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Weighted norm inequalities for multilinear fractional operators on Morrey spaces

by

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Abstract. A weighted theory describing Morrey boundedness of fractional integral operators and fractional maximal operators is developed. A new class of weights adapted to Morrey spaces is proposed and a passage to the multilinear cases is covered.

1. Introduction. The goal of this paper is to develop a theory of weights for multilinear fractional integral operators and multi(sub)linear fractional maximal operators in the framework of Morrey spaces. We give natural sufficient conditions for one-weight and two-weight inequalities to hold for these operators and, as a corollary, we obtain the Olsen inequality for multilinear fractional integral operators. The results extend to Morrey spaces those in [10] due to Moen.

We first recall some standard notation. All cubes considered are supposed to have their sides parallel to coordinate axes in $\mathbb{R}^n$. We let $\mathcal{Q}$ denote the family of all such cubes. For a cube $Q \in \mathcal{Q}$ we denote by $\ell(Q)$ its side-length and by cQ the cube with the same center as Q but with side-length $c\ell(Q)$.

Morrey spaces, named after C. Morrey, seem to describe precisely the boundedness of fractional integral operators (or Riesz potential operators). We shall show that this remains the case even in the multilinear weighted setting. Let $0 < p \le p_0 < \infty$. The *Morrey space* $\mathcal{M}_p^{p_0}(\mathbb{R}^n) = \mathcal{M}_p^{p_0}$ is defined by the norm (or quasi-norm)

$$\|f\|_{\mathcal{M}_p^{p_0}} = \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \Big(\fint_Q |f(x)|^p \, dx \Big)^{1/p}, \tag{1.1}$$

where $\fint_Q f(x)\,dx$ denotes the usual integral average of f over Q. Applying

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Hölder's inequality to (1.1), we see that

$$\|f\|_{\mathcal{M}^{p_0}_{p_1}} \geq \|f\|_{\mathcal{M}^{p_0}_{p_2}} \quad \text{for all } p_0 \geq p_1 \geq p_2 > 0. \tag{1.2}$$

This tells us that

$$L^{p_0} = \mathcal{M}^{p_0}_{p_0} \subset \mathcal{M}^{p_0}_{p_1} \subset \mathcal{M}^{p_0}_{p_2} \quad \text{for all } p_0 \geq p_1 \geq p_2 > 0. \tag{1.3}$$

Let f be a locally integrable function on $\mathbb{R}^n$, or let $f \in \mathcal{L}_+$, where $\mathcal{L}_+$ denotes the set of all positive Lebesgue measurable functions. The *fractional integral operator* (or the Riesz potential operator) $I_\alpha f(x)$, $0 < \alpha < n$, is given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}.$$

Let $1 < p \leq p_0 < \infty$ and $1 < q \leq q_0 < \infty$. There are two remarkable results on the Morrey boundedness of I_α. The first one is due to Spanne (unpublished): The inequality

$$\|I_\alpha f\|_{\mathcal{M}^{q_0}_{q}} \leq C\|f\|_{\mathcal{M}^{p_0}_{p}} \tag{1.4}$$

holds if

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

This result is recorded as [14, Theorem 5.4, p. 82].

The second one is due to Adams [1] (see also [3]): The inequality (1.4) holds if

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{1}{q} = \frac{p_0}{pq_0}. \tag{1.5}$$

This result is recorded in [2, p. 79, (3.7.2)], although the formulation is made for $(1-\Delta)^{-\alpha/2}$. The same can be said for $(-\Delta)^{-\alpha/2}$ and I_α.

A simple arithmetic shows that

$$\frac{1}{p} - \frac{\alpha}{n} = \frac{p_0}{p}\left(\frac{1}{p_0} - \frac{p}{p_0}\frac{\alpha}{n}\right) \geq \frac{p_0}{p}\left(\frac{1}{p_0} - \frac{\alpha}{n}\right) = \frac{p_0}{pq_0}.$$

This inequality together with (1.3) says that *the Spanne target space is larger than the Adams target space*. Thus, we can say that Adams improved the result of Spanne. And Olsen [13] showed by an example that the result of Adams is optimal.

Let $\vec{f} = (f_1, \ldots, f_m)$ be a collection of m locally integrable functions on $\mathbb{R}^n$. Given α with $0 < \alpha < mn$, we define the *multilinear fractional integral operator* $\mathcal{I}_\alpha(\vec{f})(x)$, $x \in \mathbb{R}^n$, by

$$\mathcal{I}_\alpha(\vec{f})(x) = \int_{y_1,\ldots,y_m \in \mathbb{R}^n} \frac{f_1(y_1)\cdots f_m(y_m)\,d\vec{y}}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}},$$

where $d\vec{y} = dy_1 \cdots dy_m$. Recently, Tang [25] extended Spanne's result to $\mathcal{I}_\alpha$.

Proposition 1.1 ([25]). *Let*

$$0 < \alpha < mn, \qquad 1 < q_i \le p_i \le \infty, \qquad i = 0, 1, \dots, m.$$

Suppose that

$$\frac{1}{p_0} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \cdots + \frac{1}{q_m} - \frac{\alpha}{n}.$$

Then

$$\|\mathcal{I}_\alpha(\vec{f})\|_{\mathcal{M}^{p_0}_{q_0}} \le C \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}.$$

We proposed in [7] the following counterpart of Adams' result for $\mathcal{I}_\alpha$:

Proposition 1.2 ([7]). *Let*

$$0 < \alpha < mn, \quad 1 < p_1, \dots, p_m \le \infty, \quad 0 < p \le p_0 < \infty, \quad 0 < q \le q_0 < \infty.$$

Suppose that

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}, \qquad \frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{1}{q} = \frac{p_0}{pq_0}.$$

Then

$$\|\mathcal{I}_\alpha(\vec{f})\|_{\mathcal{M}^{q_0}_{q}} \le C \sup_{Q \in \mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i} \Big)^{1/p_i}. \tag{1.6}$$

We named the right-hand side of (1.6) the *multi-Morrey norm* [7, (1.3)]. Denote by $\delta(x)$ the Dirac measure on the real line which concentrates unit point mass at the origin. For $m = 2$, $n = 1$, $f_1(x) = \delta(x)$ and $f_2(x) = \delta(x-1)$, and for any exponents $1 < p_1, p_2 < \infty$,

$$\|f_1\|_{\mathcal{M}^{p_1}_1} \|f_2\|_{\mathcal{M}^{p_2}_1} = \infty \times \infty = \infty \tag{1.7}$$

while

$$\sup_{a<b} (b-a)^{1/p_1 + 1/p_2} \Big(\fint_a^b f_1(x)\,dx \Big) \Big(\fint_a^b f_2(x)\,dx \Big) = 1. \tag{1.8}$$

The equalities (1.7) and (1.8) tell us that, in general, the multi-Morrey norm is strictly smaller than the m-fold product of the Morrey norms. Moreover, the following seems to be of some interest.

Remark 1.3. Under the assumptions of Proposition 1.2, we set, for $x \in \mathbb{R}^n$,

$$\frac{1}{F(x)} = \sup_{x \in Q \in \mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i} \Big)^{1/p_i}.$$

Then

$$\mathcal{I}_\alpha((f_1,\dots,f_m,F))(x) = \int_{y_1,\dots,y_m,y_{m+1}\in\mathbb{R}^n} \frac{f_1(y_1)\cdots f_m(y_m)F(y_{m+1})\,d\vec{y}}{(|x-y_1|+\cdots+|x-y_m|+|x-y_{m+1}|)^{(m+1)n-\alpha}}$$

belongs to $\mathcal{M}^{q_0}_q$. Indeed, noticing that for any $Q\in\mathcal{Q}$ and $x\in Q$,

$$\frac{1}{F(x)} \ge |Q|^{1/p_0}\prod_{i=1}^m \Big(⨍_Q |f_i|^{p_i}\Big)^{1/p_i},$$

we have

$$\sup_{Q\in\mathcal{Q}}\Big\{|Q|^{1/p_0}\prod_{i=1}^m\Big(⨍_Q |f_i|^{p_i}\Big)^{1/p_i}\cdot \sup_{x\in Q}F(x)\Big\}\le 1.$$

Applying Proposition 1.2 with m replaced by $m+1$ and with exponents $p_1,\dots,p_m,\infty$, we obtain the result.

In this paper a *weight* is simply a nonnegative measurable function w on $\mathbb{R}^n$. The Hardy–Littlewood maximal operator and its weighted norm estimates play an important role in harmonic analysis. In particular they are essential to the study of singular integral operators. Let f be a locally integrable function on $\mathbb{R}^n$. The *fractional maximal operator* $M_\alpha f(x)$, $0\le\alpha<n$, is given by

$$M_\alpha f(x) = \sup_{x\in Q\in\mathcal{Q}} \ell(Q)^\alpha ⨍_Q |f(y)|\,dy.$$

When $\alpha=0$, this is the *Hardy–Littlewood maximal operator* and we write M instead of M_0. Let $\vec{f}=(f_1,\dots,f_m)$ be a collection of m locally integrable functions on $\mathbb{R}^n$. The *multi(sub)linear fractional maximal operator* $M_\alpha(\vec{f})(x)$, $0\le\alpha<mn$ and $x\in\mathbb{R}^n$, is given by

$$M_\alpha(\vec{f})(x) = \sup_{x\in Q\in\mathcal{Q}} \ell(Q)^\alpha \prod_{i=1}^m ⨍_Q |f_i(y)|\,dy.$$

We drop the subscript 0 if $\alpha=0$. In [8], the operator $M(\vec{f})$, called the *multilinear maximal operator*, is used to obtain a precise control on multilinear singular integral operators of Calderón–Zygmund type, and the following fundamental result is proved, leading to a multilinear weighted theory for multilinear Calderón–Zygmund operators.

Proposition 1.4 ([8, Theorem 3.7]). *Let $\vec{w}=(w_1,\dots,w_m)$ be a collection of m weights on $\mathbb{R}^n$ and let $1<p_i<\infty$, $i=1,\dots,m$, and*

$$\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

Then the weighted norm inequality

$$\|M(\vec{f})w_1 \cdots w_m\|_{L^p} \le C \prod_{i=1}^{m} \|f_i w_i\|_{L^{p_i}}$$

holds if and only if

$$\sup_{Q\in\mathcal{Q}} \Big(\fint_Q (w_1 \cdots w_m)^p \Big)^{1/p} \prod_{i=1}^{m} \Big(\fint_Q w_i^{-p_i'} \Big)^{1/p_i'} < \infty,$$

where $1/p_i + 1/p_i' = 1$.

Motivated by [8], Moen [10] established a one-weight theory and two-weight theory for the multilinear fractional integral operator $\mathcal{I}_\alpha(\vec{f})$ and the multilinear fractional maximal operator $M_\alpha(\vec{f})$ on Lebesgue spaces. In this paper we consider a theory of weights related to Proposition 1.2. We give natural sufficient conditions on a weight or on two weights for a weighted version of (1.6) to hold. We shall extend the results of Moen to Morrey spaces.

