)$. The goal of the present paper is to investigate the mapping properties of the associated maximal operator $S_* : f \mapsto S_* f$,

$$S_* f(t) = \sup_{R>0} |S_R f(t)|$$

in order to establish almost everywhere convergence, $S_R f(t) \to f(t)$, for f in $L^p(d\mu)$, for a nontrivial range of p. The investigation follows [18] very closely, but several complications of a purely technical nature (the Jacobi parameters α, β not being integers, for example) will make the presentation lengthier. The philosophy is simple, however; since the functions φ_λ behave locally as a Euclidean eigenfunction (meaning a Bessel function since we always have the spherical analysis in mind), we should analyse the kernel K_R in different regions of the (t,r)-domain $\mathbb{R}_+ \times \mathbb{R}_+$ to probe similarities with as well as deviations from a purely Euclidean harmonic analysis. This will imply a decomposition of $S_R f$ as the sum $S_R f(t) = \sum_{i=1}^4 S_{i,R} f(t)$, where $S_{i,R} f(t) = \int_0^\infty K_{i,R}(t,r) f(r)\, d\mu(r)$ and $K_{i,R}(t,r) = 1_{A_i}(t,r) K_R(t,r)$, $i = 1, \ldots, 4$, where

$$A_1 = \{(t,r) : 0 \le t, r \le R_0\},$$
$$A_2 = \{(t,r) : t, r \gg R_0\},$$
$$A_3 = \{(t,r) : t \gg 1,\ 0 \le r \le R_0\},$$
$$A_4 = \{(t,r) : 0 \le t \le R_0,\ r \gg R_0\}.$$

To be more precise, the constant R_0 will be chosen as in the technical lemma below (the proof of which can be found in [25] for rank one symmetric spaces and more generally for Jacobi functions in [12]). Here $J_\mu(z)$ is the usual Bessel function of order μ and $\mathcal{J}_\mu(z)$ is the modified Bessel function defined by $\mathcal{J}_\mu(z) = 2^{\mu-1}\Gamma(1/2)\Gamma(\mu+1/2) z^{-\mu} J_\mu(z)$.

LEMMA 3.1. *Assume $\Re\alpha > 1/2$, $\Re\alpha > \Re\beta > -1/2$, and λ belongs either to a compact subset of $\mathbb{C} \setminus (-i\mathbb{N})$ or a set of the form*

$$D_{\varepsilon,\gamma} = \{\lambda \in \mathbb{C} : \gamma \ge \Im\lambda \ge -\varepsilon|\Re\lambda|\}$$

for some $\varepsilon, \gamma \ge 0$. There exist constants $R_0, R_1 \in (1, \sqrt{\pi/2})$ with $R_0^2 < R_1$ such that for every $M \in \mathbb{N}$ and every $t \in [0, R_0]$,

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\Gamma(1/2)} \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \sum_{m=0}^\infty a_m(t) t^{2m} \mathcal{J}_{m+\alpha}(\lambda t)$$

$$= \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)\Gamma(1/2)} \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \sum_{m=0}^{M} a_m(t) t^{2m} \mathcal{J}_{m+\alpha}(\lambda t) + E_{M+1}(\lambda t),$$

where $a_0(t) \equiv 1$ *and* $|a_m(t)| \le c_\alpha(t) R_1^{-(\Re\alpha+m-1/2)}$ *for all* $m \in \mathbb{N}$. *Additionally, the error term* E_{M+1} *is bounded as follows:*

$$|E_{M+1}(\lambda t)| \le \begin{cases} c_M t^{2(M+1)} & \text{if } |\lambda t| \le 1, \\ c_M t^{2(M+1)} |\lambda t|^{-(\Re\alpha+M+1)} & \text{if } |\lambda t| > 1. \end{cases}$$

In the following four subsections we will establish the mapping properties of the associated four maximal operators S_{i*} individually, and the main theorem will then follow by noting that $|S_* f(t)| \le \sum_{i=1}^4 |S_{i,*} f(t)|$. The investigation in [18] and its outcome may be summarized roughly as follows: For $f \in L^s(K\backslash G/K)$, we split the maximal operator S_* associated to the "disc multiplier" as

$$S_* f = S_{1,*} f + S_{2,*} f + S_{3,*} f + S_{4,*} f,$$

where

- $S_{1,*} f$ is bounded on $L^s(K\backslash G/K)$ for $2n/(n+1) < s < 2n/(n-1)$ (this is the "Herz range"),
- $S_{2,*} f$ is bounded into $L^2(G) + L^s(G)$ for $1 < s \le 2$,
- $S_{3,*} f$ is bounded into $L^2(G)$ for $1 < s \le 2$,
- $S_{4,*} f$ is bounded into $L^2(G)$ for $2n/(n+1) < s$.

It thus follows—and this is the main result of [18]—that $S_* f$ belongs to $L^2(G) + L^s(G)$ for $2n/(n+1) < s \le 2$ (since $2 \le 2n/(n-1)$ for all $n \in \mathbb{N}$). Our first result is a generalization thereof to the setting of Jacobi analysis. For the remainder of the paper we set

$$p_0 = \frac{4\alpha+4}{2\alpha+3} \quad \text{and} \quad p_1 = \frac{4\alpha+4}{2\alpha+1}.$$

THEOREM 3.2. *Assume* $\alpha > \beta > -1/2$. *Then*

(i) $S_{1,*}$ *is bounded on* $L^p(\mathbb{R}_+, d\mu)$ *for* $p \in (p_0, p_1)$;
(ii) $S_{2,*}$ *is bounded from* $L^p(\mathbb{R}_+, d\mu)$ *into* $L^p(\mathbb{R}, d\mu) + L^2(\mathbb{R}, d\mu)$ *for all* $p \in (1, 2]$;
(iii) $S_{3,*}$ *is bounded on* $L^p(\mathbb{R}_+, d\mu)$ *for all* $p \in (1, 2]$;
(iv) $S_{4,*}$ *is bounded from* $L^p(\mathbb{R}_+, d\mu)$ *into* $L^2(\mathbb{R}, d\mu)$ *for all* $p \in (p_0, \infty)$.

Hence S_* *is bounded from* $L^p(\mathbb{R}_+, d\mu)$ *into* $L^p(\mathbb{R}, d\mu) + L^2(\mathbb{R}, d\mu)$ *for all* $p \in (p_0, 2]$.

THEOREM 3.3. *There exists a compactly supported function* f *in* $L^{p_0}(d\mu)$ *with the property that* $\{S_R f(x)\}_{R>1}$ *diverges for almost every* $x \in \mathbb{R}_+$.

The part most closely resembling the Euclidean counterpart of the disc multiplier is $S_{1*} f$, where the kernel $K_R(t, r)$ is localized in both t and r.

The remaining three pieces of S_*f all derive their mapping properties, to some extent, from the Kunze–Stein phenomenon, perhaps most clearly seen in S_{3*}. Philosophically, the localized part S_1 of S_R (with $R=1$ for analogy) should correspond to the Euclidean disc multiplier acting on radial functions. Since the Euclidean disc multiplier is merely L^2-bounded when acting on functions not necessarily radial, we cannot expect S_1 to be bounded on $L^p(G/K)$ unless $p=2$. Note that the full operator S_R ($R=1$) is unbounded on $L^p(K\backslash G/K)$ for $p \neq 2$, since the corresponding multiplier cannot be analytically continued to the strip in the complex plane described in [3].

For the next result we must first recall the definition of Lorentz spaces.

Definition 3.4. Let (X,μ) be a measure space, $0<p<\infty$, and $0<q\le\infty$. By the Lorentz space $L^{p,q}(X,\mu)$ we understand the space of equivalence classes of measurable functions f with finite Lorentz space norm, $\|f\|_{L^{p,q}} < \infty$. Here

$$\|f\|_{L^{p,q}} = \begin{cases} \Big(q\int\limits_0^\infty [t\mu(\{x\in X : |f(x)|>t\})^{1/p}]^q\, t^{-1}\,dt\Big)^{1/q} & \text{if } q<\infty,\\ \sup\limits_{t>0} t^{1/p}\mu(\{x\in X : |f(x)|>t\}) & \text{if } q=\infty.\end{cases}$$

See [10, Chapter 1] for a summary of the properties of Lorentz spaces.

Theorem 3.5.

(i) *The maximal operator* $S_{1,*}$ *is bounded from* $L^{p_i,1}(\mathbb{R}_+,d\mu)$ *into* $L^{p_i,\infty}(\mathbb{R}_+,d\mu)$, $i=0,1$, *where* $p_0=\frac{4\alpha+4}{2\alpha+3}$ *and* $p_1=\frac{4\alpha+4}{2\alpha+1}$.
(ii) *The maximal operator* $S_{4,*}$ *is bounded from* $L^{p_0,1}(\mathbb{R}_+,d\mu)$ *into* $L^2(\mathbb{R},d\mu)$.

The maximal operator S_* *is therefore bounded from* $L^{p_0,1}(\mathbb{R},d\mu)$ *into the space* $L^2(\mathbb{R},d\mu)+L^{p_0,\infty}(\mathbb{R},d\mu)$.

This was not addressed by Meaney and Prestini but is to be seen as the Jacobi-analysis analogue of the endpoint result in [23]. As for the sharpness of the Lorentz space indices, we mention the following result.

Proposition 3.6. *The disc multiplier* S_R *is not bounded from the space* $L^{p_0,r}(\mathbb{R}_+,d\mu)$ *into* $L^{p_0,\infty}(\mathbb{R},d\mu)+L^2(\mathbb{R},d\mu)$ *for any* $r\in(1,\infty]$.

Proof. The conclusion follows at once from the observation that even the localized piece S_R^1 of the disc multiplier fails the stated mapping property, according to [4, Theorem II]. ■

4. Proof of the mapping properties for noncritical exponents. The present section contains the lengthy proof of Theorem 3.2 as well as Theorem 3.3. As already indicated, one studies S_* in four different regions of the (t,r)-plane, so we have split the proof into four subsections.

4.1. Investigation of $S_{1,*}$. We begin the lengthy examination of S_* with an analysis of the behavior of the kernel K_R when both arguments are small. We scale the corresponding operator $S_{1,R}$ slightly by writing

$$S_{1,R}f(t) = \int\limits_{A(t)} K_{1,R}(t,r)f(r)\,d\mu(r), \quad A(t) = \begin{cases} [0,R_0] & \text{if } 0 < t \le R_0/2, \\ [0,R_0/2] & \text{if } R_0/2 < t < R_0. \end{cases}$$

Moreover we assume $2(\alpha+1)$ is *not* an integer since we may otherwise copy the proofs from [18] verbatim, with $n := 2(\alpha+1)$. Recall from Lemma 3.1 that $\varphi_\lambda(t)$ may be written as

$$\varphi_\lambda(t) = c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\big(\mathcal{J}_\alpha(\lambda t) + t^2 a_1(1)\mathcal{J}_{\alpha+1}(\lambda t)\big) + E_2(\lambda,t),$$

so that

$$\begin{aligned}
&\varphi_\lambda(t)\varphi_\lambda(r) \\
&= c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\big(\mathcal{J}_\alpha(\lambda t)\mathcal{J}_\alpha(\lambda r) + \mathcal{J}_\alpha(\lambda t)r^2a_1(r)\mathcal{J}_{\alpha+1}(\lambda r) \\
&\quad + t^2a_1(t)\mathcal{J}_{\alpha+1}(\lambda t)\mathcal{J}_\alpha(\lambda r) + t^2a_1(t)\mathcal{J}_{\alpha+1}(\lambda t)r^2a_1(r)\mathcal{J}_{\alpha+1}(\lambda r)\big) \\
&\quad + c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\big(\mathcal{J}_\alpha(\lambda t) + t^2a_1(t)\mathcal{J}_{\alpha+1}(\lambda t)\big)E_2(\lambda,r) \\
&\quad + c\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\big(\mathcal{J}_\alpha(\lambda r) + r^2a_1(r)\mathcal{J}_{\alpha+1}(\lambda r)\big)E_2(\lambda,t) \\
&= c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\bigg(c_1\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}\frac{J_\alpha(\lambda r)}{(\lambda r)^\alpha} + c_2r^2a_1(r)\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}\frac{J_{\alpha+1}(\lambda r)}{(\lambda r)^{\alpha+1}} \\
&\quad + c_3t^2a_1(t)\frac{J_{\alpha+1}(\lambda t)}{(\lambda t)^{\alpha+1}}\frac{J_\alpha(\lambda r)}{(\lambda r)^\alpha} + r^2t^2a_1(t)a_1(r)\frac{J_{\alpha+1}(\lambda t)}{(\lambda t)^{\alpha+1}}\frac{J_{\alpha+1}(\lambda r)}{(\lambda r)^{\alpha+1}}\bigg) \\
&\quad + \text{negligible terms.}
\end{aligned}$$

