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Disjointification of martingale differences and conditionally independent random variables with some applications

by

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Abstract. Disjointification inequalities are proven for arbitrary martingale difference sequences and conditionally independent random variables of the form $\{f_k(s)x_k(t)\}_{k=1}^n$, where f_k's are independent and x_k's are arbitrary random variables from a symmetric space X on $[0,1]$. The main results show that the form of these inequalities depends on which side of L_2 the space X lies on. The disjointification inequalities obtained allow us to compare norms of sums of martingale differences and non-negative random variables with the norms of sums of their independent copies. The latter results can be treated as an extension of the modular inequalities proved earlier by de la Peña and Hitczenko to the setting of symmetric spaces. Moreover, using these results simplifies the proofs of some modular inequalities.

1. Introduction. In 1970, H. P. Rosenthal proved a remarkable inequality [29] from which it follows that for sequences of independent mean zero random variables in $L_p[0,1]$, $p \geq 2$, the mapping $f_k \mapsto \bar{f}_k$, where $\bar{f}_k(t) := f_k(t-k+1)\chi_{[k-1,k)}(t)$ $(t>0)$, extends to an isomorphism between the closed linear span $[f_k]_{k=1}^\infty$ (in $L_p[0,1]$) and the closed linear span $[\bar{f}_k]_{k=1}^\infty$ (in $L_p[0,\infty)\cap L_2[0,\infty)$). A significant generalization of this disjointification inequality to the class of symmetric spaces X on $[0,1]$ is due to W. B. Johnson and G. Schechtman [21]. In particular, they introduced the symmetric space Z_X^p on $[0,\infty)$ (Note: Our notation differs from that used in [21]) which can be defined as the space of all functions $f \in L_1[0,\infty)+L_\infty[0,\infty)$ such that $\|f\|_{Z_X^p} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_p < \infty$, where f^* is the non-increasing rearrangement of f (see details in Section 2.1). In [21], they showed that any sequence $\{f_k\}_{k=1}^\infty$ of independent mean zero random variables in X is equivalent to the sequence of its disjoint translates $\{\bar{f}_k\}_{k=1}^\infty$ in Z_X^2 provided that X contains an L_p-space for some $p<\infty$. Moreover, Johnson and Schecht-

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man proved analogous results for positive independent random variables but with the space Z^1_X in place of Z^2_X.

More recently, an extension of the results of Johnson and Schechtman (for mean zero and non-negative independent random variables) was obtained by introducing a new approach [7, 8]. This approach involves a positive linear operator $\mathcal{K} : L_1[0,1] \to L_1[0,1]$ whose definition is based on some probabilistic constructions of V. M. Kruglov [24] (see also M. Sh. Braverman [12]). The application of this operator enabled the extension of the results of Johnson and Schechtman to symmetric spaces with the Kruglov property, i.e., the symmetric spaces X on which the operator $\mathcal{K}$ acts boundedly (see details in Section 2.2). This is a far less restrictive condition than the assumption that $X \supset L_p$ for some $p < \infty$. For example, the exponential Orlicz space $\mathrm{Exp}(L_p)$, $0 < p \le 1$, which has the Kruglov property, does not contain any L_p-space with $p < \infty$ (see e.g. [7]).

In this paper, we replace the setting of independent random variables with more general cases: that of arbitrary martingale difference sequences and a special case of conditionally independent random variables. It should first be noted that D. L. Burkholder [14] previously derived a different generalization of Rosenthal's inequality, proving a square-function-type inequality for martingale difference sequences in L_p-spaces with $p \ge 2$. The inequalities which we present in this paper are one-sided disjointification inequalities in general symmetric spaces and may in some sense be viewed as an extension of these square function inequalities. We note, in passing, that two-sided disjointification inequalities for martingale difference sequences hold if and only if $X = L_2$ (Corollary 3.8).

In the second case, we consider conditionally independent random variables of the form $\{f_k(s)x_k(t)\}_{k=1}^n$, $s,t \in [0,1]$, where f_k's are independent and x_k's are arbitrary random variables from a symmetric space X on $[0,1]$. In this case, we fall back to the operator approach from [7, 8, 9]. This time, however, we require a modification of the operator $\mathcal{K}$ to the operator $\mathcal{K} \otimes \mathbf{1}_X$ on $L_1([0,1] \times [0,1])$ whose precise definition and properties are given in Section 4. Using this new operator, we obtain stronger one-sided disjointification inequalities than that for martingale difference sequences in the case when f_k's satisfy the condition $\sum_{k=1}^n \lambda(\operatorname{supp} f_k) \le 1$, with λ the usual Lebesgue measure (see Theorems 5.3 and 5.5).