The remainder of this paper is organized as follows: In Section 2 we first consider the question of what is the weighted norm inequality for Morrey spaces for the linear case. We propose referring to the following estimate as the *weighted norm inequality* for Morrey spaces: For any $f \in \mathcal{L}_+$, that is, for any positive Lebesgue measurable function f,

$$|Q|^{1/p_0} \Big(\fint_Q ((Mf)w)^p \Big)^{1/p} \le C \sup_{\substack{Q'\in\mathcal{Q} \\ Q'\supset Q}} |Q'|^{1/p_0} \Big(\fint_{Q'} (|f|w)^p \Big)^{1/p}.$$

In Sections 3–5, using the formulation developed in Section 2, we study the one-weight theory and two-weight theory for the multilinear fractional integral operators and the multilinear fractional maximal operators in the framework of Morrey spaces. The main results are stated in Section 3 (Theorems 3.1 and 3.3). In Section 4 we shall state and prove a principal lemma (Lemma 4.2). We feel this lemma is of independent interest. In Section 5 we complete the proofs of Theorems 3.1 and 3.3.

We will denote by $|E|$ the Lebesgue measure of $E \subset \mathbb{R}^n$. For each $1 \le p \le \infty$, p' will denote the dual exponent of p, i.e., $p' = p/(p-1)$ with the usual conventions $1' = \infty$ and $\infty' = 1$. The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as c_1, c_2, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \le A \le cB$ with some positive constant c independent of appropriate quantities.

2. What is the weighted norm inequality for Morrey spaces? As is well-known, for the Hardy–Littlewood maximal operator M and $1<p<\infty$, Muckenhoupt [11] showed that the one-weight strong type inequality

$$\|(Mf)w\|_{L^p} \le C\|fw\|_{L^p} \tag{2.1}$$

holds if and only if

$$\sup_{Q\in\mathcal{Q}} \Big(⨍_Q w^p\Big)^{1/p} \Big(⨍_Q w^{-p'}\Big)^{1/p'} < \infty. \tag{2.2}$$

For $1 < p < \infty$ one says that a weight w on $\mathbb{R}^n$ belongs to the class A_p when

$$\sup_{Q\in\mathcal{Q}} \Big(⨍_Q w\Big) \Big(⨍_Q w^{1-p'}\Big)^{p-1} < \infty. \tag{2.3}$$

That is, (2.1) holds if and only if $w^p \in A_p$. Moreover, for the fractional integral operator I_α and $1 < p < q < \infty$ with $1/q = 1/p - \alpha/n$, Muckenhoupt and Wheeden [12] showed that the one-weight strong type inequality

$$\|(I_\alpha f)w\|_{L^q} \le C\|fw\|_{L^p} \tag{2.4}$$

holds if and only if

$$[w]_{A_{p,q}} = \sup_{Q\in\mathcal{Q}} \Big(⨍_Q w^q\Big)^{1/q} \Big(⨍_Q w^{-p'}\Big)^{1/p'} < \infty. \tag{2.5}$$

To the best of our knowledge, Komori and Shirai [9] were the first to attempt extending the equivalence between (2.1) and (2.2) and the one between (2.4) and (2.5) to Morrey spaces. Actually, they showed that for $1 < p \le p_0 < \infty$ the weighted norm inequality

$$\sup_{Q\in\mathcal{Q}} w(Q)^{1/p_0-1/p}\Big(\int_Q (Mf)^p w\Big)^{1/p} \le C \sup_{Q\in\mathcal{Q}} w(Q)^{1/p_0-1/p}\Big(\int_Q |f|^p w\Big)^{1/p}$$

holds if $w \in A_p$, where $w(Q) = \int_Q w$. However, in this paper motivated by characterizing Adams type weighted norm inequality for fractional integral operators (cf. Theorem 2.4 below), we consider the weighted norm inequality

$$\|(Mf)w\|_{\mathcal{M}^{p_0}_p} \le C\|fw\|_{\mathcal{M}^{p_0}_p}. \tag{2.6}$$

We have not been able to find a necessary and sufficient condition for (2.6) to hold, however, we shall establish the following:

THEOREM 2.1. *Let $1 < p \le p_0 < \infty$ and w be a weight. Then, for every $Q \in \mathcal{Q}$, the weighted inequality*

$$|Q|^{1/p_0}\Big(⨍_Q ((Mf)w)^p\Big)^{1/p} \le C \sup_{\substack{Q'\in\mathcal{Q}\\ Q'\supset Q}} |Q'|^{1/p_0}\Big(⨍_{Q'} (|f|w)^p\Big)^{1/p} \tag{2.7}$$

holds if and only if

$$\sup_{\substack{Q,Q' \in \mathcal{Q} \\ Q \subset Q'}} \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\fint_Q w^p \right)^{1/p} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} < \infty. \tag{2.8}$$

Before we turn to the proof of Theorem 2.1, two clarifying remarks may be in order.

Remark 2.2. The weighted norm inequality (2.6) follows from (2.7) immediately. Furthermore, if $1 < p = p_0 < \infty$ (the case of Lebesgue spaces), then

$$\sup_{\substack{Q,Q' \in \mathcal{Q} \\ Q \subset Q'}} \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\fint_Q w^p \right)^{1/p} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} = \sup_{Q \in \mathcal{Q}} \left(\fint_Q w^p \right)^{1/p} \left(\fint_Q w^{-p'} \right)^{1/p'}. \tag{2.9}$$

From (2.9) we see that (2.8) extends the class A_p given by (2.2) to Morrey spaces.

Remark 2.3. The inequality (2.8) holds if w satisfies

$$[w]_{A_{p,p_0}} = \sup_{Q \in \mathcal{Q}} \left(\fint_Q w^{p_0} \right)^{1/p_0} \left(\fint_Q w^{-p'} \right)^{1/p'} < \infty. \tag{2.10}$$

Indeed, for any cubes $Q \subset Q'$ we have by Hölder's inequality

$$\begin{aligned} \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\fint_Q w^p \right)^{1/p} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} &\le \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\fint_Q w^{p_0} \right)^{1/p_0} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} \\ &\le \left(\fint_{Q'} w^{p_0} \right)^{1/p_0} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} \\ &\le [w]_{A_{p,p_0}} < \infty. \end{aligned}$$

Proof of Theorem 2.1. We first prove the "only if" part. We assume to the contrary that

$$\sup_{\substack{Q,Q' \in \mathcal{Q} \\ Q \subset Q'}} \left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\fint_Q w^p \right)^{1/p} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} = \infty. \tag{2.11}$$

By (2.11) we can select two cubes $Q \subset Q'$ such that for a large N,

$$\left(\frac{|Q|}{|Q'|} \right)^{1/p_0} \left(\fint_Q w^p \right)^{1/p} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} > N. \tag{2.12}$$

Selecting a smaller cube Q' in (2.12) (if necessary), without loss of generality we may assume that Q' is minimal in the sense that

$$\sup_{\substack{R\in\mathcal{Q}\\ Q\subset R\subset Q'}} \fint_R w^{-p'} = \fint_{Q'} w^{-p'}. \tag{2.13}$$

By noticing $1/p_0 - 1/p \le 0$, the equality (2.13) yields

$$\sup_{\substack{R\in\mathcal{Q}\\ R\supset Q}} |R|^{1/p_0}\Big(\fint_R \chi_{Q'} w^{-p'}\Big)^{1/p} = |Q'|^{1/p_0}\Big(\fint_{Q'} w^{-p'}\Big)^{1/p}. \tag{2.14}$$

It follows by applying (2.7) and (2.14) with $f = \chi_{Q'} w^{-p'}$ that

$$\begin{aligned}\Big(\fint_{Q'} w^{-p'}\Big)|Q|^{1/p_0}\Big(\fint_Q w^p\Big)^{1/p} &\le |Q|^{1/p_0}\Big(\fint_Q (M[\chi_{Q'} w^{-p'}](x)w(x))^p\,dx\Big)^{1/p}\\ &\le C\sup_{\substack{R\in\mathcal{Q}\\ R\supset Q}} |R|^{1/p_0}\Big(\fint_R \chi_{Q'} w^{-p'}\Big)^{1/p}\\ &= C|Q'|^{1/p_0}\Big(\fint_{Q'} w^{-p'}\Big)^{1/p}.\end{aligned}$$

This yields a contradiction

$$N < \Big(\frac{|Q|}{|Q'|}\Big)^{1/p_0}\Big(\fint_Q w^p\Big)^{1/p}\Big(\fint_{Q'} w^{-p'}\Big)^{1/p'} \le C.$$

We next prove the "if" part. Fix $Q \in \mathcal{Q}$ and decompose, for $x \in Q$,

$$Mf(x) \le M[\chi_{3Q} f](x) + M[\chi_{(3Q)^c} f](x). \tag{2.15}$$

For the first term on the right-hand side of (2.15), we use the fact that $w^p \in A_p$ and that (2.1) and (2.2) are equivalent. From this observation, we deduce

$$|Q|^{1/p_0}\Big(\fint_Q (M[\chi_{3Q} f]w)^p\Big)^{1/p} \le C|3Q|^{1/p_0}\Big(\fint_{3Q} (|f|w)^p\Big)^{1/p}. \tag{2.16}$$

To deal with the second term on the right-hand side of (2.15), we use a routine geometric observation: for any $x \in Q$,

$$M[\chi_{(3Q)^c} f](x) \le C \sup_{\substack{Q'\in\mathcal{Q}\\ Q'\supset Q}} \fint_{Q'} |f|. \tag{2.17}$$

By Hölder's inequality and the assumption (2.8), we have, for a cube $Q' \supset Q$,

$$\begin{aligned}
&|Q|^{1/p_0}\Big(\fint_Q w^p\Big)^{1/p}\Big(\fint_{Q'} |f|\Big) \\
&\le |Q|^{1/p_0}\Big(\fint_Q w^p\Big)^{1/p}\Big(\fint_{Q'} w^{-p'}\Big)^{1/p'}\Big(\fint_{Q'} (|f|w)^p\Big)^{1/p} \\
&= \bigg\{\Big(\frac{|Q|}{|Q'|}\Big)^{1/p_0}\Big(\fint_Q w^p\Big)^{1/p}\Big(\fint_{Q'} w^{-p'}\Big)^{1/p'}\bigg\}|Q'|^{1/p_0}\Big(\fint_{Q'} (|f|w)^p\Big)^{1/p} \\
&\le C|Q'|^{1/p_0}\Big(\fint_{Q'} (|f|w)^p\Big)^{1/p}.
\end{aligned}$$

This together with the pointwise estimate (2.17) implies

$$|Q|^{1/p_0}\Big(\fint_Q (M[\chi_{(3Q)^c} f]w)^p\Big)^{1/p} \le C \sup_{\substack{Q'\in\mathcal{Q}\\ Q'\supset Q}} |Q'|^{1/p_0}\Big(\fint_{Q'} (|f|w)^p\Big)^{1/p}. \tag{2.18}$$

The inequalities (2.16) and (2.18) yield (2.7). ■

The following is an Adams type weighted norm inequality for fractional integral operators. The result is new as far as we know.