The indicated decomposition yields a compatible decomposition of $K_{1,R}$ and $S_{1,R}f(t)$, in the sense that $K_{1,R} = \sum_{i=1}^5 K_{1,R}^i$ and $S_{1,R}f(t) = \sum_{i=1}^5 S_{1,R}^i f(t)$, where

$$K_{1,R}^1(t,r) = \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^\alpha t^\alpha}\int\limits_0^R \frac{J_\alpha(\lambda r)J_\alpha(\lambda t)}{\lambda^{2\alpha}}\,d\nu(\lambda),$$

$$K_{1,R}^2(t,r) = r^2a_1(r)\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^{\alpha+1}t^\alpha}\int\limits_0^R \frac{J_{\alpha+1}(\lambda r)J_\alpha(\lambda t)}{\lambda^{2\alpha+1}}\,d\nu(\lambda),$$

$$K_{1,R}^3(t,r) = t^2a_1(t)\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^\alpha t^{\alpha+1}}\int\limits_0^R \frac{J_\alpha(\lambda r)J_{\alpha+1}(\lambda t)}{\lambda^{2\alpha+1}}\,d\nu(\lambda),$$

$$K^4_{1,R}(t,r) = r^2 a_1(r) t^2 a_1(t) \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}} \frac{1}{r^{\alpha+1} t^{\alpha+1}} \times \int_0^R \frac{J_{\alpha+1}(\lambda r) J_{\alpha+1}(\lambda t)}{\lambda^{2(\alpha+1)}} \, d\nu(\lambda),$$

$$K^5_{1,R} = \text{sum of negligible terms}$$

(it is to be understood that all functions are extended by zero to all of $\mathbb{R}$ for t not in $[0, R_0]$), and

$$S^i_{1,R} f(t) = \begin{cases} \int_{A(t)} K^i_{1,R}(t,r) f(r) \Delta(r) \, dr & \text{for } 0 \le t \le R_0,\ i = 1, \dots, 5, \\ 0 & \text{otherwise.} \end{cases}$$

A slightly more convenient expression for $K^1_{1,R}(t,r)$ is obtained by writing

$$\begin{aligned} &K^1_{1,R}(t,r) \\ &= \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}} \frac{1}{r^\alpha} \frac{1}{t^\alpha} \left\{ \int_1^R \frac{J_\alpha(\lambda r) J_\alpha(\lambda t)}{\lambda^{2\alpha}} \, d\nu(\lambda) + \int_0^1 \frac{J_\alpha(\lambda r) J_\alpha(\lambda t)}{\lambda^{2\alpha}} \, d\nu(\lambda) \right\} \\ &= M_{1,R}(t,r) + E_1(t,r), \end{aligned}$$

where

$$|E_1(t,r)| \lesssim \left| \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}} \frac{1}{r^\alpha} \frac{1}{t^\alpha} \right| \int_0^1 \frac{|\lambda r|^\alpha |\lambda t|^\alpha}{\lambda^{2\alpha}} \, d\lambda \lesssim 1.$$

The associated operator ${}_eS^1_{1,R} f(t) = 1_{[0,R_0]}(t) \int_{A(t)} E_1(t,r) f(r) \Delta(r) \, dr$ is therefore easily estimated. We have

$$|{}_eS^1_{1,R} f(t)| \le \|1_{[0,R_0]} E_1(t,\cdot)\|_{L^{p'}(d\mu)} \|1_{[0,R_0]} f\|_{L^p(d\mu)} \lesssim \|f\|_{L^p(d\mu)},$$

where $1/p + 1/p' = 1$, $1 < p < \infty$, so the associated maximal operator $t \mapsto \sup_{R>1} |{}_eS^1_{1,R} f(t)|$ is bounded on $L^p(d\mu)$ for $1 < p < \infty$. The term $M_{1,R}(t,r)$ in our kernel decomposition $K^1_{1,R}(t,r) = M_{1,R}(t,r) + E_1(t,r)$ turns out to be fairly complicated, however.

Recall that

$$M_{1,R}(t,r) = \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}} \frac{1}{r^\alpha} \frac{1}{t^\alpha} \int_1^R J_\alpha(\lambda r) J_\alpha(\lambda t) \lambda^{-2\alpha} |\mathbf{c}(\lambda)|^{-2} \, d\lambda.$$

We need a description of not only the leading term in the asymptotic behavior of $|\mathbf{c}(\lambda)|^{-2}$ as $\lambda \to \infty$, so we use Lemma 2.1 with $M = [2\alpha + 2]$ (the integer part of $2\alpha + 2$) to write

$$|\mathbf{c}(\lambda)|^{-2} = \lambda^{2\alpha+1} + c_1 \lambda^{2\alpha} + c_2 \lambda^{2\alpha-1} + \dots + c_{M-1} \lambda^{2(\alpha+1)-[2(\alpha+1)]} + O(\lambda^{2\alpha+1-[2(\alpha+1)]}).$$

Note that $2\alpha+1 < M < 2(\alpha+1)$, since $2(\alpha+1)$ is not an integer. Correspondingly the asymptotic expansion of $|\mathbf{c}(\lambda)|^{-2}$ still takes the form $|\mathbf{c}(\lambda)|^{-2} = P(\lambda) + E(\lambda)$, but P is not a polynomial anymore. At any rate

$$\begin{aligned}(4.1)\quad & M_{1,R}(t,r) \\ &= \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^\alpha}\frac{1}{t^\alpha}\Big\{\int_1^R J_\alpha(\lambda r)J_\alpha(\lambda t)\Big(\lambda + c_1 + \frac{c_2}{\lambda} + \cdots + \frac{c_{M-1}}{\lambda^{M-2}}\Big)\,d\lambda \\ &\qquad + \int_1^R J_\alpha(\lambda r)J_\alpha(\lambda t)E(\lambda)\,d\lambda\Big\}.\end{aligned}$$

We thus need to consider separately a host of new operators, like

$$\begin{aligned}M_{1,R}^d(t,r) &= \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^\alpha}\frac{1}{t^\alpha}\int_1^R J_\alpha(\lambda r)J_\alpha(\lambda t)\lambda^d\,d\lambda, \quad d = 2-M,\ldots,1, \\ &= \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^\alpha}\frac{1}{t^\alpha}\Big\{\int_0^R \cdots \lambda^d\,d\lambda - \int_0^1 \cdots \lambda^d\,d\lambda\Big\}\end{aligned}$$

where the latter piece $\cdots \int_0^1 \cdots \lambda\,d\lambda$ (at least for $d = 1$) gives rise to an operator comparable with ${}_eS_{1,R}^1 f$ considered above. We will keep the constants c_k, as they never influence the estimates. The conclusion, as before, is that the "error" term gives rise to a maximal operator that is bounded for the full range $p \in (1,\infty)$, hence uninteresting as far as the ongoing proof is concerned.

The first piece in the above-mentioned decomposition of $M_{1,R}^1$ gives rise to the operator

$$\begin{aligned}S_{1,R}^{M_{1,R}^1} f(t) &= \int_{A(t)} M_{1,R}^1(t,r)f(r)\Delta(r)\,dr \\ &= \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{1}{t^\alpha}\int_0^{R_0}\Big\{\int_0^R J_\alpha(\lambda r)J_\alpha(\lambda t)\lambda\,d\lambda\Big\} f(r)\sqrt{\Delta(r)}r^{1/2}\,dr.\end{aligned}$$

Since $0 \le r,t \le R_0$, we can introduce the approximation $\sqrt{\Delta(r)} \sim r^{\alpha+1/2}$, $\sqrt{\Delta(t)} \sim t^{\alpha+1/2}$ in order to arrive at the favorable estimate

$$\begin{aligned}|S_{1,R}^{M_{1,R}^1} f(t)| &\lesssim \frac{1}{t^\alpha}\Big|\int_0^{R_0}\Big\{\int_0^R J_\alpha(\lambda r)J_\alpha(\lambda t)\lambda\,d\lambda\Big\} f(r)r^{\alpha+1}\,dr\Big| \\ &\le \frac{1}{t^\alpha}\Big|\int_0^{\infty}\Big\{\int_0^R J_\alpha(\lambda r)J_\alpha(\lambda t)\lambda\,d\lambda\Big\} f(r)r^{\alpha+1}\,dr\Big| = cT_R f(t),\end{aligned}$$

where T_R is formally the standard partial sum operator for Euclidean Fourier integrals, f being viewed as a radial function on $\mathbb{R}^n$, with $n := 2(\alpha+1)$. More precisely, T_R is indeed the partial sum operator for the *Hankel transform* $\mathcal{H}_\alpha$, and by [15] the maximal operator $t \mapsto \sup_{R>1} |S_{1,R}^{M_{1,R}^1} f(t)|$ is bounded on $L^p(\mathbb{R}_+, x^{2\alpha+1}dx)$ for $\frac{4\alpha+4}{2\alpha+3} < p < \frac{4\alpha+4}{2\alpha+1}$. This also explains the appearance of the Herz range. The result of Kanjin applies to $S_{1,R}^{M_{1,R}^1}$ as well, since we have localized its integral kernel in both arguments. We must still consider the remaining pieces $S_{1,R}^{M_{1,R}^0} f(t), S_{1,R}^{M_{1,R}^{-1}} f(t), \ldots$ and $S_{1,R}^{M_{1,R}^E} f(t)$.

Next on the list is the piece (of the kernel $M_{1,R}$)

$$M_{1,R}^0(t,r) = c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\,\frac{1}{t^\alpha}\,\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\,\frac{1}{r^\alpha}\int_1^R J_\alpha(\lambda t) J_\alpha(\lambda r)\,d\lambda.$$

Recall that we DO allow the possibility that $r \geq t$ in this analysis. So let us assume that $r \geq t$ and choose a smooth partition of unity $1 = g_{(1)} + g_{(2)} + g_{(3)}$ on $(0,\infty)$, indicated schematically by

$$[1,R] = \underset{(1)}{[1,1/r]} \cup \underset{(2)}{[1/r,1/t]} \cup \underset{(3)}{[1/t,R]}$$

where only the piece $g_{(3)} =: g_t$ will be important. Here g_t is taken to be identically 1 on $(1/t,R]$ and supported in $(1/r,R]$ (we will later choose g_t more carefully). The corresponding expansion of $M_{1,R}^0$ will be written

$$\begin{aligned}(4.2)\qquad & M_{1,R}^0(t,r)\\ &= c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\,\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\,\frac{1}{t^\alpha}\,\frac{1}{r^\alpha}\Big\{\int_1^{1/r}\cdots d\lambda + \int_{1/r}^{1/t}\cdots d\lambda + \int_{1/t}^{R}\cdots g_t(\lambda)\,d\lambda\Big\}\\ &=: M_{1,R}^{0,(1)}(t,r) + M_{1,R}^{0,(2)}(t,r) + M_{1,R}^{0,(3)}(t,r).\end{aligned}$$