THEOREM 2.4. *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$, $1 < q \le q_0 < \infty$ and let w be a weight. Suppose that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Then the weighted norm inequality

$$\|(I_\alpha f)w\|_{\mathcal{M}^{q_0}_q} \le C\|fw\|_{\mathcal{M}^{p_0}_p} \tag{2.19}$$

holds if there exists a number $a > 1$ such that

$$\sup_{\substack{Q,Q'\in\mathcal{Q}\\ Q\subset Q'}} \Big(\frac{|Q|}{|Q'|}\Big)^{1/aq_0}\Big(\fint_Q w^q\Big)^{1/q}\Big(\fint_{Q'} w^{-p'}\Big)^{1/p'} < \infty. \tag{2.20}$$

REMARK 2.5. The inequality (2.20) holds if the weight w satisfies

$$[w]_{A_{p,q_0}} = \sup_{Q\in\mathcal{Q}}\Big(\fint_Q w^{q_0}\Big)^{1/q_0}\Big(\fint_Q w^{-p'}\Big)^{1/p'} < \infty. \tag{2.21}$$

Thus, when $q = q_0$ and $p = p_0$ (the case of Lebesgue spaces), Theorem 2.4 recovers the result due to Muckenhoupt and Wheeden [12]. Assuming (2.21), we see that $w^{q_0} \in A_t$ with $t = 1 + q_0/p'$ and, by the reverse Hölder inequality, there exists $a > 1$ such that, for any $Q' \in \mathcal{Q}$,

$$\left(\fint_{Q'} w^{aq_0} \right)^{1/aq_0} \le C \left(\fint_{Q'} w^{q_0} \right)^{1/q_0}. \tag{2.22}$$

By Hölder's inequality and (2.22) we have, for any pair of cubes $Q \subset Q'$,

$$\begin{aligned}
&\left(\frac{|Q|}{|Q'|} \right)^{1/aq_0} \left(\fint_Q w^q \right)^{1/q} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} \\
&\qquad \le \left(\frac{|Q|}{|Q'|} \right)^{1/aq_0} \left(\fint_Q w^{aq_0} \right)^{1/aq_0} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} \\
&\qquad \le \left(\fint_{Q'} w^{aq_0} \right)^{1/aq_0} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} \\
&\qquad \le C \left(\fint_{Q'} w^{q_0} \right)^{1/q_0} \left(\fint_{Q'} w^{-p'} \right)^{1/p'} \le C[w]_{A_{p,q_0}} < \infty.
\end{aligned}$$

Remark 2.6. As the example $w(x) = |x|^{-n/q_0}$, $f(x) = |x|^{-\alpha}$ shows, we cannot choose $a = 1$ in Theorem 2.4. Indeed, we have $I_\alpha f \equiv \infty$ but $\|fw\|_{\mathcal{M}^{p_0}_p} \le C$. This means the bound (2.19) is false. Meanwhile, a direct calculation shows that the quantity

$$\sup_{0<b<c<\infty} \left(\frac{b}{c} \right)^{n/aq_0} \left(\fint_{|y| \le b} |x|^{-nq/q_0} \, dx \right)^{1/q} \left(\fint_{|y| \le c} |x|^{np'/q_0} \, dx \right)^{1/p'}$$

is equivalent to

$$\sup_{0<b<c<\infty} \frac{b^{n/aq_0}}{b^{n/q_0}} \frac{c^{n/q_0}}{c^{n/aq_0}},$$

which is finite if and only if $a = 1$.

To prove Theorem 2.4 we need the following lemma, the proof of which will be given in Section 4.

Lemma 2.7. *Let $1 < q < \infty$ and w be a weight. Then, for any $Q_0 \in \mathcal{Q}$ and $a > 1$, we have*

$$\|I_\alpha[\chi_{3Q_0} f] w\|_{L^q(Q_0)} \le C \|\widetilde{M}^{aq}_\alpha(\chi_{3Q_0} f, w)\|_{L^q(Q_0)},$$

where

$$\widetilde{M}^{aq}_\alpha(f, w)(x) = \sup_{x \in Q \in \mathcal{Q}} \ell(Q)^\alpha \left(\fint_Q |f(y)| \, dy \right) \left(\fint_Q w(y)^{aq} \, dy \right)^{1/aq}. \tag{2.23}$$

Proof of Theorem 2.4. Without loss of generality we may assume that f is nonnegative. Fix a cube Q_0 in $\mathcal{Q}$. Then by a standard argument we have, for $x \in Q_0$,

$$I_\alpha f(x) \le I_\alpha[\chi_{3Q_0} f](x) + CC_\infty,$$

where

$$C_\infty = \sum_{k=0}^{\infty} \ell(2^k Q_0)^\alpha \fint_{2^k Q_0} f.$$

We have to estimate the quantities

$$|Q_0|^{1/q_0 - 1/q} \Big(\int_{Q_0} (I_\alpha[\chi_{3Q_0} f](x) w(x))^q \, dx \Big)^{1/q} \tag{2.24}$$

and

$$|Q_0|^{1/q_0} \Big(\fint_{Q_0} w^q \Big)^{1/q} C_\infty. \tag{2.25}$$

The estimate of (2.24). For the proof it suffices by Lemma 2.7 to estimate

$$|Q_0|^{1/q_0 - 1/q} \| \widetilde{M}_\alpha^{aq}(\chi_{3Q_0} f, w) \|_{L^q(Q_0)}. \tag{2.26}$$

By (2.20) we notice that, with $t = 1 + q/p'$, $w^q \in A_t$ and $w^{-p'} \in A_{t'}$. Hence, by the reverse Hölder inequality, there exists $a > 1$ such that, for any $Q \in \mathcal{Q}$,

$$\Big(\fint_Q w^{aq} \Big)^{1/aq} \le C \Big(\fint_Q w^q \Big)^{1/q} \quad \text{and} \quad \Big(\fint_Q w^{-(p/a)'} \Big)^{1/(p/a)'} \le C \Big(\fint_Q w^{-p'} \Big)^{1/p'}. \tag{2.27}$$

It follows from Hölder's inequality, (2.27) and (2.20) with $Q = Q'$ that, for any cube $Q \subset 3Q_0$,

$$\begin{aligned} \Big(\fint_Q f \Big) \Big(\fint_Q w^{aq} \Big)^{1/aq} &\le \Big(\fint_Q w^{aq} \Big)^{1/aq} \Big(\fint_Q w^{-(p/a)'} \Big)^{1/(p/a)'} \Big(\fint_Q (fw)^{p/a} \Big)^{a/p} \\ &\le C \Big(\fint_Q w^q \Big)^{1/q} \Big(\fint_Q w^{-p'} \Big)^{1/p'} \Big(\fint_Q (fw)^{p/a} \Big)^{a/p} \\ &\le C \Big(\fint_Q (fw)^{p/a} \Big)^{a/p}. \end{aligned}$$

In view of (2.23), we obtain a pointwise estimate

$$\widetilde{M}_\alpha^{aq}(\chi_{3Q_0} f, w)(x) \le C M_{\alpha p/a}[(fw)^{p/a}](x)^{a/p},$$

and hence we can control (2.26):

$$
\begin{aligned}
(2.28)\qquad & |Q_0|^{1/q_0-1/q}\|\widetilde{M}_\alpha^{aq}(\chi_{3Q_0}f,w)\|_{L^q(Q_0)} \\
&\le C|Q_0|^{1/q_0}\Big(\fint_{Q_0} M_{\alpha p/a}[(fw)^{p/a}](x)^{aq/p}\,dx\Big)^{1/q} \\
&= C\Big\{|Q_0|^{p/aq_0}\Big(\fint_{Q_0} M_{\alpha p/a}[(fw)^{p/a}](x)^{aq/p}\,dx\Big)^{p/aq}\Big\}^{a/p} \\
&\le C\{\|M_{\alpha p/a}[(fw)^{p/a}]\|_{\mathcal{M}^{aq_0/p}_{aq/p}}\}^{a/p}.
\end{aligned}
$$

We now deduce from the Adams theorem for fractional maximal operators (cf. (1.5)) that $M_{\alpha p/a}$ is bounded from $\mathcal{M}^{ap_0/p}_a$ to $\mathcal{M}^{aq_0/p}_{aq/p}$. Consequently,

$$
(2.29)\qquad \|M_{\alpha p/a}[(fw)^{p/a}]\|_{\mathcal{M}^{aq_0/p}_{aq/p}} \le C\|fw\|^{p/a}_{\mathcal{M}^{p_0}_p}.
$$

If we combine (2.28) and (2.29), then we obtain

$$
(2.24) \le C\|fw\|_{\mathcal{M}^{p_0}_p}.
$$

The estimate of (2.25). The matter is not so delicate as for (2.24) and a crude estimate by using Hölder's inequality suffices. For any $Q\in\mathcal{Q}$ we have

$$
\begin{aligned}
\ell(Q)^\alpha\fint_Q f &\le \ell(Q)^\alpha\Big(\fint_Q w^{-p'}\Big)^{1/p'}\Big(\fint_Q (fw)^p\Big)^{1/p} \\
&\le |Q|^{\alpha/n-1/p_0}\Big(\fint_Q w^{-p'}\Big)^{1/p'}\|fw\|_{\mathcal{M}^{p_0}_p} \\
&= |Q|^{-1/q_0}\Big(\fint_Q w^{-p'}\Big)^{1/p'}\|fw\|_{\mathcal{M}^{p_0}_p}.
\end{aligned}
$$

It follows from this inequality and (2.20) that

$$
\begin{aligned}
(2.25) &\le \|fw\|_{\mathcal{M}^{p_0}_p}\sum_{k=0}^\infty\Big(\frac{|Q_0|}{|2^kQ_0|}\Big)^{1/q_0}\Big(\fint_{Q_0} w^q\Big)^{1/q}\Big(\fint_{2^kQ_0} w^{-p'}\Big)^{1/p'} \\
&= \|fw\|_{\mathcal{M}^{p_0}_p}\sum_{k=0}^\infty\Big(\frac{|Q_0|}{|2^kQ_0|}\Big)^{1/q_0-1/aq_0} \\
&\quad\times\Big(\frac{|Q_0|}{|2^kQ_0|}\Big)^{1/aq_0}\Big(\fint_{Q_0} w^q\Big)^{1/q}\Big(\fint_{2^kQ_0} w^{-p'}\Big)^{1/p'} \\
&\le C\|fw\|_{\mathcal{M}^{p_0}_p}\sum_{k=0}^\infty(2^{-nk})^{1/q_0-1/aq_0} = C\|fw\|_{\mathcal{M}^{p_0}_p},
\end{aligned}
$$

where we have used $1/q_0 - 1/aq_0 > 0$ for the last line. Thus, the estimate for (2.25) is now valid and the proof is finished. ∎

3. Multilinear fractional operators. We can extend Theorems 2.1 and 2.4 to multilinear fractional operators. The proofs are postponed to Section 5.

THEOREM 3.1. *Let $\vec{w} = (w_1, \dots, w_m)$ be a collection of m weights on $\mathbb{R}^n$ and let*

$$0 \le \alpha < mn, \quad \vec{P} = (p_1, \dots, p_m), \quad 1 < p_1, \dots, p_m < \infty, \quad 0 < p \le p_0 < \infty,$$
$$0 < q \le q_0 < \infty, \qquad a > 1,$$

where p denotes the number determined by the Hölder relationship

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}.$$

Suppose that

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Then there exists a constant C independent of $\vec{f}$ such that the following one-weight norm inequalities hold:

(a) *If $\alpha > 0$ and*

$$[\vec{w}]_{q,\vec{P}}^{aq_0} = \sup_{\substack{Q,Q' \in \mathcal{Q} \\ Q \subset Q'}} \left(\frac{|Q|}{|Q'|} \right)^{1/aq_0} \left(\fint_Q (w_1 \cdots w_m)^q \right)^{1/q} \prod_{i=1}^m \left(\fint_{Q'} w_i^{-p_i'} \right)^{1/p_i'} < \infty, \tag{3.1}$$

then

$$\|\mathcal{I}_\alpha(\vec{f}) w_1 \cdots w_m\|_{\mathcal{M}_q^{q_0}} \le C \sup_{Q \in \mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \left(\fint_Q (|f_i| w_i)^{p_i} \right)^{1/p_i}.$$

(b) *If $\alpha \ge 0$ and*

$$[\vec{w}]_{q,\vec{P}}^{q_0} = \sup_{\substack{Q,Q' \in \mathcal{Q} \\ Q \subset Q'}} \left(\frac{|Q|}{|Q'|} \right)^{1/q_0} \left(\fint_Q (w_1 \cdots w_m)^q \right)^{1/q} \prod_{i=1}^m \left(\fint_{Q'} w_i^{-p_i'} \right)^{1/p_i'} < \infty, \tag{3.2}$$

then

$$\|M_\alpha(\vec{f}) w_1 \cdots w_m\|_{\mathcal{M}_q^{q_0}} \le C \sup_{Q \in \mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \left(\fint_Q (|f_i| w_i)^{p_i} \right)^{1/p_i}.$$

REMARK 3.2. When $m = 1$, Theorem 3.1(a) corresponds to Theorem 2.4. When $m = 1$ and $\alpha = 0$, Theorem 3.1(b) corresponds to Theorem 2.1. In the same manner as in Remarks 2.3 and 2.5, by using Lemma 5.2 below, the inequalities (3.1) and (3.2) hold if

$$\sup_{Q\in\mathcal{Q}}\Big(\fint_Q (w_1\cdots w_m)^{q_0}\Big)^{1/q_0}\prod_{i=1}^m\Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'}<\infty. \tag{3.3}$$

Thus, when $p=p_0$ and $q=q_0$ (the case of Lebesgue spaces), Theorem 3.1 recovers the one-weight results due to Moen [10] and, when $\alpha=0$, it recovers Proposition 1.4. In this case Moen proved that the condition is also necessary.

The following is a two-weight norm inequality for multilinear fractional operators in the framework of Morrey spaces.

Theorem 3.3. *Let v be a weight on $\mathbb{R}^n$ and $\vec{w}=(w_1,\dots,w_m)$ be a collection of m weights on $\mathbb{R}^n$. Let*

$$0\le\alpha<mn,\qquad \vec{P}=(p_1,\dots,p_m),\qquad 1<p_1,\dots,p_m<\infty,$$
$$0<p\le p_0<\infty,\qquad 0<q\le q_0<r_0\le\infty$$

and $1<a<\min(r_0/q_0,p_1,\dots,p_m)$. Here, p is given by

$$\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

Suppose that

$$\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}\quad\text{and}\quad\frac{q}{q_0}=\frac{p}{p_0}.$$

Then there exists a constant C independent of $\vec{f}$, v and $\vec{w}$ such that the following two-weight norm inequalities hold:

(a) *For $\alpha>0$, $q>1$ set*

$$\begin{aligned}&[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}\\&=\sup_{\substack{Q,Q'\in\mathcal{Q}\\Q\subset Q'}}\Big(\frac{|Q|}{|Q'|}\Big)^{1/aq_0}|Q'|^{1/r_0}\Big(\fint_Q v^{aq}\Big)^{1/aq}\prod_{i=1}^m\Big(\fint_{Q'} w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'}.\end{aligned}\tag{3.4}$$

If $[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}<\infty$, then

$$\|\mathcal{I}_\alpha(\vec{f})v\|_{\mathcal{M}^{q_0}_q}\le C[v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}\sup_{Q\in\mathcal{Q}}|Q|^{1/p_0}\prod_{i=1}^m\Big(\fint_Q(|f_i|w_i)^{p_i}\Big)^{1/p_i}.$$

(b) *Let $\alpha>0$, $0<q\le1$ and define*

$$\begin{aligned}&[v,\vec{w}]^{r_0,aq_0}_{q,\vec{P}/a}\\&=\sup_{\substack{Q,Q'\in\mathcal{Q}\\Q\subset Q'}}\Big(\frac{|Q|}{|Q'|}\Big)^{1/aq_0}|Q'|^{1/r_0}\Big(\fint_Q v^{q}\Big)^{1/q}\prod_{i=1}^m\Big(\fint_{Q'} w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'}.\end{aligned}\tag{3.5}$$

If $[v,\vec{w}]^{r_0,aq_0}_{q,\vec{P}/a}<\infty$, *then*

$$\|\mathcal{I}_\alpha(\vec{f})v\|_{\mathcal{M}^{q_0}_q}\le C[v,\vec{w}]^{r_0,aq_0}_{q,\vec{P}/a}\sup_{Q\in\mathcal{Q}}|Q|^{1/p_0}\prod_{i=1}^m\Big(⨍_Q(|f_i|w_i)^{p_i}\Big)^{1/p_i};$$

(c) *If*

$$[v,\vec{w}]^{r_0,q_0}_{q,\vec{P}/a}=\sup_{\substack{Q,Q'\in\mathcal{Q}\\ Q\subset Q'}}\Big(\frac{|Q|}{|Q'|}\Big)^{1/q_0}|Q'|^{1/r_0}\Big(⨍_Q v^q\Big)^{1/q}\prod_{i=1}^m\Big(⨍_{Q'}w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'}\tag{3.6}$$

is finite, then

$$\|M_\alpha(\vec{f})v\|_{\mathcal{M}^{q_0}_q}\le C[v,\vec{w}]^{r_0,q_0}_{q,\vec{P}/a}\sup_{Q\in\mathcal{Q}}|Q|^{1/p_0}\prod_{i=1}^m\Big(⨍_Q(|f_i|w_i)^{p_i}\Big)^{1/p_i}.$$

REMARK 3.4. In the same manner as in Remark 2.3, the inequalities (3.4) and (3.5) hold if

$$\sup_{Q\in\mathcal{Q}}|Q|^{1/r_0}\Big(⨍_Q v^{aq_0}\Big)^{1/aq_0}\prod_{i=1}^m\Big(⨍_Q w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'}<\infty,\tag{3.7}$$

and the inequality (3.6) holds if

$$\sup_{Q\in\mathcal{Q}}|Q|^{1/r_0}\Big(⨍_Q v^{q_0}\Big)^{1/q_0}\prod_{i=1}^m\Big(⨍_Q w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'}<\infty.\tag{3.8}$$

Thus, when $p=p_0$, $q=q_0$ and $q>1$ (the case of Lebesgue spaces), Theorem 3.3 recovers the two-weight results due to Moen [10]. When $p=p_0$, $q=q_0$, $q>1$ and $m=1$, Theorem 3.3 goes back to Sawyer and Wheeden [22].

The following is the Olsen inequality for multilinear fractional operators (see [4, 5, 13, 18–21, 23, 24] for the linear cases).

COROLLARY 3.5. *Let v be a weight on $\mathbb{R}^n$ and let*

$$0\le\alpha<mn,\quad 1<p_1,\dots,p_m<\infty,\quad 0<p\le p_0<\infty,$$
$$0<q\le q_0<r_0\le\infty,\quad a>1,$$

where p is given by

$$\frac{1}{p}=\frac{1}{p_1}+\dots+\frac{1}{p_m}.$$

Suppose that

$$\frac{1}{q_0}=\frac{1}{p_0}+\frac{1}{r_0}-\frac{\alpha}{n}\quad\text{and}\quad\frac{q}{q_0}=\frac{p}{p_0}.$$

(a) *If $\alpha > 0$ and $q > 1$, then*

$$\|\mathcal{I}_\alpha(\vec{f})v\|_{\mathcal{M}^{q_0}_q} \le C\|v\|_{\mathcal{M}^{r_0}_{aq}} \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i} \Big)^{1/p_i}.$$

(b) *If $\alpha > 0$ and $0 < q \le 1$, then*

$$\|\mathcal{I}_\alpha(\vec{f})v\|_{\mathcal{M}^{q_0}_q} \le C\|v\|_{\mathcal{M}^{r_0}_{q}} \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i} \Big)^{1/p_i}.$$

(c) *We have*

$$\|M_\alpha(\vec{f})v\|_{\mathcal{M}^{q_0}_q} \le C\|v\|_{\mathcal{M}^{r_0}_{q}} \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i} \Big)^{1/p_i}.$$

Proof. This follows from Theorem 3.3 by letting $w_1, \dots, w_m \equiv 1$ and by noticing that, for every $Q \subset Q'$,

$$\Big(\frac{|Q|}{|Q'|}\Big)^{1/aq_0} |Q'|^{1/r_0} = |Q|^{1/aq_0}|Q'|^{1/r_0 - 1/aq_0} \le |Q|^{1/r_0}, \tag{3.9}$$

$$\Big(\frac{|Q|}{|Q'|}\Big)^{1/q_0} |Q'|^{1/r_0} = |Q|^{1/q_0}|Q'|^{1/r_0 - 1/q_0} \le |Q|^{1/r_0}. \tag{3.10}$$

The inequalities (3.9) and (3.10) can be deduced from the facts that

$$\frac{1}{r_0} - \frac{1}{aq_0} < 0 \quad \text{and} \quad \frac{1}{r_0} - \frac{1}{q_0} < 0,$$

respectively, which follow from $q_0 < aq_0 < r_0$. ∎

The following is the Fefferman–Stein type dual inequality for multilinear fractional operators in the framework of Morrey spaces.