We first analyze the range $1 \leq \lambda \leq 1/r$, corresponding to the function $M_{1,R}^{0,(1)}$. Here we estimate

$$\begin{aligned}|M_{1,R}^{0,(1)}(t,r)| &\lesssim \frac{1}{t^\alpha r^\alpha}\int_1^{1/r} |J_\alpha(\lambda t)|\,|J_\alpha(\lambda r)|\,d\lambda \lesssim \frac{1}{t^\alpha r^\alpha}\int_1^{1/r} (\lambda t)^{-1/2}\,d\lambda\\ &= \frac{1}{t^{\alpha+1/2} r^\alpha}[\lambda^{1/2}]_1^{1/r} = \frac{1}{t^{\alpha+1/2} r^\alpha}\Big(\frac{1}{r^{1/2}} - 1\Big) \lesssim \frac{1}{t^{\alpha+1/2} r^{\alpha+1/2}},\end{aligned}$$

where we have used that $1/r \geq 1$. It thus follows by the Hölder inequality (with $1/p + 1/p' = 1$) that

$$|S_{1,R}^{M_{1,R}^{0,(1)}} f(t)| \lesssim \frac{1}{t^{\alpha+1/2}}\int_0^{R_0} \frac{|f(r)|}{r^{\alpha+1/2}}\Delta(r)\,dr$$

$$= \frac{1}{t^{\alpha+1/2}} \Big(\int_0^{R_0} |f(r)|^p \Delta(r)\, dr \Big)^{1/p} \Big(\int_0^{R_0} r^{-(\alpha+1/2)p'} \Delta(r)\, dr \Big)^{1/p'}$$

$$\lesssim \frac{1}{t^{\alpha+1/2}} \|f\|_{L^p(d\mu)} \Big(\int_0^{R_0} r^{2\alpha+1-(\alpha+\frac{1}{2})p'}\, dr \Big)^{1/p'},$$

which is finite precisely when $2\alpha+1-(\alpha+1/2)p' > -1$, that is, $\frac{4\alpha+4}{2\alpha+1} > p'$ or equivalently when $p > \frac{4\alpha+4}{2\alpha+3}$. The associated maximal operator $S_{1,*}^{M_{1,R}^{0,(1)}}$ therefore satisfies the estimate

$$|S_{1,*}^{M_{1,R}^{0,(1)}} f(t)| \lesssim t^{-(\alpha+1/2)} \|_{L^p} \qquad \text{for } p > \tfrac{4\alpha+4}{2\alpha+3} \text{ and } 0 \le t \le R_0.$$

It follows that

$$\|S_{1,*}^{M_{1,R}^{0,(1)}} f\|_{L^p}^p \lesssim \|f\|_{L^p}^p \int_0^{R_0} t^{-(\alpha+1/2)p} \Delta(t)\, dt \lesssim \|f\|_{L^p}^p \int_0^{R_0} t^{-(\alpha+1/2)p+2\alpha+1}\, dt,$$

which is finite precisely when $-(\alpha+1/2)p+2\alpha+1 > -1$, that is, when $p < \frac{4\alpha+4}{2\alpha+1}$. We have therefore established that $S_{1,*}^{M_{1,R}^{0,(1)}}$ is bounded on L^p precisely when $\frac{4\alpha+4}{2\alpha+3} < p < \frac{4\alpha+4}{2\alpha+1}$.

The range $1/r \le \lambda \le 1/t$, corresponding to the piece $M_{1,R}^{0,(2)}$, is just as easily handled: The standard estimates $|J_\mu(\lambda r)| \lesssim (\lambda r)^{-1/2}$ and $|J_\mu(\lambda t)| \lesssim 1$ imply that

$$|M_{1,R}^{0,(2)}(t,r)| \lesssim \frac{1}{t^\alpha r^\alpha} \int_{1/r}^{1/t} \frac{1}{\lambda^{1/2}}\, d\lambda \lesssim \frac{1}{t^{\alpha+1/2} r^{\alpha+1/2}}.$$

Prior analysis shows that the associated maximal operator $S_{1,*}^{M_{1,R}^{0,(2)}}$ is L^p-bounded for $\frac{4\alpha+4}{2\alpha+3} < p < \frac{4\alpha+4}{2\alpha+1}$.

It turns out to be more difficult to estimate the piece $M_{1,R}^{0,(3)}$, corresponding to the range $1/t \le \lambda \le R$. In order to get started we use the more precise Bessel function estimate

$$J_\alpha(t) = c\frac{\cos(t - \alpha\pi/2 - \pi/4)}{t^{1/2}} + O(t^{-3/2})$$

from [26, p. 199] to write

$$M_{1,R}^{0,(3)}(t,r)$$
$$= c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}} \frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}} \frac{1}{t^\alpha} \frac{1}{r^\alpha} \int_{1/t}^{R} \frac{\cos\big(\lambda t - \frac{2\alpha+1}{4}\pi\big)}{(\lambda t)^{1/2}} \frac{\cos\big(\lambda r - \frac{2\alpha+1}{4}\pi\big)}{(\lambda r)^{1/2}} g_t(\lambda)\, d\lambda + E$$

$$= c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{t^\alpha}\frac{1}{r^\alpha}\int_{1/t}^{R}\frac{\cos\big(\lambda t-\frac{2\alpha+1}{4}\pi\big)\cos\big(\lambda r-\frac{2\alpha+1}{4}\pi\big)}{\lambda}g_t(\lambda)\,d\lambda+E$$

$$= c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{t^\alpha}\frac{1}{r^\alpha}\sum_{\epsilon,\nu=\pm1}\int_{1/t}^{R}\frac{e^{i\lambda\epsilon(t+\nu r)}}{\lambda}g_t(\lambda)\,d\lambda+E$$

where E is an error term. Observe that

$$\int_0^R\frac{e^{i\lambda\varepsilon(t+\nu r)}}{\lambda}g_t(\lambda)\,d\lambda = \Big(\lambda\mapsto 1_{[0,R]}(\lambda)\frac{g_t(\lambda)}{\lambda}\Big)^{\wedge}(\varepsilon(t+\nu r))$$
$$= \Big(\Big(\lambda\mapsto\frac{g_t(\lambda)}{\lambda}\Big)^{\wedge}\star\Big(x\mapsto\frac{e^{iRx}}{x}\Big)\Big)(\varepsilon+\nu r),$$

where $(\cdots)^{\wedge}$ designates the *Euclidean* Fourier transform. Let σ_t denote the Fourier transform of $\lambda\mapsto g_t(\lambda)/\lambda$ and set $\sigma_*(y)=\sup_t|\sigma_t(y)|$. Choosing g_t as in [18, Lemma 1], it follows that σ_* is Lebesgue integrable on $\mathbb{R}$, so that—apart from the error term E—we can estimate $S_{1,*}^{M_{1,R}^{0,(3)}}f(t)$ as follows:

$$|S_{1,*}^{M_{1,R}^{0,(3)}}f(t)|$$
$$\lesssim\frac{1}{t^{\alpha+1/2}}\sum_{\varepsilon,\nu=\pm1}\sup_{t,R}\Big|\int\Big(\sigma_t\star\Big(x\mapsto\frac{e^{iRx}}{x}\Big)\Big)(\varepsilon(t+\nu r))f(r)\sqrt{\Delta(r)}\,dr\Big|$$
$$=\frac{1}{t^{\alpha+1/2}}\sum_{\varepsilon,\nu=\pm1}\sup_{t,R}\Big|\int\sigma_t(\varepsilon(t+\nu r))\Big(\frac{e^{iRx}}{x}\star(f(x)\sqrt{\Delta(x)})\Big)(r)\,dr\Big|$$
$$\le\frac{1}{t^{\alpha+1/2}}\sum_{\varepsilon,\nu=\pm1}\int\sigma_*(\varepsilon(t+\nu r))\sup_{R>0}\Big|\Big(\frac{e^{iRx}}{x}\star(f(x)\sqrt{\Delta(x)})\Big)(r)\Big|\,dr$$

where we recognize the Carleson operator

$$C(f\sqrt{\Delta})(r)=\sup_{R>0}\Big(\frac{e^{iRx}}{x}\star(f(x)\sqrt{\Delta(x)})\Big)(r)$$

applied to the function $f\sqrt{\Delta}$. Since convolution with the L^1-function σ_* is an L^p-bounded operation, it follows from the weighted estimates for the Carleson operator, developed in [21] (see also [22]), that

$$\|S_{1,*}^{M_{1,R}^{0,(3)}}f\|_{L^p(d\mu)}\le c\|f\|_{L^p(d\mu)} \tag{4.3}$$

for $\frac{4\alpha+4}{2\alpha+3}<p<\frac{4\alpha+4}{2\alpha+1}$. As for the error term E in our decomposition of

$M^{0,(3)}_{1,R}$, we notice that

$$|E| \le \frac{1}{r^\alpha}\,\frac{1}{t^\alpha}\,\frac{1}{r^{1/2}}\,\frac{1}{t^{3/2}}\int_{1/t}^{R}\frac{1}{\lambda^2}\,d\lambda \le \frac{1}{r^{\alpha+1/2}t^{\alpha+1/2}}$$

since $1/r < 1/t$ and $1/R < t$. It thus follows, as above, that the error term E gives rise to a maximal operator that is L^p-bounded for the same range of p.

It could, however, also happen that $R < 1/r$, in which case we do not decompose $[0, R]$ but rather estimate $M^0_{1,R}$ directly: trivially $|M^0_{1,R}(r,t)| \le r^{-(\alpha+1/2)}t^{-(\alpha+1/2)}$, so once again we end up with a maximal operator that is L^p-bounded for the stated range of p.

The last remaining case is $1/r < R < 1/t$, where we use the decomposition $[0, R] = [0, 1/r] \cup [1/r, R]$ and carry out the exact same type of estimate as before. This completes our analysis of $M^0_{1,R}$.

Now consider

$$M^{-1}_{1,R}(t,r) = \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\,\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\,\frac{1}{t^\alpha}\,\frac{1}{r^\alpha}\int_1^R \frac{J_\alpha(\lambda t)J_\alpha(\lambda r)}{\lambda}\,d\lambda$$

and assume without loss of generality that $r \ge t$. Decompose the domain of integration smoothly as

$$[1, R] = \underset{(1)}{[1, 1/r]} \cup \underset{(2)}{[1/r, 1/t]} \cup \underset{(3)}{[1/t, R]}.$$

The resulting pieces $M^{-1,(1)}_{1,R}$ and $M^{-1,(2)}_{1,R}$ are trivially estimated since they happen to resemble the error terms appearing in the analysis of $M^0_{1,R}$ above. The final piece $M^{-1,(3)}_{1,R}$ is also easily handled, since

$$\Big|\int_{1/t}^{R}\frac{J_\alpha(\lambda r)J_\alpha(\lambda t)}{\lambda}\,d\lambda\Big| \le \frac{1}{r^{1/2}t^{1/2}}\int_{1/t}^{R}\frac{1}{\lambda^2}\,d\lambda \lesssim \frac{1}{r^{1/2}t^{1/2}}.$$

For the remaining terms $M^i_{1,R}$, $i = 2-M, \dots, -2$, and $M^E_{1,R}$ we employ the trivial estimate $|J_\mu(t)| \le c$ for all t to see that

$$|M^{-2}_{1,R}(t,r) + \dots + M^{2-M}_{1,R}(t,r) + M^E_{1,R}(t,r)| \lesssim \frac{1}{t^{\alpha+1/2}}\,\frac{1}{r^{\alpha+1/2}},$$

hence the associated maximal operator is L^p-bounded for $p \in (p_0, p_1)$. We have thus finished the proof for the operator $S^1_{1,R}$.

Regarding $S^2_{1,R}$, we first recall that the associated kernel is

$$K^2_{1,R}(t,r) = \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\,\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\,\frac{1}{t^\alpha}\,\frac{a_1(r)r^2}{r^\alpha}\int_0^R \frac{J_{\alpha+1}(\lambda r)J_\alpha(\lambda t)}{\lambda^{2\alpha+1}}|\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

We once more break up the domain of integration as $[0,R]=[0,1]\cup[1,R]$ and use the basic estimate $|J_\mu(t)|\le ct^\mu$ to estimate in the range $\lambda\in[0,1]$ as follows:

$$\Big|\int_0^1 \frac{J_{\alpha+1}(\lambda r)J_\alpha(\lambda t)}{\lambda^{2\alpha+1}}|\mathbf{c}(\lambda)|^{-2}\,d\lambda\Big| \lesssim \int_0^1 \frac{(\lambda r)^{\alpha+1}(\lambda t)^\alpha}{\lambda^{2\alpha+1}}\,d\lambda \le r^{\alpha+1}t^\alpha.$$

In the range $\lambda\in[1,R]$ we once again use the asymptotic expansion of $|\mathbf{c}(\lambda)|^{-2}$ from Lemma 2.1 to decompose the integral

$$\int_1^R \frac{J_{\alpha+1}(\lambda r)J_\alpha(\lambda t)}{\lambda^{2\alpha+1}}|\mathbf{c}(\lambda)|^{-2}\,d\lambda$$

further, giving rise to integrals of the sort encountered in $M_{1,R}$.

By symmetry in t and r, the same estimates hold for $K^3_{1,R}(t,r)$, so it remains to investigate the kernels $K^4_{1,R}$ and $K^5_{1,R}$ (the error term) together with the associated operators $S^4_{1,R}$ and $S^5_{1,R}$. To this end recall that

$$K^4_{1,R}(t,r)=\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\,\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\,\frac{r^2a_1(r)}{r^{\alpha+1}}\,\frac{t^2a_1(t)}{t^{\alpha+1}}\int_0^R \frac{J_{\alpha+1}(\lambda r)J_{\alpha+1}(\lambda t)}{\lambda^{2(\alpha+1)}}|\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

For $\lambda\in[0,1]$ in the domain of integration we may proceed as above, and for $\lambda\in[1,R]$ we once again use Lemma 2.1, leading us to consider terms already analysed for $M^3_{1,R}$.

A favorable estimate for the maximal operator associated with the error term $K^5_{1,R}$ also follows from earlier considerations: If $r>t$, we decompose $[0,R]=[0,1]\cup[1,1/r]\cup[1/r,1/t]\cup[1/t,R]$ smoothly, thereby breaking $K^5_{1,R}$ into four pieces. Using that $|E_2(\lambda,t)|\lesssim t^4$ for $|\lambda t|\le 1$ and $|E_2(\lambda,t)|\lesssim t^2(\lambda t)^{2-(\alpha+1/2)}$ for $|\lambda t|>1$ (cf. Lemma 3.1), it follows that $|K^5_{1,R}(t,r)|\lesssim r^{-(\alpha+1/2)}t^{-(\alpha+1/2)}$. The associated maximal operator $S^5_{1,*}$ is therefore L^p-bounded for the usual range $p\in(p_0,p_1)$.

By piecing together all the estimates in the present section we have thus finally proved that $S_{1,*}$ is bounded on $L^p(\mathbb{R}_+,d\mu)$ if and only if $\frac{4\alpha+4}{2\alpha+3}<p<\frac{4\alpha+4}{2\alpha+1}$. We have also seen that the reason for this restricted range is purely Euclidean. At this stage in the analysis, the curved geometry of the underlying symmetric space is not strong enough for non-Euclidean phenomena to overpower the Euclidean structure.

4.2. Investigation of $S_{2,*}$. From now on the analysis of S_R will involve the behavior of the Jacobi function $\varphi_\lambda(t)$ when t tends to infinity, and Lemma 3.1 is not applicable in this region. As in the case of symmetric spaces, this investigation requires sharp bounds on the $\mathbf{c}$-function, a close study of the Harish-Chandra series for φ_λ, and an analogue of the Gangolli estimates in the Jacobi setting. Recall that $\varphi_\lambda(t)=\mathbf{c}(\lambda)e^{(i\lambda-\rho)t}\phi_\lambda(t)+$

$\mathbf{c}(-\lambda)e^{(-i\lambda-\rho)t}\phi_{-\lambda}(t)$, where we now formally expand $\phi_\lambda(t)$ as a power series (the "Harish-Chandra series"),

$$\phi_\lambda(t)=\sum_{k=0}^{\infty}\Gamma_k(\lambda)e^{-2kt}.$$

Since ϕ_λ is a solution to $\mathcal{L}_{\alpha,\beta}\varphi+(\lambda^2+\rho^2)\varphi=0$, the $\Gamma_k(\lambda)$ are given recursively—according to [25, formula (3.4)]—by $\Gamma_0(\lambda)\equiv 1$,

$$\begin{aligned}(k+1)(k+1-i\lambda)\Gamma_{k+1}(\lambda)&=(\alpha-\beta)\sum_{j=0}^{k}(\rho+2j-i\lambda)\Gamma_j(\lambda)\\&\quad+(\beta+1/2)\sum_{j=1}^{[(k+1)/2]}(\rho+2(k+1-2j)-i\lambda)\Gamma_{k+1-2j}(\lambda),\end{aligned}$$

where $[x]$ is the integer part of x. In fact, $\Gamma_{k+1}=a_k\Gamma_k+\sum_{j=0}^{k-1}b_j^k\Gamma_j$, where (by [25, Corollary 3.4])

$$\begin{aligned}a_k&=1+\frac{\alpha-\beta-1}{k+1}+\frac{\alpha-\beta-1+\frac{1}{k+1}(\alpha(\alpha-1)-\beta(\beta-1)+1)}{k+1-i\lambda},\\b_j^k&=(-1)^{k+j+1}\frac{2\beta+1}{k+1}\biggl(1+\frac{\rho+2j-1}{k+1-i\lambda}\biggr).\end{aligned}$$

Lemma 4.1 (Gangolli estimates). *Let D be either a compact subset of $\mathbb{C}\setminus(-i\mathbb{N})$ or a set of the form $D=\{\lambda=\xi+i\eta\in\mathbb{C}\mid\eta\geq-\varepsilon|\xi|\}$ for some $\varepsilon\geq 0$. There exist positive constants K,d such that*

$$|\Gamma_k(\lambda)|\leq K(1+k)^d\quad\textit{for all } k\in\mathbb{Z}_+,\ \lambda\in D. \tag{4.4}$$

Proof. See [8, Lemma 7]. ∎

It follows that the expansion for $\phi_\lambda(t)$ converges uniformly on sets of the form $\{(t,\lambda)\in[c,\infty)\times D\}$, where c is a positive constant. More precisely, if $\lambda\in D$, and $c>0$ is fixed, we see that

$$\forall t\geq c:\quad|\phi_\lambda(t)|\leq\sum_{k=0}^{\infty}K(1+k)^de^{-2kt}\lesssim\sum_{k=0}^{\infty}(1+k)^de^{-2ck}\lesssim 1,$$

that is, $\phi_\lambda(t)$ is bounded uniformly in $\lambda\in D$ for $t\geq c>0$. We will take $c=R_0$ in later applications. Since $\lambda\mapsto\phi_\lambda(t)$ is analytic in a strip containing the real axis, it follows as in the proof of [18, Lemma 7] that derivatives of ϕ_λ in λ are bounded independently of λ as well.

Remark 4.2. It is easy to prove that $\bigl|\frac{\partial^k}{\partial\lambda^k}\phi_\lambda(t)\bigr|\leq c_k$ for all $t\geq R_0$ and $\lambda\in[0,2\rho]$. This was done for symmetric spaces in [18, Lemma 7], whereas a more general statement in the context of Jacobi analysis was obtained in [13, Lemma 4.1].

The asymptotic behavior of $\varphi_\lambda(t)$ as t increases can now be investigated. The result is formally the same as the analogues in [25] and [24], and the proof will even work for complex parameters α, β.

THEOREM 4.3.

(i) *For every* $M \geq 0$, $0 \leq m \leq M$ *and* $\lambda \in \mathbb{C}$ *with* $\Im\lambda \geq 0$, *there exist polynomials* f_{l_m} *in* λ *of degree* m *such that*

$$\Gamma_k(\lambda) = \sum_{m=0}^{M} \gamma_m^k + E_{M+1}^k,$$

where γ_m^k *is a sum of terms* $1/f_{l_m}$, *and where*

$$|\gamma_m^k(\lambda)| \leq A\frac{|\rho|^m e^{2k}}{|\Re\lambda|^m}, \qquad |D_{\Re\lambda}^a \gamma_m^k| \leq 2^a A\frac{|\rho|^m e^{2k}}{|\Re\lambda|^{m+a}},$$

$$|E_{M+1}^k| \leq A\frac{|\rho|^{M+1} e^{2k}}{|\Re\lambda|^{M+1}};$$

the constant A *is independent of* M *and* λ.

(ii) *Let* $\Lambda_m(\lambda, t) = \sum_{j=0}^{\infty} \gamma_m^{m+j}(\lambda)e^{-2jt}$. *There exists a function* $\mathcal{E}_{M+1}$ *such that, for every* $M \geq 0$, $t \geq R_0$, *and* $\lambda \in \mathbb{C}$ *with* $\Im\lambda \geq 0$,

$$\begin{aligned}\phi_\lambda(t) &= \sum_{m=0}^{\infty} \Lambda_m(\lambda, t)e^{-2mt} \\ &= \sum_{m=0}^{M} \Lambda_m(\lambda, t)e^{-2mt} + e^{-2(M+1)t}\mathcal{E}_{M+1}(\lambda, t),\end{aligned}$$

where

$$|D_\lambda^a D_t^b \Lambda_m| \leq 2^{a+b} A\frac{|\rho|^m e^{2m}}{|\Re\lambda|^{m+a}} G_b(t),$$

$$|D_t^b \mathcal{E}_{M+1}| \leq 2^b A\frac{e^{2(M+1)}|\rho|^{M+1}}{|\Re\lambda|^{M+1}} G_b(t),$$

with $G_k(t) := \sum_{j=0}^{\infty} j^k e^{2k(1-t)}$.

Proof. The algebraic properties of the Harish-Chandra series are investigated in [25, Section 3], along with the estimates in part (i) of the theorem, and it is an arduous (yet elementary) matter to redo the proofs for complex parameters α, β instead. The improved statement in (ii) via the presence of the exponential factor in $e^{-2(M+1)t}\mathcal{E}_{M+1}(\lambda, t)$ was established in [18, Lemma 6], the proof of which need not be repeated. ∎

PROPOSITION 4.4. *For* $t \geq 1$ *consider the operator*

$$Uf(t) = e^{-\rho t} \sup_{R>1} \left| \int_1^\infty \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\, dr \right|.$$

Then U maps $L^p(\mathbb{R}_+, d\mu)$ boundedly into $L^2(\mathbb{R}, d\mu)+L^p(\mathbb{R}, d\mu)$ for $1<p\leq 2$.

Proof. The proof is similar to the one for [18, Theorem 3] and immediate for $p = 2$. For $1 < p < 2$ fixed and $k \in \mathbb{N}$ write $\varphi_k(t) = 1_{[-k,k]}(t)$, $\varphi_k(t) + \psi_k(t) \equiv 1$ for all $t > 1$, and correspondingly

$$
\begin{aligned}
e^{-\rho t}\int_1^\infty \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\,dr &= e^{-\rho t}\sum_{k=1}^\infty \int_k^{k+1} \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\,dr \\
&= e^{-\rho t}\sum_{k=1}^\infty (\varphi_k(t)+\psi_k(t)) \int_k^{k+1} \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\,dr \\
&= \sum_{k=1}^\infty (A_{k,R}f(t) + B_{k,R}f(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{k,R}f(t) &= e^{-\rho t}\varphi_k(t) \int_k^{k+1} \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\,dr, \\
B_{k,R}f(t) &= e^{-\rho t}\psi_k(t) \int_k^{k+1} \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\,dr.
\end{aligned}
$$

Then

$$
Uf(t) \leq \sum_{k=1}^\infty A_{k,*}f(t) + \sup_{R>1}\Big|\sum_{h=1}^\infty B_{h,R}f(t)\Big|
$$

with

$$
\begin{aligned}
A_{k,*}f(t) &= e^{-\rho t}\varphi_k(t) \sup_{R>1}\Big|\int_k^{k+1} \frac{e^{iR(t-r)}}{t-r} f(r)\Delta(r)e^{-\rho r}\,dr\Big| \\
&= e^{-\rho t}\varphi_k(t) C(g_k(\cdot))(t),
\end{aligned}
$$

$C(g_k)$ being the Carleson operator applied to the function

$$
g_k(r) = f(r)\Delta(r)e^{-\rho r}1_{[k,k+1)}(r).
$$

For $p \in (1,\infty)$ it follows that

$$
\begin{aligned}
\Big\|\sum_{h=1}^\infty A_{k,*}f\Big\|^p_{L^p(\mathbb{R},d\mu)} &\leq \sum_{k=1}^\infty \|A_{k,*}f\|^p_{L^p} = \sum_{k=1}^\infty \|e^{-\rho t}\varphi_k(t)C(g_k)(t)\|^p_{L^p} \\
&= \sum_{k=1}^\infty \int_{k-1}^{k+2} e^{-\rho p t}\varphi_k(t)^p (C(g_k)(t))^p \Delta(t)\,dt \\
&\lesssim \sum_{k=1}^\infty e^{-\rho p(k-1)} \int_{k-1}^{k+2} (C(g_k)(t))^p \Delta(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(\sharp)}{\lesssim} \sum_{k=1}^{\infty} e^{\rho(k-1)(2-p)} \int_{k-1}^{k+2} |g_k(r)|^p \, dr \\
&= \sum_{k=1}^{\infty} e^{\rho(k-1)(2-p)} \int_{1}^{\infty} |g_k(r)|^p \, dr \\
&\lesssim \sum_{k=1}^{\infty} e^{\rho(k-1)(2-p)} \int_{k}^{k+1} |f(r)|^p e^{\rho p r} \, dr \\
&= \sum_{k=1}^{\infty} e^{\rho(k-1)(2-p)} \int_{k}^{k+1} |f(r)|^p e^{2\rho r} e^{\rho p r - 2\rho r} \, dr \\
&\sim c_p \sum_{k=1}^{\infty} \int_{k}^{k+1} |f(r)|^p e^{2\rho r} \, dr \sim c_p \|f\|_{L^p}^p.
\end{aligned}
$$

Here $c_p = e^{\rho(p-2)}$ for $p < 2$ and $c_p = e^{2\rho(p-2)}$ for $p \geq 2$. The actual value of c_p is immaterial of course, but it should be noted that it can become very large. At $(\sharp)$ we have used the classical weighted estimates for the Carleson operator corresponding to the weight $w \equiv 1$.