Corollary 3.6. *Suppose that the parameters $0 < q_i < r_i \le \infty$, $i = 1, \dots, m$, satisfy*

$$\frac{1}{aq_0} = \sum_{i=1}^m \frac{1}{q_i}, \qquad \frac{1}{r_0} = \sum_{i=1}^m \frac{1}{r_i}.$$

Then, for any collection of m weights $w_1, \dots, w_m$, we have

$$\|\mathcal{I}_\alpha(\vec{f})w_1 \cdots w_m\|_{\mathcal{M}^{q_0}_q} \le C \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q (|f_i|W_i)^{p_i} \Big)^{1/p_i},$$

where

$$W_i(x) = \sup_{x\in Q\in\mathcal{Q}} |Q|^{1/r_i} \Big(\fint_Q w_i^{q_i} \Big)^{1/q_i} \qquad \textit{for } i = 1, \dots, m.$$

Proof. We need only verify the inequality (3.7) with $v = w_1 \cdots w_m$ and $w_i = W_i$, $i = 1, \ldots, m$. It follows from Hölder's inequality that

$$|Q|^{1/r_0} \Big(\fint_Q (w_1 \cdots w_m)^{aq_0} \Big)^{1/aq_0} \le |Q|^{1/r_0} \prod_{i=1}^m \Big(\fint_Q w_i^{q_i} \Big)^{1/q_i} = \prod_{i=1}^m |Q|^{1/r_i} \Big(\fint_Q w_i^{q_i} \Big)^{1/q_i}.$$

Corollary 3.6 follows immediately from the inequality

$$W_i(x) \ge |Q|^{1/r_i} \Big(\fint_Q w_i^{q_i} \Big)^{1/q_i} \quad \text{for all } x \in Q. \ ■$$

Theorem 3.3 is new even in the linear case, that is, when $m = 1$. For the convenience of the readers we restate Theorem 3.3 for this case.

COROLLARY 3.7. *Let v, w be weights on $\mathbb{R}^n$. Let*

$$0 \le \alpha < n, \quad 1 < p \le p_0 < \infty, \quad 0 < q \le q_0 < r_0 \le \infty$$

and $1 < a < \min(r_0/q_0, p)$. Suppose that

$$\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Then there exists a constant C independent of f, v and w such that the following two-weight norm inequalities hold:

(a) *If $\alpha > 0$, $q > 1$ and*

$$[v, w]^{r_0, aq_0}_{aq, p/a} = \sup_{\substack{Q, Q' \in \mathcal{Q} \\ Q \subset Q'}} \Big(\frac{|Q|}{|Q'|} \Big)^{1/aq_0} |Q'|^{1/r_0} \Big(\fint_Q v^{aq} \Big)^{1/aq} \Big(\fint_{Q'} w^{-(p/a)'} \Big)^{1/(p/a)'} < \infty,$$

then

$$\|(I_\alpha f) v\|_{\mathcal{M}^{q_0}_q} \le C [v, w]^{r_0, aq_0}_{aq, p/a} \|f w\|_{\mathcal{M}^{p_0}_p}.$$

(b) *If $\alpha > 0$, $0 < q \le 1$ and*

$$[v, w]^{r_0, aq_0}_{q, p/a} = \sup_{\substack{Q, Q' \in \mathcal{Q} \\ Q \subset Q'}} \Big(\frac{|Q|}{|Q'|} \Big)^{1/aq_0} |Q'|^{1/r_0} \Big(\fint_Q v^{q} \Big)^{1/q} \Big(\fint_{Q'} w^{-(p/a)'} \Big)^{1/(p/a)'} < \infty,$$

then

$$\|(I_\alpha f) v\|_{\mathcal{M}^{q_0}_q} \le C [v, w]^{r_0, aq_0}_{q, p/a} \|f w\|_{\mathcal{M}^{p_0}_p}.$$

(c) *If*

$$[v,w]_{q,p/a}^{r_0,q_0} = \sup_{\substack{Q,Q'\in\mathcal{Q}\\ Q\subset Q'}} \left(\frac{|Q|}{|Q'|}\right)^{1/q_0} |Q'|^{1/r_0} \Big(\fint_Q v^q\Big)^{1/q} \Big(\fint_{Q'} w^{-(p/a)'}\Big)^{1/(p/a)'} < \infty,$$

then

$$\|(M_\alpha f)v\|_{\mathcal{M}_q^{q_0}} \le C[v,w]_{q,p/a}^{r_0,q_0} \|fw\|_{\mathcal{M}_p^{p_0}}.$$

4. Principal lemma. We shall state and prove a principal lemma (Lemma 4.2). Our key tool is the following multilinear maximal operator.

Definition 4.1. Let $0 \le \alpha < mn$ and $0 < q < \infty$. Let v be a weight on $\mathbb{R}^n$ and let $\vec{f} = (f_1, \ldots, f_m)$ be a collection of m locally integrable functions on $\mathbb{R}^n$. Then define a multilinear maximal operator $\widetilde{M}_\alpha^q(\vec{f}, v)(x)$, $x \in \mathbb{R}^n$, by

$$\widetilde{M}_\alpha^q(\vec{f}, v)(x) = \sup_{x\in Q\in\mathcal{Q}} \ell(Q)^\alpha \Big(\prod_{i=1}^m \fint_Q |f_i(y)|\,dy\Big) \Big(\fint_Q v(y)^q\,dy\Big)^{1/q}.$$

The following is our principal lemma, which seems to be of interest on its own.

Lemma 4.2. *Let v be a weight on $\mathbb{R}^n$. For $\vec{f} = (f_1, \ldots, f_m)$ and $Q_0 \in \mathcal{Q}$, set*

$$\vec{f^0} = (\chi_{3Q_0} f_1, \ldots, \chi_{3Q_0} f_m).$$

Then there exists a constant C independent of v, $\vec{f}$ and Q_0 such that the following inequalities hold:

(a) *Let $0 < \alpha < mn$. If $0 < q \le 1$, then*

$$\|\mathcal{I}_\alpha(\vec{f^0})v\|_{L^q(Q_0)} \le C\|\widetilde{M}_\alpha^q(\vec{f^0}, v)\|_{L^q(Q_0)}. \tag{4.1}$$

(b) *Let $0 < \alpha < mn$ and $a > 1$. If $1 < q < \infty$, then*

$$\|\mathcal{I}_\alpha(\vec{f^0})v\|_{L^q(Q_0)} \le C\|\widetilde{M}_\alpha^{aq}(\vec{f^0}, v)\|_{L^q(Q_0)}. \tag{4.2}$$

(c) *Let $0 \le \alpha < mn$. If $0 < q < \infty$, then*

$$\|M_\alpha(\vec{f^0})v\|_{L^q(Q_0)} \le C\|\widetilde{M}_\alpha^q(\vec{f^0}, v)\|_{L^q(Q_0)}. \tag{4.3}$$

In the rest of this section we shall prove Lemma 4.2. Since $\mathcal{I}_\alpha$ is a positive operator, without loss of generality we may assume that $f_1, \ldots, f_m$ are nonnegative. For simplicity, we will use the notation

$$m_Q(\vec{f}) = \prod_{i=1}^m \fint_Q f_i(y)\,dy.$$

Fix a cube $Q_0 \in \mathcal{Q}$. Let $\mathcal{D}(Q_0)$ be the collection of all dyadic subcubes of Q_0, that is, all those cubes obtained by dividing Q_0 into 2^n congruent cubes of half its side-length, dividing each of those into 2^n congruent cubes, and so on. By convention, Q_0 itself belongs to $\mathcal{D}(Q_0)$. To prove Lemma 4.2(a)&(b), we need the following estimate.

LEMMA 4.3. *For $x \in Q_0$,*

$$\mathcal{I}_\alpha(\vec{f^0})(x) \le C \sum_{Q \in \mathcal{D}(Q_0)} \ell(Q)^\alpha m_{3Q}(\vec{f})\chi_Q(x).$$

Proof. Following [15–17], we have

$$\begin{aligned}
&\mathcal{I}_\alpha(\vec{f^0})(x) \\
&\quad = \sum_{k \in \mathbb{Z}} \int_{(3Q_0)^m \cap \{2^{k-1} < |x-y_1|+\cdots+|x-y_m| \le 2^k\}} \frac{f_1(y_1)\cdots f_m(y_m)\, d\vec{y}}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}} \\
&\quad \le C \sum_{k \in \mathbb{Z}} (2^k)^{\alpha-mn} \int_{(3Q_0)^m \cap \{2^{k-1} < |x-y_1|+\cdots+|x-y_m| \le 2^k\}} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} \\
&\quad \le C \sum_{k \in \mathbb{Z}} (2^k)^{\alpha-mn} \int_{(3Q_0)^m \cap \{|x-y_1|+\cdots+|x-y_m| \le 2^k\}} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} \\
&\quad \le C \sum_{x \in Q \in \mathcal{D}(Q_0)} \ell(Q)^{\alpha-mn} \prod_{i=1}^m \int_{3Q} f_i(y)\, dy = C \sum_{x \in Q \in \mathcal{D}(Q_0)} \ell(Q)^\alpha m_{3Q}(\vec{f}).
\end{aligned}$$

The proof is therefore complete. ∎

Following [10], we observe the following.

Letting $\gamma_0 = m_{3Q_0}(\vec{f})$ and $A = (2m18^n)^m$, we set, for $k = 1, 2, \ldots,$

$$D_k = \bigcup \{Q : Q \in \mathcal{D}(Q_0),\ m_{3Q}(\vec{f}) > \gamma_0 A^k\}.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D_k = \bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\} \subset \mathcal{D}(Q_0)$ are nonoverlapping. By the maximality of $Q_{k,j}$ we see that

$$\gamma_0 A^k < m_{3Q_{k,j}}(\vec{f}) \le 2^{mn}\gamma_0 A^k. \tag{4.4}$$

Let

$$E_0 = Q_0 \setminus D_1, \quad E_{k,j} = Q_{k,j} \setminus D_{k+1}.$$

We need the following properties: $\{E_0\} \cup \{E_{k,j}\}$ is a disjoint family of sets which decomposes Q_0 and satisfies

$$|Q_0| \le 2|E_0|, \quad |Q_{k,j}| \le 2|E_{k,j}|. \tag{4.5}$$

The inequalities (4.5) can be verified as follows:

For fixed $Q_{k,j}$ we set

$$A_i = \Big(\prod_{l=1}^{m} \int_{3Q_{k,j}} f_l(y)\,dy\Big)^{-1/m} (\gamma_0 A^{k+1})^{1/m} \int_{3Q_{k,j}} f_i(y)\,dy. \tag{4.6}$$

Observe that $\prod_{i=1}^m A_i = \gamma_0 A^{k+1}$. By (4.4) we see that

$$\begin{aligned} Q_{k,j} \cap D_{k+1} &\subset \{x \in Q_{k,j} : M((\chi_{3Q_{k,j}} f_1, \ldots, \chi_{3Q_{k,j}} f_m))(x) > \gamma_0 A^{k+1}\} \\ &\subset \Big\{x \in Q_{k,j} : \prod_{i=1}^m M(\chi_{3Q_{k,j}} f_i)(x) > \gamma_0 A^{k+1}\Big\} \\ &\subset \bigcup_{i=1}^m \{x : M(\chi_{3Q_{k,j}} f_i)(x) > A_i\}. \end{aligned}$$

Using the weak-$(1,1)$ boundedness of M, we have

$$\begin{aligned} |Q_{k,j} \cap D_{k+1}| &\le \sum_{i=1}^m |\{x : M(\chi_{3Q_{k,j}} f_i)(x) > A_i\}| \\ &\le \sum_{i=1}^m \frac{3^n}{A_i} \int_{3Q_{k,j}} f_i(y)\,dy \\ &= m3^n \Big(\frac{1}{\gamma_0 A^{k+1}} \prod_{i=1}^m \int_{3Q_{k,j}} f_i(y)\,dy\Big)^{1/m}, \end{aligned}$$

where we have used (4.6). From (4.4) we further have

$$|Q_{k,j} \cap D_{k+1}| = m3^n \Big(\frac{1}{\gamma_0 A^{k+1}} m_{3Q_{k,j}}(\vec{f}\,)\Big)^{1/m} |3Q_{k,j}| \le \frac{1}{2} |Q_{k,j}|. \tag{4.7}$$

Similarly, we see that

$$|D_1| \le \frac{1}{2} |Q_0|. \tag{4.8}$$

Clearly, (4.7) and (4.8) imply (4.5).