It now suffices to establish the estimate

$$
\Big\| \sup_{R>1} \Big| \sum_{k=1}^{\infty} B_{k,R} f(t) \Big| \Big\|_{L^2(\mathbb{R},d\mu)} \leq c_p \|f\|_{L^p(\mathbb{R},d\mu)} \quad \text{for } 1 < p \leq 2. \tag{4.5}
$$

For $p = 2$ this follows from the easy estimate $\sup_{R>1} \big| \sum_{k=1}^{\infty} B_{k,R} f \big| \leq Uf + \sum_{k=1}^{\infty} A_{k,*} f$, since U is trivially L^2-bounded. It remains to establish a restricted (L^p, L^2)-estimate for $1 < p < 2$ by the interpolation theorem of Marcinkiewicz. Fix a measurable subset $E \subset [1, \infty)$. If $\|1_E\|_{L^2(\mathbb{R},d\mu)} \geq 1$, then

$$
\Big\| \sum_{k} B_{k,*} 1_E \Big\|_{L^2(\mathbb{R},d\mu)} \lesssim \|1_E\|_{L^2(\mathbb{R},d\mu)} \lesssim \|1_E\|_{L^p(\mathbb{R},d\mu)}
$$

for $p \leq 2$.

On the other hand, if $\|1_E\|_{L^p(\mathbb{R},d\mu)} < 1$,

$$
\begin{aligned}
|B_{k,*} 1_E(t)| &\leq e^{-\rho t} \Big(\int_{k}^{k+1} \frac{1}{|t-r|} 1_E(r) \varDelta(r) e^{-\rho r} \, dr \Big) \psi_k(t) \\
&\leq \frac{e^{-\rho t}}{|t-h|} e^{\rho k} |E_k| \psi_k(t),
\end{aligned}
$$

where $|E_k|$ denotes the μ-measure of the set $E_k = E \cap [k, k+1)$. But then

$$
\Big\| \sum_{k=1}^{\infty} B_{k,*} 1_E \Big\|_{L^2(\mathbb{R},d\mu)} \leq \sum_{k=1}^{\infty} \|B_{k,*} 1_E\|_{L^2(\mathbb{R},d\mu)} \leq c \sum_{k=1}^{\infty} e^{\rho k} |E_k|
$$

$$\leq c\sum_{k=1}^{\infty} e^{2\rho k}|E_k| = c\int_1^{\infty} 1_E(r)\Delta(r)\,dr \leq c\|1_E\|^p_{L^p(\mathbb{R},d\mu)},$$

thus proving the restricted (L^p, L^2)-estimate in this case as well. This also finishes the proof of the proposition. ∎

LEMMA 4.5. *Let T be either the Carleson operator, the Hilbert transform on $\mathbb{R}$, or a convolution operator with Lebesgue integrable kernel on $\mathbb{R}$. The maximal operator associated with the function $t \mapsto e^{-\rho t}T(f(r)\Delta(r)e^{-\rho r})(t)$ maps $L^p(\mathbb{R}_+, d\mu)$ boundedly into $L^p(\mathbb{R}, d\mu) + L^2(\mathbb{R}, d\mu)$ for $1 < p \leq 2$.*

Proof. The corresponding statement in [18] was established by using [21], hence cast in terms of radial Fourier analysis on $\mathbb{R}^n$. Prestini later generalized her weighted estimates to a setting that applies to Jacobi analysis, witness the paper [22]. Hence there is nothing new to prove. ∎

THEOREM 4.6. *The maximal operator $S_{2,*}$ associated with the operator*

$$S_{2,R}f(t) = \int_{R_0}^{\infty} K_{2,R}(t,r)f(r)\Delta(r)\,dr$$

is bounded from $L^p(\mathbb{R}_+, d\mu)$ into $L^p(\mathbb{R}, d\mu) + L^2(\mathbb{R}, d\mu)$ for $1 < p \leq 2$ (where it is implicitly understood that we are in the range $t \geq R_0$).

The clever technique of proof—originating in [18]—is to bound $S_{2,*}$ by means of maximal operators associated with the Carleson operator and the Hilbert transform, using Proposition 4.4 and Lemma 4.5. This is a standard technique when working on spaces of homogeneous type, but for weighted measures where the volume of large balls grows exponentially more care is needed. The above technical results are designed to deal with this problem.

Proof of Theorem 4.6. Adopting an earlier idea we decompose smoothly the domain of integration appearing in the definition of $K_{2,R}$ as $[0, R] = [0, 2\rho] \cup [2\rho, R]$ by means of a partition of unity $g_1 + g_2 \equiv 1$ on $[0,\infty)$, where $g_2 \in C^{\infty}_{\text{even}}(\mathbb{R})$ is chosen so that $g_2(\lambda) \equiv 1$ for $\lambda > 2\rho$, and $\operatorname{supp} g_2 \subset [\rho, \infty)$. The kernel $K_{2,R}$ decomposes accordingly as $K_{2,R} = K^E_{2,R} + K^M_{2,R}$, where $K^E_{2,R}$—to be regarded as an error term—is

$$\begin{aligned}2K^E_{2,R}(t,r) &= \int_{\mathbb{R}} e^{(i\lambda-\rho)(r+t)}\phi_\lambda(t)\phi_\lambda(r)g_1(\lambda)\frac{\mathbf{c}(\lambda)^2}{|\mathbf{c}(\lambda)|^2}\,d\lambda \\ &\quad + \int_{\mathbb{R}} e^{i\lambda(t-r)-\rho(t+r)}\phi_\lambda(t)\phi_{-\lambda}(r)g_1(\lambda)\underbrace{\frac{\mathbf{c}(\lambda)\mathbf{c}(-\lambda)}{|\mathbf{c}(\lambda)|^2}}_{=1}\,d\lambda \\ &\quad + \int_{\mathbb{R}} e^{i\lambda(r-t)-\rho(t+r)}\phi_{-\lambda}(t)\phi_\lambda(r)g_1(\lambda)\frac{\mathbf{c}(-\lambda)\mathbf{c}(\lambda)}{|\mathbf{c}(\lambda)|^2}\,d\lambda\end{aligned}$$

$$+\int_{\mathbb{R}} e^{(-i\lambda-\rho)(r+t)}\phi_{-\lambda}(t)\phi_{-\lambda}(r)g_1(\lambda)\frac{\mathbf{c}(-\lambda)^2}{|\mathbf{c}(\lambda)|^2}\,d\lambda.$$

Since $|\mathbf{c}(\pm\lambda)^2/|\mathbf{c}(\lambda)|^2|=1$, all four terms are estimated in a similar manner, as follows: Let $\mathcal{F}_\lambda$ denote the Euclidean Fourier transform in λ. Then

$$\Big|\int_{\mathbb{R}} e^{i\lambda x}\phi_{\pm}\lambda(t)\phi_{\pm}\lambda(r)g_1(\lambda)\,d\lambda\Big| = |\mathcal{F}_\lambda(\phi_{\pm\lambda}(t)\phi_{\pm\lambda}(r)g_1(\lambda))(x)| \le \begin{cases} c & \text{if } |x|<1,\\ 1/x^2 & \text{if } |x|>1,\end{cases}$$

that is, the functions $\int_{\mathbb{R}} e^{i\lambda(t-r)}\phi_{\pm\lambda}(t)\phi_{\pm\lambda}(r)g_1(\lambda)\,d\lambda$ are Lebesgue integrable on $\mathbb{R}$ with respect to t and r separately, with L^1-norm independent of either t or r. Additionally,

$$\Big|\int_{\mathbb{R}} e^{i\lambda(t+r)}\phi_{\pm\lambda}(t)\phi_{\pm\lambda}(r)\frac{\mathbf{c}(\mp\lambda)^2}{|\mathbf{c}(\lambda)|^2}g_1(\lambda)\,d\lambda\Big| \le \frac{c}{t+r},$$

so that the maximal operator associated with K_R^E is well behaved.

To avoid notational clutter we will now indicate how to proceed with estimates for integrands of the form $\phi_\pm(t)\phi_\pm(r)e^{i\lambda(t\pm r)}g_1(\lambda)\mathbf{c}(\mp\lambda)^2/|\mathbf{c}(\lambda)|^2$; here we allow all possible combinations of signs on λ and r. Use the first few terms in the Harish-Chandra series expansion for ϕ_λ to write $\phi_\lambda(t) = \Lambda_0(\lambda,t)+\Lambda_1(\lambda,t)e^{-2t}+\mathcal{E}_2(\lambda,t)e^{-4t}$. Strictly speaking we would obtain 12 terms in the expansion of $K_{2,R}^M$ upon inserting the Harish-Chandra series for $\phi_\pm(\lambda)$. By abuse of notation we simply write the decomposition of $K_{2,R}^M$ as $K_{2,R}^M = K_{2,R}^{M,0}+K_{2,R}^{M,1}+\mathrm{E}$, where E is whatever remains. More precisely

$$K_{2,R}^{M,0}(t,r) = e^{-\rho(t+r)}\Big[\int_{-R}^{R} e^{i\lambda(t-r)}g_2(\lambda)\Lambda_0(\lambda,t)\Lambda_0(\lambda,r)\,d\lambda + \int_{-R}^{R} e^{i\lambda(t+r)}g_2(\lambda)\Lambda_0(\lambda,t)\Lambda_0(\lambda,r)\frac{\mathbf{c}(\mp\lambda)^2}{|\mathbf{c}(\lambda)|^2}\,d\lambda + \text{similar terms}\Big].$$

Since $\Lambda_0(\lambda,t)=1+\sum_{k=1}^\infty \gamma_0^k(\lambda)e^{-2kt}$, one has

$$\int_{-R}^{R} e^{i\lambda(t-r)}g_2(\lambda)\,d\lambda = -\int_{\mathbb{R}} e^{i\lambda(t-r)}g_1(\lambda)\,d\lambda + \int_{-R}^{R} e^{i\lambda(t-r)}\,d\lambda = -\widehat{g_1}(t-r)+\frac{e^{iR(t-r)}}{t-r}-\frac{e^{-iR(t-r)}}{t-r},$$

where $\widehat{g_1}$ is Lebesgue integrable on $\mathbb{R}$. Proposition 4.4 is therefore applicable. As for the remaining terms in $\Lambda_0(\lambda,t)\Lambda_0(\lambda,r)$, we use the fact that $\gamma_0^k(\lambda)$ is in fact a constant (cf. [25, p. 262]). Since all these terms decay exponentially fast, their associated maximal operators will be L^s-bounded in the full range $1 < s < \infty$ and therefore uninteresting.

Moreover, by integration by parts,

$$\int_{-R}^{R} e^{i\lambda(t+r)} G_2(\lambda)\, d\lambda = 2\int_{\rho}^{R} e^{i\lambda(t+r)} G_2(\lambda)\, d\lambda$$

$$= \left[\frac{e^{i\lambda(t+r)}}{t+r} G_2(\lambda)\right]_{\rho}^{R} + \left[\frac{e^{i\lambda(t+r)}}{(t+r)^2} G_2'(\lambda)\right]_{\rho}^{R} + \frac{1}{(t+r)^2}\int_{\rho}^{R} e^{i\lambda(t+r)} G_2''(\lambda)\, d\lambda$$

where

$$G_2(\lambda) = \frac{\mathbf{c}(\mp\lambda)^2}{|\mathbf{c}(\lambda)|^2} g_2(\lambda) \varLambda_0(\lambda, t) \varLambda_0(\lambda, r).$$

For $\lambda \geq \rho$ we have $|G_2(\lambda)| \leq c$, $|G_2'(\lambda)| \leq c/\lambda$, and $|G_\lambda''| \leq c/\lambda^2$, meaning that

$$\Big|\int_{-R}^{R} e^{i\lambda(t+r)} G_2(\lambda)\, d\lambda\Big| \leq \frac{c}{t+r}.$$

The remaining piece of $K_{2,R}$ is slightly more troublesome, but since the γ_0^k in the expansion $\varLambda_0(\lambda, t) = 1 + \sum_k \gamma_0^k(\lambda) e^{-2kt}$ are constants, we can simplify the investigation at hand by writing

$$K_{2,R}^{M,1}(t, r) = e^{-\rho(t+r)}\Bigg[\int_{-R}^{R} e^{i\lambda(t-r)} H_-(t, r, \lambda)\, d\lambda + \int_{-R}^{R} e^{i\lambda(t+r)} H_+(t, r, \lambda) \frac{\mathbf{c}(\mp\lambda)^2}{|\mathbf{c}(\lambda)|^2}\, d\lambda + \text{similar terms}\Bigg],$$

with

$$H_\pm(t, r, \lambda) = \{\varLambda_1(\pm\lambda, r) e^{-2r} + \varLambda_1(\lambda, t) e^{-2t} + \varLambda_1(\pm\lambda, r)\varLambda_1(\lambda, t) e^{-2r} e^{-2t}\} g_2(\lambda).$$

The kernels associated with the indicated three pieces of, say, H_-, are all estimated in the same manner, so let us simply consider the first term; it gives rise to the kernel

$$e^{-2r}\int_{-R}^{R} e^{i\lambda(t-r)} \varLambda_1(-\lambda, r) g_2(\lambda)\, d\lambda = e^{-2r}\sum_{j=0}^{\infty}\int_{-R}^{R} e^{i\lambda(t-r)} \gamma_1^{1+j}(\lambda) g_2(\lambda)\, d\lambda$$

$$= e^{-2r}\sum_{j=0}^{\infty} e^{-2jr}\big(\widehat{1_{[-R,R]}} * (\gamma_1^{1+j} g_2)^\wedge\big)(t - r),$$

where we use the estimate $|\gamma_1^{1+j}(\lambda)| \le c(e^{2(1+j)}/\lambda)$ for $\lambda \ge \rho$ to conclude that the Euclidean Fourier transform of $\gamma_1^{1+j} g_2$ is Lebesgue integrable on $\mathbb{R}$, whereas the Fourier transform of $1_{[-R,R]}$ is e^{iRx}/x. Again Proposition 4.4 applies.

The kernel associated with H_+ is slightly different, but the kernel associated with each of the three terms in H_+ satisfies

$$\left| e^{-2r} \int_{-R}^{R} e^{i\lambda(t+r)} \Lambda_1(\lambda, r) \frac{\mathbf{c}(\mp\lambda)^2}{|\mathbf{c}(\lambda)|^2} g_2(\lambda)\, d\lambda \right| \le \frac{e^{-2r}}{t+r}.$$

The final remaining piece $K_{2,R}^M - K_{2,R}^{M,0} - K_{2,R}^{M,1}$ is easily bounded by $\int_\rho^R \lambda^{-2}\, d\lambda \le c$, for all $R > 1$, thereby completing the proof. ■

4.3. Investigation of $S_{3,*}$. Recall that we are concerned with the operator

$$S_{3,R} f(t) = \int K_{3,R}(t,r) f(r) \Delta(r)\, dr$$

in the region where $r > R_0$ and $t < R_0/2$.

Lemma 4.7. *For $r > R_0$, $t < R_0/2$, and $R > 1$ we have*

$$|K_{3,R}(t,r)| \lesssim \frac{e^{-\rho t}}{t^{\alpha+1/2}} \frac{1}{r}.$$

We shall prove the lemma in a moment but first we observe that it leads to the desired bound on the relevant maximal function $S_{3,*}$. Indeed, by Lemma 4.7,

$$\begin{aligned}
|S_{3,R} f(t)| &\lesssim \frac{1}{t^{\alpha+1/2}} \int_{R_0}^{\infty} |f(r)| \frac{e^{\rho r}}{r}\, dr \lesssim \frac{\|f\|_{L^p(d\mu)}}{t^{\alpha+1/2}} \int_{R_0}^{\infty} \frac{e^{\rho r(1-2/p)p'}}{r^{p'}}\, dr \\
&\le \frac{c_p}{t^{\alpha+1/2}} \|f\|_{L^p(d\mu)}
\end{aligned}$$

for $1/p + 1/p' = 1$, $1 < p \le 2$. By typical arguments we conclude that $\|S_{3,*} f\|_{L^p(\mathbb{R},d\mu)} \le c_p \|f\|_{L^p(\mathbb{R}_+,d\mu)}$ for $1 < p \le 2$.

Proof of Lemma 4.7. Observing that $1/|t-r| \lesssim 1/r$ and $1/(t+r) \lesssim 1/r$, we decompose the kernel $K_{3,R}(t,s)$ as $K_{3,R} = K_{3,R}^{(1)} + K_{3,R}^{(2)} + K_{3,R}^{(3)}$ where

$$K_{3,R}^{(1)}(t,r) = \int_0^{2\rho} \varphi_\lambda(t) \varphi_\lambda(r) |\mathbf{c}(\lambda)|^{-2}\, d\lambda,$$

$$K_{3,R}^{(2)}(t,r) = \int_{2\rho}^{1/t} \varphi_\lambda(t) \varphi_\lambda(r) |\mathbf{c}(\lambda)|^{-2}\, d\lambda,$$

$$K^{(3)}_{3,R}(t,r) = \int_{1/t}^{R} \varphi_\lambda(t)\varphi_\lambda(r)|\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

Here

$$K^{(1)}_{3,R}(t,r) = \int_0^{2\rho} \varphi_\lambda(t)\big(\mathbf{c}(\lambda)e^{(i\lambda-\rho)r}\phi_\lambda(r) + \mathbf{c}(-\lambda)e^{(-i\lambda-\rho)r}\phi_{-\lambda}(r)\big)|\mathbf{c}(\lambda)|^{-2}\,d\lambda$$

$$= e^{-\rho r}\left[\frac{1}{ir}e^{i\lambda r}\frac{\varphi_\lambda(t)\phi_\lambda(r)}{\mathbf{c}(-\lambda)}\right]_0^{2\rho} - \frac{1}{ir}e^{-\rho r}\int_0^{2\rho} e^{i\lambda r}\frac{d}{d\lambda}\left(\frac{\varphi_\lambda(t)\phi_\lambda(r)}{\mathbf{c}(-\lambda)}\right)d\lambda$$

$$+ e^{-\rho r}\left[-\frac{1}{ir}e^{-i\lambda r}\frac{\varphi_\lambda(t)\phi_{-\lambda}(r)}{\mathbf{c}(\lambda)}\right]_0^{2\rho} + \frac{1}{ir}e^{-\rho r}\int_0^{2\rho} e^{-i\lambda r}\frac{d}{d\lambda}\left(\frac{\varphi_\lambda(t)\phi_{-\lambda}(r)}{\mathbf{c}(\lambda)}\right)d\lambda$$

by integration by parts. By Lemma 2.1 and Remark 4.2, the derivatives $\frac{d}{d\lambda}\big(\frac{\varphi_\lambda(t)\phi_\lambda(r)}{\mathbf{c}(-\lambda)}\big)$ and $\frac{d}{d\lambda}\big(\frac{\varphi_\lambda(t)\phi_{-\lambda}(r)}{\mathbf{c}(\lambda)}\big)$ are bounded for $0 \le \lambda \le 2\rho$, so $|K^{(1)}_{3,R}(t,r)| \lesssim e^{-\rho r}/r$. Note that this estimate is stronger than what was stated in Lemma 4.7, since for small t the factor $1/t^{\alpha+1/2}$ would become large, implying a very poor kernel estimate.

Integration by parts, now for $K^{(2)}_{3,R}$, shows that

$$K^{(2)}_{3,R}(t,r) = e^{-\rho r}\left[\frac{1}{ir}e^{i\lambda r}\frac{\varphi_\lambda(t)\phi_\lambda(r)}{\mathbf{c}(-\lambda)}\right]_{2\rho}^{1/t} - \frac{1}{ir}e^{-\rho r}\int_{2\rho}^{1/t} e^{i\lambda r}\frac{d}{d\lambda}\left(\frac{\varphi_\lambda(t)\phi_\lambda(r)}{\mathbf{c}(-\lambda)}\right)d\lambda$$

$$+ e^{-\rho r}\left[-\frac{1}{ir}e^{-i\lambda r}\frac{\varphi_\lambda(t)\phi_{-\lambda}(r)}{\mathbf{c}(\lambda)}\right]_{2\rho}^{1/t} + \frac{1}{ir}e^{-\rho r}\int_{2\rho}^{1/t} e^{-i\lambda r}\frac{d}{d\lambda}\left(\frac{\varphi_\lambda(t)\phi_{-\lambda}(r)}{\mathbf{c}(\lambda)}\right)d\lambda$$

where we utilize the estimates $|\varphi_\lambda(t)| \le c$, $|\phi_{\pm\lambda}(r)| \le c$, $|1/\mathbf{c}(\pm\lambda)| \le c\lambda^{\alpha+1/2}$, $|\varphi'_\lambda(t)| \le c/\lambda$, $|\phi'_{\pm\lambda}(r)| \le ce^{-2r}/\lambda$, and $\big|\frac{d}{d\lambda}(\varphi_\lambda(t)\phi_{\pm\lambda}(r)/\mathbf{c}(\pm\lambda))\big| \le c\lambda^{\alpha-1/2}$ to conclude that

$$|K^{(2)}_{3,R}(t,r)| \le c\frac{e^{-\rho r}}{t^{\alpha+1/2}}\,\frac{1}{r}.$$

It remains to study $K^{(3)}_{3,R}$ but since λ is allowed to become either very large (when R is large) or small (less than one, at least), here we have to be slightly more careful in the estimates, especially since the proof in [18] leaves out most terms in the calculation (similar to what happened in the analysis of $K_{2,R}$). First write

$$\varphi_\lambda(t) = \frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\left[\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha} + E_1(\lambda,t)\right],$$

so that

$$K^{(3)}_{3,R}(t,r) = e^{-\rho r}\frac{t^{1/2}}{\sqrt{\Delta(t)}}\int_{1/t}^{R}\frac{J_\alpha(\lambda t)}{\lambda^\alpha}\phi_\lambda(r)e^{i\lambda r}\frac{d\lambda}{\mathbf{c}(-\lambda)}$$
$$+ e^{-\rho r}\frac{t^{1/2}}{\sqrt{\Delta(t)}}\int_{1/t}^{R}\frac{J_\alpha(\lambda t)}{\lambda^\alpha}\phi_{-\lambda}(r)e^{-i\lambda r}\frac{d\lambda}{\mathbf{c}(\lambda)}$$
$$+ e^{-\rho r}\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\int_{1/t}^{R}E_1(\lambda,t)\phi_\lambda(r)e^{i\lambda r}\frac{d\lambda}{\mathbf{c}(-\lambda)}$$
$$+ e^{-\rho r}\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\int_{1/t}^{R}E_1(\lambda,t)\phi_{-\lambda}(r)e^{-i\lambda r}\frac{d\lambda}{\mathbf{c}(\lambda)}$$

where the first two terms satisfy the same estimates. We therefore concentrate on the first one. To this end recall that $J_\alpha(t) \sim t^{-1/2}\cos\big(t - \frac{2\alpha+1}{4}\pi\big) + O(t^{-3/2})$ for $t \to \infty$, so that

$$\int_{1/t}^{R}\frac{J_\alpha(\lambda)}{\lambda^\alpha}\phi_\lambda(r)e^{i\lambda r}\frac{d\lambda}{\mathbf{c}(-\lambda)} = \gamma_1 t^{-1/2}\int_{1/t}^{R}\frac{e^{i\lambda(r+t)}}{\lambda^{\alpha+1/2}}\phi_\lambda(r)\frac{d\lambda}{\mathbf{c}(-\lambda)}$$
$$+ \gamma_{-1}t^{-1/2}\int_{1/t}^{R}\frac{e^{i\lambda(r-t)}}{\lambda^{\alpha+1/2}}\phi_\lambda(r)\frac{d\lambda}{\mathbf{c}(-\lambda)} + \int_{1/t}^{R}\frac{E(\lambda)}{\lambda^{\alpha+1/2}}e^{i\lambda r}\phi_\lambda(r)\frac{d\lambda}{\mathbf{c}(-\lambda)}$$

where $\gamma_{\pm 1} = \exp\big(\pm i\frac{2\alpha+1}{4}\pi\big)$. Since $E(\lambda) \sim O(\lambda^{-3/2})$ and $\mathbf{c}(-\lambda)^{-1} \sim \lambda^{\alpha+1/2}$, the third integral is easily bounded. As for the first two integrals, we have

$$\int_{1/t}^{R}\frac{e^{i\lambda(r\pm t)}}{\lambda^{\alpha+1/2}}\phi_\lambda(r)\frac{d\lambda}{\mathbf{c}(\mp\lambda)} = \left[\frac{e^{i\lambda(r\pm t)}}{i(r\pm t)}\frac{\phi_\lambda(r)}{\lambda^{\alpha+1/2}\mathbf{c}(\mp\lambda)}\right]_{1/t}^{R}$$
$$- \frac{1}{i(r\pm t)}\int_{1/t}^{R}e^{i\lambda(r\pm t)}\frac{d}{d\lambda}\left(\frac{\phi_\lambda(r)}{\lambda^{\alpha+1/2}\mathbf{c}(\mp\lambda)}\right)d\lambda.$$