We set

$$\begin{aligned} \mathcal{D}_0(Q_0) &= \{Q \in \mathcal{D}(Q_0) : m_{3Q}(\vec{f}\,) \le \gamma_0 A\}, \\ \mathcal{D}_{k,j}(Q_0) &= \{Q \in \mathcal{D}(Q_0) : Q \subset Q_{k,j},\ \gamma_0 A^k < m_{3Q}(\vec{f}\,) \le \gamma_0 A^{k+1}\}. \end{aligned}$$

Then we obtain

$$\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \bigcup_{k,j} \mathcal{D}_{k,j}(Q_0). \tag{4.9}$$

Proof of Lemma 4.2(a). By Lemma 4.3, it suffices to estimate

$$\int_{Q_0} (F(x) v(x))^q \, dx, \tag{4.10}$$

where

$$F(x) = \sum_{Q \in \mathcal{D}(Q_0)} \ell(Q)^{\alpha} m_{3Q}(\vec{f}\,) \chi_Q(x). \tag{4.11}$$

It follows that

$$(4.10) = \sum_{Q \in \mathcal{D}(Q_0)} \ell(Q)^{\alpha} m_{3Q}(\vec{f}\,) \int_Q F(x)^{q-1} v(x)^q \, dx.$$

First, based upon (4.9) we estimate

$$\sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^{\alpha} m_{3Q}(\vec{f}\,) \int_Q F(x)^{q-1} v(x)^q \, dx. \tag{4.12}$$

By use of $q - 1 \le 0$ and the definition of $F(x)$, we have

$$F(x)^{q-1} \le (\ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,))^{q-1} \quad \text{for all } x \in Q_{k,j}. \tag{4.13}$$

If we combine (4.12) and (4.13), then we obtain

$$\begin{aligned}
(4.12) &\le \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,))^{q-1} \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^{\alpha} m_{3Q}(\vec{f}\,) \int_Q v(x)^q \, dx \\
&\le (\ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,))^{q-1} \gamma_0 A^{k+1} \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \ell(Q)^{\alpha} \int_Q v(x)^q \, dx.
\end{aligned}$$

By using a geometric property of $\mathcal{D}$ and the assumption $\alpha > 0$, we have

$$(4.12) \le C(\ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,))^{q-1} \gamma_0 A^{k+1} \ell(Q_{k,j})^{\alpha} \int_{Q_{k,j}} v(x)^q \, dx. \tag{4.14}$$

From (4.4), (4.5) and (4.14), we conclude that

$$\begin{aligned}
(4.12) &\le C \Big\{ \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,) \Big(\fint_{Q_{k,j}} v^q \Big)^{1/q} \Big\}^q |Q_{k,j}| \\
&\le C \Big\{ \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,) \Big(\fint_{Q_{k,j}} v^q \Big)^{1/q} \Big\}^q |E_{k,j}| \\
&\le C \int_{E_{k,j}} \widetilde{M}^q_{\alpha}(\vec{f^0}, v)(x)^q \, dx.
\end{aligned} \tag{4.15}$$

Similarly,

$$\sum_{Q \in \mathcal{D}_0(Q_0)} \ell(Q)^{\alpha} m_{3Q}(\vec{f}\,) \int_Q F(x)^{q-1} v(x)^q \, dx \le C \int_{E_0} \widetilde{M}^q_{\alpha}(\vec{f^0}, v)(x)^q \, dx. \tag{4.16}$$

Summing up (4.15) and (4.16), we obtain

$$(4.10) \le C \int_{Q_0} \widetilde{M}^q_{\alpha}(\vec{f^0}, v)(x)^q \, dx.$$

This is our desired inequality (4.1).

Proof of Lemma 4.2(b). Again by Lemma 4.3, with F defined by (4.11), it suffices to estimate, for $q>1$,

$$\Big(\int_{Q_0}(F(x)v(x))^q\,dx\Big)^{1/q}. \tag{4.17}$$

We shall estimate (4.17) by a duality argument. To this end we take a weight w satisfying $\|w\|_{L^{q'}(Q_0)}=1$ and estimate

$$\sum_{Q\in\mathcal{D}(Q_0)}\ell(Q)^{\alpha}m_{3Q}(\vec{f}\,)\int_Q v(x)w(x)\,dx. \tag{4.18}$$

Going through the same argument as before, we shall estimate

$$\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\ell(Q)^{\alpha}m_{3Q}(\vec{f}\,)\int_Q v(x)w(x)\,dx. \tag{4.19}$$

It follows from the definition of $\mathcal{D}_{k,j}(Q_0)$, (4.4) and the same manipulation as above that

$$\begin{aligned}(4.19)&\le\gamma_0A^{k+1}\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\ell(Q)^{\alpha}\int_Q v(x)w(x)\,dx\\&=C\gamma_0A^{k+1}\ell(Q_{k,j})^{\alpha}\int_{Q_{k,j}}v(x)w(x)\,dx\\&\le C\ell(Q_{k,j})^{\alpha}m_{3Q_{k,j}}(\vec{f}\,)\int_{Q_{k,j}}v(x)w(x)\,dx.\end{aligned}$$

By Hölder's inequality and (4.5) we see that (4.19) does not exceed

$$\begin{aligned}&C\ell(Q_{k,j})^{\alpha}m_{3Q_{k,j}}(\vec{f}\,)\Big(\fint_{Q_{k,j}}v(x)^{aq}\,dx\Big)^{1/aq}\Big(\fint_{Q_{k,j}}w(x)^{(aq)'}\,dx\Big)^{1/(aq)'}|Q_{k,j}|\\&\quad\le C\ell(Q_{k,j})^{\alpha}m_{3Q_{k,j}}(\vec{f}\,)\Big(\fint_{Q_{k,j}}v(x)^{aq}\,dx\Big)^{1/aq}\Big(\fint_{Q_{k,j}}w(x)^{(aq)'}\,dx\Big)^{1/(aq)'}|E_{k,j}|.\end{aligned}$$

From the definition of the maximal operators we obtain

$$(4.19)\le C\int_{E_{k,j}}\widetilde{M}^{aq}_{\alpha}(\vec{f^0},v)(x)M[w^{(aq)'}](x)^{1/(aq)'}\,dx. \tag{4.20}$$

Similarly,

$$\begin{aligned}&\sum_{Q\in\mathcal{D}_0(Q_0)}\ell(Q)^{\alpha}m_{3Q}(\vec{f}\,)\int_Q v(x)w(x)\,dx\\&\qquad\qquad\le C\int_{E_0}\widetilde{M}^{aq}_{\alpha}(\vec{f^0},v)(x)M[w^{(aq)'}](x)^{1/(aq)'}\,dx.\end{aligned} \tag{4.21}$$

Summing up (4.20) and (4.21), we obtain

$$
\begin{aligned}
(4.18) &\leq C \int_{Q_0} \widetilde{M}_\alpha^{aq}(\vec{f}^0, v)(x) M[w^{(aq)'}](x)^{1/(aq)'}\,dx \\
&\leq C\Big(\int_{Q_0} \widetilde{M}_\alpha^{aq}(\vec{f}^0, v)(x)^q\,dx\Big)^{1/q}\Big(\int_{Q_0} M[w^{(aq)'}](x)^{q'/(aq)'}\,dx\Big)^{1/q'} \\
&\leq C\Big(\int_{Q_0} \widetilde{M}_\alpha^{aq}(\vec{f}^0, v)(x)^q\,dx\Big)^{1/q},
\end{aligned}
$$

where we have invoked the $L^{q'/(aq)'}$-boundedness of M. This is our desired inequality (4.2).

Proof of Lemma 4.2(c). First of all, for $x \in Q_0$ we notice that the following pointwise equivalence holds:

$$
M_\alpha(\vec{f}^0)(x) \approx \hat{M}_\alpha(\vec{f})(x), \tag{4.22}
$$

where

$$
\hat{M}_\alpha(\vec{f})(x) = \sup_{x\in Q\in\mathcal{D}(Q_0)} \ell(Q)^\alpha m_{3Q}(\vec{f}).
$$

We shall repeat the above argument with some modifications.