The first term is dominated by c/r, which is what we need, whereas the second term is controlled by an additional integration by parts. This gives rise to the additional contribution

$$\left[\frac{e^{i\lambda(r\pm t)}}{(r\pm t)^2}\frac{d}{d\lambda}\left(\frac{\phi_\lambda(r)}{\lambda^{\alpha+1/2}\mathbf{c}(\mp\lambda)}\right)\right]_{1/t}^{R} + \frac{1}{(r\pm t)^2}\int_{1/t}^{R}e^{i\lambda(r\pm t)}\frac{d^2}{d\lambda^2}\left(\frac{\phi_\lambda(r)}{\lambda^{\alpha+1/2}\mathbf{c}(\mp\lambda)}\right)d\lambda,$$

where the first term is bounded by c/r^2 and where the integral is bounded by

$$\frac{c}{r^2}\int_{1/t}^{R}\frac{1}{\lambda^2}\,d\lambda \le \frac{c}{r^2}.$$

Collecting powers in t (noting that $\sqrt{\Delta(t)} \sim t^{\alpha+1/2}$ for $t < R_0$ and that we gained the factor $t^{-1/2}$ when estimating $\int_{1/t}^{R} J_\alpha(\lambda t)\lambda^{-\alpha}\phi_\lambda(r)e^{i\lambda r}\mathbf{c}(-\lambda)^{-1}d\lambda$), the required kernel estimate follows. The remaining terms in the decomposition of $K^{(3)}_{3,R}$ are treated analogously. ∎

Remark 4.8. The proof is as in [18] but it must be pointed out that the proof in [18] has a technical gap, in that the authors ignore $\phi_{-\lambda}$ and $\mathbf{c}(-\lambda)$ in the estimates. The results on asymptotic properties of φ_λ and $|\mathbf{c}(\lambda)|^{-2}$ are stated under the assumption that λ is nonnegative, so one must be more careful. Moreover, as we do not complex conjugate anywhere, and since the original proof was a bit short, we have filled out the gaps along the way. Lemma 2.1 and the results on the asymptotic behavior of $\varphi_\lambda(t)$ have been stated and proved in a way that repairs this small deficiency.

4.4. Investigation of $S_{4,*}$. It remains to analyze $S_{4,*}$ but the required estimates follow at once from those for $S_{3,*}$ once we interchange t and r. The resulting kernel estimate is

$$|K_{4,R}(t,r)| \le c\frac{e^{-\rho t}}{t}\frac{1}{r^{\alpha+1/2}} \quad \text{for } r < R_0/2 \text{ and } t > R_0, \tag{4.6}$$

which implies that

$$\begin{aligned}|S_{4,R}f(t)| &\lesssim \int_0^{R_0/2}\frac{e^{-\rho t}}{t}\frac{1}{r^{\alpha+1/2}}|f(r)|\Delta(r)\,dr\\ &\lesssim \frac{e^{-\rho t}}{t}\|f\|_{L^p}\Big(\int_0^{R_0/2} r^{-(\alpha+1/2)p'+2\alpha+1}\,dr\Big)^{1/p'},\end{aligned}$$

which is finite whenever $p' < \frac{4\alpha+4}{2\alpha+1}$. Unlike the operator $S_{1,*}$, the L^p-boundedness of $S_{4,*}$ does not require an additional constraint on the range of p. Indeed $|S_{4,*}f(t)| \lesssim \frac{e^{-\rho t}}{t}\|f\|_{L^p}$, whence

$$\begin{aligned}\|S_{4,*}f\|^2_{L^2(d\mu)} &= \int_{R_0}^{\infty}|S_{4,*}f(t)|^2\Delta(t)\,dt\\ &\lesssim \int_{R_0}^{\infty}\frac{e^{-2\rho t}}{t^2}\|f\|^2_{L^p}\Delta(t)\,dt \lesssim \|f\|^2_{L^p}\int_{R_0}^{\infty}\frac{dt}{t^2} \lesssim \|f\|^p_{L^p}.\end{aligned}$$

In other words $\|S_{4,*}f\|_{L^2(d\mu)} \le c\|f\|_{L^p}$ for $\frac{4\alpha+4}{2\alpha+3} < p$.

4.5. Divergence at $p = p_0$. We will presently prove Theorem 3.3 regarding the existence of a particularly unpleasant function $f \in L^{p_0}(d\mu)$. The technique is an easy extension of the one used to establish [18, Theorem 4], which we review for the sake of completeness. It was already used in [17], which in turn was an extension of the classical Cantor–Lebesgue Lemma (for trigonometric series) to the setting of Jacobi polynomials on $[-1,1]$.

In the following, one should think of the parameter α (which Meaney and Kanjin use for the Hankel transform) as our Jacobi parameter α; the point is that we may ignore the other Jacobi parameter β when we are merely interested in the local (Euclidean) behavior of the Jacobi functions. So assume $\alpha \geq -1/2$, $p \in [1,\infty)$, and $0 \leq a < b \leq \infty$. Let $L^p_\alpha((a,b))$ denote the space of all measurable functions g on $\mathbb{R}_+$ for which

$$\|g\|_{\alpha,p} = \Big(\int_a^b |g(t)|^p t^{2\alpha+1}\,dt\Big)^{1/p} < \infty.$$

Lemma 4.9. *Assume $\frac{4\alpha+2}{2\alpha+3} \leq p \leq 2$, and that $F \in L^{p'}_\alpha((1,\infty))$ has the property that*

$$\lim_{R\to\infty} \int_R^{R+h} F(\lambda)\Big(\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}\Big)\lambda^{2\alpha+1}\,d\lambda = 0$$

uniformly in $h \in [0,1]$. Then

$$\lim_{R\to\infty} \int_R^{R+h} F(\lambda)\lambda^{\alpha+1/2}\,d\lambda = 0$$

uniformly in $h \in [0,1]$.

We refer to [15] for a proof. The lemma will be applied to $F = \widehat{f}$, which is permissible since the Hausdorff–Young inequality implies that $\|\widehat{f}\|_{L^{p'}(d\nu)} \lesssim \|f\|_{L^p(d\mu)}$ whenever f belongs to $L^p(d\mu)$. Since $|\mathbf{c}(\lambda)|^{-2} \sim \lambda^{2\alpha+1}$ for $\lambda \to \infty$, it thus follows that $\widehat{f}|_{[1,\infty)} \in L^{p'}_\alpha((1,\infty))$.

Lemma 4.10. *Assume that $p \in \big[\frac{4\alpha+2}{2\alpha+3}, 2\big]$ and $f \in L^p(\mathbb{R}_+, d\mu)$ has the property that $\lim_{R\to\infty} S_R f(t)$ exists for every t in a subset $E \subset [0,1]$ of positive measure. Then*

$$\lim_{R\to\infty} \int_R^{R+h} \widehat{f}(\lambda)|\mathbf{c}(\lambda)|^{-1}\,d\lambda = 0 \tag{4.7}$$

uniformly in $h \in [0,1]$.

The proof of Theorem 3.3 will be completed once we produce a function $f \in L^{p_0}(\mathbb{R}_+, d\mu)$ that violates the conclusion (4.7).

Proof. We may assume without loss of generality that E is contained in an interval of the form $[\varepsilon, 1]$, $\lambda > 1/\varepsilon$. Using that

$$\varphi_\lambda(t) = c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\left(\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha} + t^2 a_1(t)\frac{J_{\alpha+1}(\lambda t)}{(\lambda t)^{\alpha+1}}\right) + E_2(\lambda, t)$$

with $|E_2(\lambda,t)| \lesssim t^{2-(\alpha+1/2)}\lambda^{-(2\alpha+5)/2}$ for $|\lambda t| > 1$, it suffices (due to the fact that $|\mathbf{c}(\lambda)|^{-2}/\lambda^{2\alpha+1} \lesssim 1$ for $\lambda \to \infty$) to show that *if*

$$\lim_{R\to\infty}\int_R^{R+h} \widehat{f}(\lambda)\varphi_\lambda(t)|\mathbf{c}(\lambda)|^{-2}\,d\lambda = 0$$

for $\varepsilon < t < 1$, then

$$\lim_{R\to\infty}\int_R^{R+h} \widehat{f}(\lambda)\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}\lambda^{2\alpha+1}\,d\lambda = 0,$$

since Lemma 4.9 is then applicable. Note that the conclusion is not automatic, since the integrands are not always positive. There could be lots of oscillation going on that would prevent the requirement in Lemma 4.9 to be satisfied. We must therefore prove the statements

(i) $$\lim_{R\to\infty}\int_R^{R+h} \widehat{f}(\lambda)\frac{J_{\alpha+1}(\lambda t)}{(\lambda t)^{\alpha+1}}\lambda^{2\alpha+1}\,d\lambda = 0,$$

(ii) $$\lim_{R\to\infty}\int_R^{R+h} \widehat{f}(\lambda)E_2(\lambda,t)\lambda^{2\alpha+1}\,d\lambda = 0.$$

For (i), it follows from the usual Bessel function estimate $|J_\mu(x)| \lesssim x^{-1/2}$ for large x that

$$\left|\int_R^{R+h} \widehat{f}(\lambda)\frac{J_{\alpha+1}(\lambda t)}{(\lambda t)^{\alpha+1}}\lambda^{2\alpha+1}\,d\lambda\right| \le ct^{-(2\alpha+3)/2}\int_R^{R+h} |\widehat{f}(\lambda)|\lambda^{-(2\alpha+3)/2}\lambda^{2\alpha+1}\,d\lambda$$
$$\le ct^{-(2\alpha+3)/2}\Big(\int_R^{R+h} |\widehat{f}(\lambda)|^{p'}\lambda^{2\alpha+1}\,d\lambda\Big)^{1/p'}\Big(\int_R^{R+h} \lambda^{-(2\alpha+3)/2p}\lambda^{2\alpha+1}\,d\lambda\Big)^{1/p}$$

where the first integral is bounded by $\|\widehat{f}\|_{L^{p'}_\alpha((1,\infty))}$. The second integral is roughly of size $(hR^{-(2\alpha+3)p/3+2\alpha+1})^{1/p}$, which tends to zero as $R\to\infty$, since the assumption that p be larger than $\frac{4\alpha+2}{2\alpha+3}$ implies that $2\alpha+1-(\alpha+3/2) < 0$. Therefore (i) holds; the proof of (ii) is just as easy. ■