Letting $\gamma_1 = \ell(Q_0)^\alpha m_{3Q_0}(\vec{f})$ and $A = (2m18^n)^m$, we set, for $k = 1, 2, \ldots,$

$$
D_k = \bigcup\{Q : Q \in \mathcal{D}(Q_0),\ \ell(Q)^\alpha m_{3Q}(\vec{f}) > \gamma_1 A^k\}.
$$

Considering the maximal cubes with respect to inclusion, we can write, with nonoverlapping cubes,

$$
D_k = \bigcup_j Q_{k,j}.
$$

By the maximality of $Q_{k,j}$ we have

$$
\gamma_1 A^k < \ell(Q_{k,j})^\alpha m_{3Q_{k,j}}(\vec{f}) \leq 2^{mn}\gamma_1 A^k. \tag{4.23}
$$

Let $E_0 = Q_0 \setminus D_1$ and $E_{k,j} = Q_{k,j} \setminus D_{k+1}$ as before. Then we have to check

$$
|Q_0| \leq 2|E_0|, \qquad |Q_{k,j}| \leq 2|E_{k,j}|. \tag{4.24}
$$

We set, for fixed $Q_{k,j}$,

$$
B_i = \Big(\prod_{l=1}^m \int_{3Q_{k,j}} f_l(y)\,dy\Big)^{-1/m}\Big(\frac{\gamma_1 A^{k+1}}{\ell(Q_{k,j})^\alpha}\Big)^{1/m} \int_{3Q_{k,j}} f_i(y)\,dy \tag{4.25}
$$

and observe $\prod_{i=1}^m B_i = \gamma_1 A^{k+1}/\ell(Q_{k,j})^\alpha$. Notice that, if $Q_{k+1,j'} \subset Q_{k,j}$, then

$$
\gamma_1 A^{k+1} < \ell(Q_{k+1,j'})^\alpha m_{3Q_{k+1,j'}}(\vec{f}) < \ell(Q_{k,j})^\alpha m_{3Q_{k+1,j'}}(\vec{f}). \tag{4.26}
$$

The inequalities (4.23) and (4.26) give

$$\begin{aligned}
Q_{k,j}\cap D_{k+1} &\subset \left\{x\in Q_{k,j} : M((\chi_{3Q_{k,j}}f_1,\dots,\chi_{3Q_{k,j}}f_m))(x) > \frac{\gamma_1 A^{k+1}}{\ell(Q_{k,j})^{\alpha}}\right\}\\
&\subset \left\{x\in Q_{k,j} : \prod_{i=1}^m M(\chi_{3Q_{k,j}}f_i)(x) > \frac{\gamma_1 A^{k+1}}{\ell(Q_{k,j})^{\alpha}}\right\}\\
&\subset \bigcup_{i=1}^m \{x : M(\chi_{3Q_{k,j}}f_i)(x) > B_i\}.
\end{aligned}$$

Using the weak-$(1,1)$ boundedness of M, we have

$$\begin{aligned}
|Q_{k,j}\cap D_{k+1}| &\le \sum_{i=1}^m |\{x : M(\chi_{3Q_{k,j}}f_i)(x) > B_i\}|\\
&\le \sum_{i=1}^m \frac{3^n}{B_i}\int_{3Q_{k,j}} f_i(y)\,dy = m3^n\left(\frac{\ell(Q_{k,j})^{\alpha}}{\gamma_1 A^{k+1}}\prod_{i=1}^m\int_{3Q_{k,j}} f_i(y)\,dy\right)^{1/m},
\end{aligned}$$

where we have used (4.25). It follows from (4.23) that

$$|Q_{k,j}\cap D_{k+1}| \le m3^n\left(\frac{\ell(Q_{k,j})^{\alpha}}{\gamma_1 A^{k+1}} m_{3Q_{k,j}}(\vec{f}\,)\right)^{1/m}|3Q_{k,j}| \le \frac{1}{2}|Q_{k,j}|. \tag{4.27}$$

Similarly, we can show that

$$|D_1| \le \frac{1}{2}|Q_0|. \tag{4.28}$$

We consequently deduce (4.24) from (4.27) and (4.28).

We now return to the proof of Lemma 4.2 (c). Using the usual notation for weights, we write $v^q(E) = \int_E v(x)^q\,dx$. It follows that

$$\begin{aligned}
&\int_{Q_0} \hat{M}_{\alpha}(\vec{f}\,)(x)^q v(x)^q\,dx\\
&\qquad = \int_{E_0} \hat{M}_{\alpha}(\vec{f}\,)(x)^q v(x)^q\,dx + \sum_{k,j}\int_{E_{k,j}} \hat{M}_{\alpha}(\vec{f}\,)(x)^q v(x)^q\,dx\\
&\qquad \le v^q(E_0)(\gamma_1 A)^q + \sum_{k,j} v^q(E_{k,j})(\gamma_1 A^{k+1})^q\\
&\qquad \le C\left\{\ell(Q_0)^{\alpha} m_{3Q_0}(\vec{f}\,)\left(\fint_{Q_0} v^q\right)^{1/q}\right\}^q |Q_0|\\
&\qquad\quad + C\sum_{k,j}\left\{\ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(\vec{f}\,)\left(\fint_{Q_{k,j}} v^q\right)^{1/q}\right\}^q |Q_{k,j}|.
\end{aligned}$$

Using (4.24), we see further that

$$\int_{Q_0} \hat{M}_\alpha(\vec{f})(x)^q v(x)^q\,dx \le C\Big(\int_{E_0} \widetilde{M}^q_\alpha(\vec{f}^0, v)(x)^q\,dx + \sum_{k,j}\int_{E_{k,j}} \widetilde{M}^q_\alpha(\vec{f}^0, v)(x)^q\,dx\Big) \le C\int_{Q_0} \widetilde{M}^q_\alpha(\vec{f}^0, v)(x)^q\,dx.$$

This is our desired inequality (4.3) in view of (4.22).

5. Proofs of theorems. We now prove Theorems 3.1 and 3.3. To this end we need two more lemmas.

Let $0 \le \alpha < mn$. For a vector $\vec{f} = (f_1, \dots, f_m)$ of locally integrable functions and a vector $\vec{R} = (r_1, \dots, r_m)$ of exponents, define a maximal operator

$$M_{\alpha,\vec{R}}(\vec{f})(x) = \sup_{x\in Q\in\mathcal{Q}} \ell(Q)^\alpha \prod_{i=1}^m \Big(\fint_Q |f_i(y)|^{r_i}\,dy\Big)^{1/r_i}. \tag{5.1}$$

Lemma 5.1. *Let $0 \le \alpha < mn$. Set $\vec{P}=(p_1, \dots, p_m)$ and $\vec{R}=(r_1, \dots, r_m)$. Assume in addition that $0 < r_i < p_i < \infty$, $i = 1, \dots, m$. If $0 < q \le q_0 < \infty$ and $0 < p \le p_0 < \infty$ satisfy*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0} = \frac{p}{p_0}, \tag{5.2}$$

where p is given by

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m},$$

then

$$\|M_{\alpha,\vec{R}}(\vec{f})\|_{\mathcal{M}^{q_0}_q} \le C \sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i}\Big)^{1/p_i}.$$

Proof. A normalization allows us to assume that

$$\sup_{Q\in\mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q |f_i|^{p_i}\Big)^{1/p_i} = 1. \tag{5.3}$$

First, we shall prove the case $\alpha = 0$. Fix a cube $Q_0 \in \mathcal{Q}$. If we define

$$c_\infty = \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}} \prod_{i=1}^m \Big(\fint_Q |f_i|^{r_i}\Big)^{1/r_i}, \tag{5.4}$$

then, for $x \in Q_0$,

$$M_{0,\vec{R}}(\vec{f})(x) \le M_{0,\vec{R}}((\chi_{3Q_0} f_1, \dots, \chi_{3Q_0} f_m))(x) + Cc_\infty. \tag{5.5}$$

This implies

$$
(5.6)\qquad |Q_0|^{1/p_0}\Big(\fint_{Q_0} M_{0,\vec R}(\vec f)(x)^p\,dx\Big)^{1/p}
\le |Q_0|^{1/p_0}\Big(\fint_{Q_0} M_{0,\vec R}((\chi_{3Q_0}f_1,\dots,\chi_{3Q_0}f_m))(x)^p\,dx\Big)^{1/p}+C|Q_0|^{1/p_0}c_\infty.
$$

It follows from Hölder's inequality and the L^{p_i/r_i}-boundedness of M that

$$
(5.7)\qquad \Big(\fint_{Q_0} M_{0,\vec R}((\chi_{3Q_0}f_1,\dots,\chi_{3Q_0}f_m))(x)^p\,dx\Big)^{1/p}
\le \prod_{i=1}^m\Big(\fint_{Q_0} M[\chi_{3Q_0}f_i^{r_i}](x)^{p_i/r_i}\,dx\Big)^{1/p_i}
\le C\prod_{i=1}^m\Big(\fint_{3Q_0} f_i(x)^{p_i}\,dx\Big)^{1/p_i}\le C|Q_0|^{-1/p_0},
$$

where we have used (5.3). Since $0<r_i<p_i<\infty$, we can apply Hölder's inequality to obtain

$$
(5.8)\qquad |Q_0|^{1/p_0}c_\infty\le \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}}|Q|^{1/p_0}\prod_{i=1}^m\Big(\fint_Q |f_i(y)|^{r_i}\,dy\Big)^{1/r_i}\le 1,
$$

where we have used (5.3) again. Estimates (5.6)–(5.8) imply

$$
(5.9)\qquad |Q_0|^{1/p_0}\Big(\fint_{Q_0} M_{0,\vec R}(\vec f)(x)^p\,dx\Big)^{1/p}\le C.
$$

Taking the supremum over all cubes $Q_0\in\mathcal{Q}$ on the left-hand side of (5.9), we complete the proof of the case $\alpha=0$.

Next, we verify the case $\alpha>0$. For any $x\in\mathbb{R}^n$ and $\varepsilon>0$, we see from the definition of the maximal operator given by (5.1) that

$$
\begin{aligned}
M_{\alpha,\vec R}(\vec f)(x)&\le \sup_{\substack{x\in Q\in\mathcal{Q}\\ \ell(Q)<\varepsilon}}\ell(Q)^\alpha\prod_{i=1}^m\Big(\fint_Q |f_i(y)|^{r_i}\,dy\Big)^{1/r_i}\\
&\quad+\sup_{\substack{x\in Q\in\mathcal{Q}\\ \ell(Q)\ge\varepsilon}}\ell(Q)^\alpha\prod_{i=1}^m\Big(\fint_Q |f_i(y)|^{r_i}\,dy\Big)^{1/r_i}\\
&\le \varepsilon^\alpha M_{0,\vec R}(\vec f)(x)+\varepsilon^{\alpha-n/p_0}\sup_{\substack{x\in Q\in\mathcal{Q}\\ \ell(Q)\ge\varepsilon}}|Q|^{1/p_0}\prod_{i=1}^m\Big(\fint_Q |f_i(y)|^{r_i}\,dy\Big)^{1/r_i}\\
&\le \varepsilon^\alpha M_{0,\vec R}(\vec f)(x)+\varepsilon^{\alpha-n/p_0}\le 2M_{0,\vec R}(\vec f)(x)^{1-\alpha p_0/n},
\end{aligned}
$$

where we have used (5.3) and a minimizing argument. Using the assumption (5.2), we see that $1 - \alpha p_0/n = p_0/q_0 = p/q$ and $q/q_0 p = 1/p_0$. Hence,

$$\|M_{\alpha,\vec{R}}(\vec{f})\|_{\mathcal{M}^{q_0}_q} \le C\|M_{0,\vec{R}}(\vec{f})\|^{p/q}_{\mathcal{M}^{p_0}_p} \le C.$$

This proves Lemma 5.1. ∎

We also need the following equivalences.