Proof of Theorem 3.3. Let

$$F_R(f) = \int_R^{R+1} \widehat{f}(\lambda)|\mathbf{c}(\lambda)|^{-1}\,d\lambda = \int_0^1\Big\{\int_R^{R+1} \varphi_\lambda(t)|\mathbf{c}(\lambda)|^{-1}\,d\lambda\Big\}f(t)\Delta(t)\,dt$$

for $R > 0$ and $f \in L^p(d\mu)$ with $\operatorname{supp} f \subset [0,1]$. It is seen that the operator norm of F_R is precisely

$$\|F_R\| = \Big(\int_0^1 \Big|\int_R^{R+1} \varphi_\lambda(t)|\mathbf{c}(\lambda)|^{-1}\,d\lambda\Big|^{p'} \Delta(t)\,dt\Big)^{1/p'},$$

which in turn is the norm of $t \mapsto \int_R^{R+1} \varphi_\lambda(t)|\mathbf{c}(\lambda)|^{-1}\,d\lambda$ in $L^{p'}([0,1],\Delta(t)dt)$. Keeping in mind that

$$\varphi_\lambda(t) = c\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha} + E_1(\lambda,t)$$

with $|E_1(\lambda,t)| \lesssim t^2(\lambda t)^{-(2\alpha+3)/2}$ for $|\lambda t| > 1$, we infer from the proof of [15, Lemma 2] (see also the proof of [14, Lemma 1]) that

$$\Big\|t \mapsto \int_R^{R+1} \frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}\lambda^{\alpha+1/2}\,d\lambda\Big\|_{L^{p'}([0,1],\Delta(t)dt)} \sim \Big\|t \mapsto \int_R^{R+1} \frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha}\lambda^{\alpha+1/2}\,d\lambda\Big\|_{L^{p'}([0,1],t^{2\alpha+1}dt)} \gtrsim (\log R)^{1/p'}.$$

Moreover

$$\int_{1/R}^1 \Big|\int_R^{R+1} E_1(\lambda,t)\lambda^{\alpha+1/2}\,d\lambda\Big|^{p'} \Delta(t)\,dt$$

is uniformly bounded in R, when we take $p = p_0$. Indeed,

$$\begin{aligned}\int_{1/R}^1 \Big|\int_R^{R+1} E_1(\lambda,t)\lambda^{\alpha+1/2}\,d\lambda\Big|^{p'} \Delta(t)\,dt &\le \int_{1/r}^1 \Big|\int_R^{R+1} c_1 t^2(\lambda t)^{-(2\alpha+3)/2}\lambda^{\alpha+1/2}\,d\lambda\Big|^{p'} \Delta(t)\,dt\\ &\le \int_{1/R}^1 \Big|\int_R^{R+1} c_1\lambda^{-1}\,d\lambda\Big|^{p'} t^{p'(2-(2\alpha+3)/2)}t^{2\alpha+1}\,dt\\ &= c\Big(\log\frac{R+1}{R}\Big)^{p'} \int_{1/R}^1 t^{\frac{2\alpha+3}{2\alpha+1}}\,dt = c'\Big(\log\frac{R+1}{R}\Big)^{p'} \big[t^{\frac{4\alpha+4}{2\alpha+1}}\big]_{1/R}^1\\ &= c'\Big(\log\frac{R+1}{R}\Big)^{\frac{4\alpha+4}{2\alpha+1}} R^{-\frac{4\alpha+4}{2\alpha+1}} = o(1)\end{aligned}$$

for $R \to \infty$. By the Banach–Steinhaus theorem there exists a function $f \in$

$L^{p_0}_\alpha((0,1))$ so that

$$\limsup_{R\to\infty}\Big|\int_R^{R+1}\widehat{f}(\lambda)|\mathbf{c}(\lambda)|^{-1}\,d\lambda\Big|=0.$$

It thus follows from Lemma 4.10 that $\{S_R f(t)\}_R$ diverges for almost every $t\in[0,1]$. ∎

5. Proof of the mapping properties for critical exponents. We now prove Theorem 3.5. Since $S_{2,*}$ and $S_{3,*}$ do not behave worse on L^{p_0} than on other L^p-spaces, it suffices to establish the endpoint mapping properties of $S_{1,*}$ and $S_{4,*}$. The endpoint mapping property of $S_{4,*}$ is stated below as Lemma 5.1, so we shall presently concentrate on $S_{1,*}$.

Recall from Subsection 4.1 that we decomposed the integral kernel $K_{1,R}$ for the localized piece $S_{1,R}$ of the disc multiplier into a large collection of pieces. The contributions $K^2_{1,R},\dots,K^5_{1,R}$ are easily handled, so we begin with those:

For $K^1_{1,R}$, we introduced a further decomposition $K^1_{1,R}(t,r)=M_{1,R}(t,r)+E_1(t,r)$, where $M_{1,R}$ was decomposed even further (cf. (4.1)) into functions of the form

$$M^d_{1,R}(t,r)=\frac{t^{\alpha+1/2}}{\sqrt{\Delta(t)}}\frac{r^{\alpha+1/2}}{\sqrt{\Delta(r)}}\frac{1}{r^\alpha}\frac{1}{t^\alpha}\int_1^R J_\alpha(\lambda r)J_\alpha(\lambda t)\lambda^d\,d\lambda,\qquad d=2-M,\dots,1.$$

The operator $S^{M^1_{1,R}}_{1,R}$ associated with $M^1_{1,R}$ was already seen to be controlled by the spherical summation operator for the Hankel transform, so the associated maximal operator has the stated mapping property according to [4]. The case $d=0$ (cf. (4.2)) entails an analysis of three pieces, $M^{0,(1)}_{1,R}$, $M^{0,(2)}_{1,R}$, and $M^{0,(3)}_{1,R}$, corresponding to a suitable smooth partition of the interval $[1,R]$.

The piece $M^{0,(1)}_{1,R}(t,r)$ was seen to satisfy the estimate $|M^{0,(1)}_{1,R}(t,r)|\lesssim t^{-(\alpha+1/2)}r^{-(\alpha+1/2)}$, whence

$$\begin{aligned}|S^{M^{0,(1)}_{1,R}}_{1,R}f(t)|&\lesssim\int_0^{R_0}|M^{0,(1)}_{1,R}(t,r)|\,|f(r)|\Delta(r)\,dr\lesssim\frac{1}{t^{\alpha+1/2}}\int_0^{R_0}\frac{|f(r)|}{r^{\alpha+1/2}}\Delta(r)\,dr\\&\lesssim\frac{1}{t^{\alpha+1/2}}\|f\|_{L^{p_0,1}([0,R_0],d\mu)}\cdot\|r\mapsto r^{-(\alpha+1/2)}\|_{L^{p_1,\infty}([0,R_0],d\mu)}\\&\lesssim\frac{1}{t^{\alpha+1/2}}\|f\|_{L^{p_0,1}(\mathbb{R}_+,d\mu)}.\end{aligned}$$

The relevant level function for $S_{1,*}^{M_{1,R}^{0,(1)}}$ therefore satisfies the estimate

$$d(\lambda)=\mu(\{t\in[0,R_0]:|S_{1,*}^{M_{1,R}^{0,(1)}}f(t)|>\lambda\})\le\frac{1}{\lambda^{p_0}}\int_0^{R_0}|S_{1,*}^{M_{1,R}^{0,(1)}}f(t)|^{p_0}\Delta(t)\,dt$$
$$\lesssim\frac{\|f\|_{L^{p_0,1}}^{p_0}}{\lambda^{p_0}}\int_0^{R_0}t^{2\alpha+1-(\alpha+1/2)p_0}\,dt$$

where the integral is finite since

$$2\alpha+1-\left(\alpha+\frac{1}{2}\right)p_0=2\alpha+1-\left(\alpha+\frac{1}{2}\right)\frac{4\alpha+4}{2\alpha+3}=\frac{2\alpha+1}{2\alpha+3}>0>-1$$

($\alpha>-1/2$ by the standing assumption), implying $\|S_{1,*}^{M_{1,R}^{0,(1)}}f\|_{L^{p_0,\infty}(\mathbb{R}_+,d\mu)}\lesssim\|f\|_{L^{p_0,1}(\mathbb{R}_+,d\mu)}$ as claimed.

The mapping properties of the maximal operator associated with $M_{1,R}^{0,(2)}$ are the same as for $M_{1,R}^{0,(1)}$ since $|M_{1,R}^{0,(2)}(t,r)|\lesssim t^{-(\alpha+1/2)}r^{-(\alpha+1/2)}$.

The most difficult piece, $M_{1,R}^{0,(3)}$, gave rise to an operator that was controlled by the Carleson operator applied to the function $f\sqrt{\Delta}$, the upshot being the estimate (4.3). It is seen by close inspection of the argument on top of page 75 in [4] that the Carleson maximal operator is even bounded from $L^{p_0,1}$ into $L^{p_0,\infty}$ (the underlying measure space now being $\mathbb{R}_+$ with weighted Lebesgue measure $x^{2\alpha+1}dx$), so it follows that the maximal operator $S_{1,R}^{M_{1,R}^{0,(3)}}$ is bounded from $L^{p_0,1}(\mathbb{R}_+,d\mu)$ into $L^{p_0,\infty}(\mathbb{R}_+,d\mu)$. Recall here that $S_{1,R}^{M_{1,R}^{0,(3)}}f(t)$ is only considered for $0\le t\le R_0$, hence the stated result on the Carleson operator is applicable. Hence $S_{1,R}^{M_{1,R}^{0}}$ enjoys the stated endpoint mapping property.

The cases $d=2-M,\dots,-2,-1$ now follow at once; above we have merely used that the relevant kernels were dominated by $ct^{-(\alpha+1/2)}r^{-(\alpha+1/2)}$. Since all the remaining operators satisfy the same estimates, we are effectively done; the maximal operator $S_{1,*}$ is bounded from $L^{p_0,1}(\mathbb{R}_+,d\mu)$ into $L^{p_0,\infty}(\mathbb{R}_+,d\mu)$.

As for $S_{4,*}$ we will state the precise result as a lemma:

LEMMA 5.1. *The maximal operator $S_{4,*}$ is bounded from $L^{p_0,1}(\mathbb{R}_+,d\mu)$ into $L^2(\mathbb{R},d\mu)$.*

Proof. Recall that the level function of a function $f\in L^{p_0}(\mathbb{R}_+,d\mu)$ is defined by $d_f(\lambda) = \mu(\{t : |f(t)| > \lambda\})$, hence

$$d_{S_{4,*}f}(\lambda) \le \frac{1}{\lambda^2} \int_{R_0}^{\infty} |S_{4,*}f(t)|^2 \Delta(t)\,dt$$

$$\lesssim \frac{1}{\lambda^2} \int_{R_0}^{\infty} \left(\frac{e^{-\rho t}}{t}\right)^2 \left| \int_0^{R_0/2} \frac{|f(r)|}{r^{\alpha+1/2}} \Delta(r)\,dr \right|^2 \Delta(t)\,dt.$$

Observe that $\Delta(t)$ grows as $e^{\rho t}$ for $t \to \infty$, so that the t-integrand is dominated by $1/t^2$ on $[R_0, \infty)$. For the inner integral, we intend to use the Lorentz space version of the Hölder inequality, that is (with $p_0 = \frac{4\alpha+4}{2\alpha+3}$),

$$\left| \int_0^{R_0/2} f(r) \cdot r^{-(\alpha+1/2)}\,d\mu(r) \right| \le \|f\|_{L^{p_0,1}([0,R_0/2],d\mu)} \|r^{-(\alpha+1/2)}\|_{L^{p_0',\infty}([0,R_0/2],d\mu)},$$

to which end it suffices to show that $g : [0, R_0/2] \to \mathbb{R}$, $r \mapsto r^{-(\alpha+1/2)}$, belongs to $L^{p_0',\infty}([0, R_0/2], d\mu)$. This is easy: It follows from the estimate

$$\begin{aligned} d_g(\gamma) &= \mu(\{r \in [0, R_0/2] : r^{-(\alpha+1/2)} > \gamma\}) \le \mu(\{r \ge 0 : r^{\alpha+1/2} < 1/\gamma\}) \\ &= \mu(r \in \mathbb{R}_+ : r < \gamma^{-\frac{1}{\alpha+1/2}}) \le (\gamma^{-\frac{1}{\alpha+1/2}})^{2\alpha+2} = \gamma^{-\frac{2\alpha+2}{\alpha+1/2}} \end{aligned}$$

that $d_g(\gamma)^{1/p_0'} \le (\gamma^{-\frac{2\alpha+2}{\alpha+1/2}})^{\frac{2\alpha+1}{4\alpha+4}} = \gamma^{-1}$, and therefore

$$\|g\|_{L^{p_0',\infty}([0,R_0/2],d\mu)} = \sup_{\gamma>0} \gamma d_g(\gamma)^{1/p_0'} \le 1. \ \blacksquare$$

This completes the proof of Theorem 3.5.

Remark 5.2. Lemma 5.1 should be seen as a "non-Euclidean" analogue of the result from [23] and the statement is new even for rank one symmetric spaces.

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