LEMMA 5.2 ([6]). *Let $1 < p_1, \dots, p_m < \infty$ and $q \ge p$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}.$$

Then, for m weights $w_1, \dots, w_m$, the inequality

$$\sup_{Q \in \mathcal{Q}} \Big(\fint_Q (w_1 \cdots w_m)^q \Big)^{1/q} \prod_{i=1}^m \Big(\fint_Q w^{-p_i'} \Big)^{1/p_i'} < \infty$$

holds if and only if

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q(m-1/p)}, \\ w_i^{-p_i'} \in A_{p_i'(1/q+m-1/p)}, \quad i = 1, \dots, m. \end{cases}$$

Proof of Theorems 3.1 and 3.3. In what follows we always assume that $f_1, \dots, f_m$ are nonnegative and

$$\sup_{Q \in \mathcal{Q}} |Q|^{1/p_0} \prod_{i=1}^m \Big(\fint_Q (f_i w_i)^{p_i} \Big)^{1/p_i} = 1 \tag{5.10}$$

by normalization. To prove the theorems we have to estimate, for an arbitrary cube $Q_0 \in \mathcal{Q}$, the quantities

$$\begin{cases} |Q_0|^{1/q_0} \Big(\fint_{Q_0} (\mathcal{I}_\alpha(\vec{f}) w_1 \cdots w_m)^q \Big)^{1/q}, & |Q_0|^{1/q_0} \Big(\fint_{Q_0} (\mathcal{I}_\alpha(\vec{f}) v)^q \Big)^{1/q}, \\ |Q_0|^{1/q_0} \Big(\fint_{Q_0} (M_\alpha(\vec{f}) w_1 \cdots w_m)^q \Big)^{1/q}, & |Q_0|^{1/q_0} \Big(\fint_{Q_0} (M_\alpha(\vec{f}) v)^q \Big)^{1/q}. \end{cases} \tag{5.11}$$

Fix a cube $Q_0 \in \mathcal{Q}$ and recall that $\vec{f^0} = (\chi_{3Q_0} f_1, \dots, \chi_{3Q_0} f_m)$. Then by a standard argument we have, for $x \in Q_0$,

$$\mathcal{I}_\alpha(\vec{f})(x) \le \mathcal{I}_\alpha(\vec{f^0})(x) + CC_\infty, \tag{5.12}$$

where

$$C_\infty = \sum_{k=0}^\infty \ell(2^k Q_0)^\alpha m_{2^k Q_0}(\vec{f}), \tag{5.13}$$

and

$$M_\alpha(\vec{f})(x) \le M_\alpha(\vec{f^0})(x) + Cc_\infty, \tag{5.14}$$

where

$$c_\infty = \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}} \ell(Q)^\alpha m_Q(\vec{f}). \tag{5.15}$$

First step. Keeping in mind (5.11)–(5.15), we now estimate, for Theorem 3.3,

$$|Q_0|^{1/q_0}\Big(\fint_{Q_0} v^q\Big)^{1/q} C_\infty \quad \text{and} \quad |Q_0|^{1/q_0}\Big(\fint_{Q_0} v^q\Big)^{1/q} c_\infty.$$

By (3.4), (3.6) and Hölder's inequality we have

$$c_1 = \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}} \left(\frac{|Q_0|}{|Q|}\right)^{1/aq_0} |Q|^{1/r_0}\Big(\fint_{Q_0} v^q\Big)^{1/q} \prod_{i=1}^m \Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'} \le [v,\vec{w}]_{aq,\vec{P}/a}^{r_0,aq_0},$$

$$c_2 = \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}} \left(\frac{|Q_0|}{|Q|}\right)^{1/q_0} |Q|^{1/r_0}\Big(\fint_{Q_0} v^q\Big)^{1/q} \prod_{i=1}^m \Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'} \le [v,\vec{w}]_{q,\vec{P}/a}^{r_0,q_0}.$$

From Hölder's inequality, (5.10) and the fact that

$$\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$$

it follows that

$$\begin{aligned}
C_\infty &\le \sum_{k=0}^\infty \ell(2^kQ_0)^\alpha \Big(\prod_{i=1}^m \Big(\fint_{2^kQ_0} (f_iw_i)^{p_i}\Big)^{1/p_i}\Big)\Big(\prod_{i=1}^m \Big(\fint_{2^kQ_0} w_i^{-p_i'}\Big)^{1/p_i'}\Big)\\
&= \sum_{k=0}^\infty \Big(\prod_{i=1}^m \Big(\fint_{2^kQ_0} w_i^{-p_i'}\Big)^{1/p_i'}\Big) |2^kQ_0|^{\alpha/n-1/p_0}|2^kQ_0|^{1/p_0}\\
&\quad \times \Big(\prod_{i=1}^m \Big(\fint_{2^kQ_0} (f_iw_i)^{p_i}\Big)^{1/p_i}\Big)\\
&\le \sum_{k=0}^\infty |2^kQ_0|^{1/r_0-1/q_0}\Big(\prod_{i=1}^m \Big(\fint_{2^kQ_0} w_i^{-p_i'}\Big)^{1/p_i'}\Big).
\end{aligned}$$

This yields

$$|Q_0|^{1/q_0}\Big(\fint_{Q_0} v^q\Big)^{1/q} C_\infty \le c_1 \sum_{k=0}^\infty \left(\frac{|Q_0|}{|2^kQ_0|}\right)^{(1-1/a)/q_0} = Cc_1,$$

where we have used $1-1/a>0$. Similarly,

$$
\begin{aligned}
|Q_0|^{1/q_0}\Big(\fint_{Q_0} v^q\Big)^{1/q} c_\infty \le |Q_0|^{1/q_0}\Big(\fint_{Q_0} v^q\Big)^{1/q}& \\
\times \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}} \ell(Q)^\alpha \Big(\prod_{i=1}^m\Big(\fint_Q (f_iw_i)^{p_i}\Big)^{1/p_i}\Big)\Big(\prod_{i=1}^m\Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'}\Big)& \\
= \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}}\Big(\frac{|Q_0|}{|Q|}\Big)^{1/q_0}|Q|^{1/r_0+1/p_0}& \\
\times\Big(\fint_{Q_0} v^q\Big)^{1/q}\Big(\prod_{i=1}^m\Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'}\Big)\Big(\prod_{i=1}^m\Big(\fint_Q (f_iw_i)^{p_i}\Big)^{1/p_i}\Big)& \\
\le \sup_{\substack{Q\in\mathcal{Q}\\ Q\supset Q_0}}\Big(\frac{|Q_0|}{|Q|}\Big)^{1/q_0}|Q|^{1/r_0}\Big(\fint_{Q_0} v^q\Big)^{1/q}\Big(\prod_{i=1}^m\Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'}\Big) \le c_2.&
\end{aligned}
$$

Just as for Theorem 3.1, we can estimate

$$
|Q_0|^{1/q_0}\Big(\fint_{Q_0}(w_1\cdots w_m)^q\Big)^{1/q}C_\infty \le C[\vec{w}]^{q_0}_{q,\vec{P}},
$$

$$
|Q_0|^{1/q_0}\Big(\fint_{Q_0}(w_1\cdots w_m)^q\Big)^{1/q}c_\infty \le [\vec{w}]^{q_0}_{q,\vec{P}}.
$$

Second step. To prove Theorem 3.3(1) we shall estimate, for $q>1$,

$$
|Q_0|^{1/q_0}\Big(\fint_{Q_0}(\mathcal{I}_\alpha(\vec{f}^0)v)^q\Big)^{1/q}.
$$

By (3.4) we have

$$
c_3=\sup_{Q\in\mathcal{Q}}|Q|^{1/r_0}\Big(\fint_Q v^{aq}\Big)^{1/aq}\prod_{i=1}^m\Big(\fint_Q w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'} \le [v,\vec{w}]^{r_0,aq_0}_{aq,\vec{P}/a}.
$$

To apply Lemma 4.2(b) we now compute, for any $Q\in\mathcal{Q}$,

$$
\begin{aligned}
\Big(\prod_{i=1}^m\fint_Q f_i\Big)\Big(\fint_Q v^{aq}\Big)^{1/aq}& \\
\le\Big(\prod_{i=1}^m\Big(\fint_Q (f_iw_i)^{p_i/a}\Big)^{a/p_i}\Big)\Big(\prod_{i=1}^m\Big(\fint_Q w_i^{-(p_i/a)'}\Big)^{1/(p_i/a)'}\Big)\Big(\fint_Q v^{aq}\Big)^{1/aq}& \\
\le c_3|Q|^{-1/r_0}\prod_{i=1}^m\Big(\fint_Q (f_iw_i)^{p_i/a}\Big)^{a/p_i}.&
\end{aligned}
$$

This implies, for $x \in Q_0$,

$$\widetilde{M}_{\alpha}^{aq}(\vec{f^0}, v)(x) \le c_3 M_{\alpha-n/r_0,\vec{P}/a}(\vec{f})(x) \tag{5.16}$$

(see (5.1)). The inequality (5.16), Lemma 4.2(b) and Lemma 5.1 yield

$$|Q_0|^{1/q_0}\Big(\fint_{Q_0} \mathcal{I}_{\alpha}(\vec{f^0})(x)^q v(x)^q\Big)^{1/q} \le Cc_3 \|M_{\alpha-n/r_0,\vec{P}/a}(\vec{f})\|_{\mathcal{M}_q^{q_0}} \le Cc_3,$$

where we have used the assumption

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha - n/r_0}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}$$

and (5.10). The case $q \le 1$ is similar. Just use Lemma 4.2(a) instead of Lemma 4.2(b).

Third step. Next, we shall estimate, for Theorem 3.1,

$$|Q_0|^{1/q_0}\Big(\fint_{Q_0} (\mathcal{I}_{\alpha}(\vec{f^0}) w_1 \cdots w_m)^q\Big)^{1/q}.$$

By assumption we have

$$c_4 = \sup_{Q \in \mathcal{Q}} \Big(\fint_Q (w_1 \cdots w_m)^q\Big)^{1/aq} \prod_{i=1}^m \Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'} < \infty.$$

Then we can deduce from Lemma 5.2 and the reverse Hölder inequality that there is a constant $\theta \in (1, \min_i p_i)$ such that, for any cube $Q \in \mathcal{Q}$,

$$\Big(\fint_Q (w_1 \cdots w_m)^{\theta q}\Big)^{1/\theta q} \le C\Big(\fint_Q (w_1 \cdots w_m)^q\Big)^{1/q} \tag{5.17}$$

and, for each $i = 1, \ldots, m$,

$$\Big(\fint_Q w_i^{-(p_i/\theta)'}\Big)^{1/(p_i/\theta)'} \le C\Big(\fint_Q w_i^{-p_i'}\Big)^{1/p_i'}. \tag{5.18}$$

Going through a similar argument as above with (5.17) and (5.18), we obtain

$$|Q_0|^{1/q_0}\Big(\fint_{Q_0} (\mathcal{I}_{\alpha}(\vec{f^0}) w_1 \cdots w_m)^q \, dx\Big)^{1/q} \le C.$$

Consequently, Theorems 3.1 and 3.3 are proved.

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