The inequalities obtained have some interesting applications, given in Section 6. Firstly, they allow us to compare martingale difference sequences and non-negative random variables to their independent copies. The latter results can be treated as an extension of modular inequalities proved earlier by de la Peña in [17] and Hitczenko in [18] to the setting of symmetric spaces. Furthermore, these results then provide a simple method for proving modular inequalities similar to those established in [17] and [18]. In particular, we eliminate the need for the Hoffmann–Jörgensen inequality required in [18], where some comparison results for non-negative random variables in symmetric spaces were obtained by using the concept of tangent sequences of random variables.

2. Preliminaries

2.1. Symmetric spaces. In this paper, we shall denote by $S(\Omega)$ $(= S(\Omega,\mu))$ the linear space of all measurable finite a.e. functions on a given measure space (Ω,μ) equipped with the topology of convergence locally in measure. In particular, the interval $[0,\alpha)$, $0<\alpha\le\infty$, will be considered with the usual Lebesgue measure λ.

Definition 2.1. A Banach space $(X, \|\cdot\|_X)$ of real-valued Lebesgue-measurable functions (with identification λ-almost everywhere) on the interval $[0,\alpha)$, $0<\alpha\le\infty$, will be called *symmetric* if

(1) X is an *ideal lattice*, i.e., whenever $y\in X$ and $x\in S[0,\alpha)$ with $0\le |x|\le |y|$, then $x\in X$ and $\|x\|_X\le \|y\|_X$;
(2) whenever $x,y\in S[0,\alpha)$ are such that

$$\lambda(\{s\in[0,\alpha): |x(s)|>t\}) = \lambda(\{s\in[0,\alpha): |y(s)|>t\}) \quad (t>0)$$

(we will say in this case that the functions $|x|$ and $|y|$ are *equimeasurable*) and $y\in X$, then $x\in X$ and $\|x\|_X=\|y\|_X$.

Also, if X is a symmetric space on $I=[0,1]$ then $X(I\times I)$ is the corresponding symmetric space on the square with the norm $\|x\|_{X(I\times I)} = \|x^*\|_X$. Here, x^* denotes the non-increasing, left-continuous rearrangement of x, which is equimeasurable with $|x|$ and given by

$$x^*(t)=\inf\{\tau>0 : \lambda_2(\{(u,v): |x(u,v)|>\tau\})<t\}, \qquad t>0,$$

where λ_2 is the 2-dimensional Lebesgue measure on $I\times I$.

Important examples of symmetric spaces are Lorentz (including $L_{p,q}$ spaces) and Orlicz spaces, which will be denoted by Λ_ψ and L_M respectively. For more detailed information about these spaces and other basic properties of symmetric spaces, we refer to [25, 23, 12].

The following symmetric space Z_X^p introduced in [19] (see also [21]) will play an important role in this paper.

Definition 2.2. For an arbitrary symmetric space X on $[0,1]$ and any $p\in[1,\infty]$, we define the function space Z_X^p on $[0,\infty)$ by

$$Z_X^p := \{f\in L_1[0,\infty)+L_\infty[0,\infty) : \|f\|_{Z_X^p} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_p < \infty\}.$$

The *Köthe dual* X' of a symmetric space X on the interval $[0,\alpha)$ consists of all measurable functions y for which

$$\|y\|_{X'} = \sup\left\{\int_0^\alpha |x(t)y(t)|\,dt : x\in X,\ \|x\|_X \le 1\right\} < \infty.$$

Basic properties of Köthe duality can be found in [25] and [23].

We say that X has

(1) *order continuous norm* if from $\{x_k\}_{k=1}^\infty \subseteq X$, $x_k \downarrow_k 0$ a.e. on $[0,\alpha)$ it follows that $\|x_k\|_X \to 0$;
(2) *order semicontinuous norm* if from $\{x_k\}_{k=1}^\infty \subseteq X$, $x\in X$ and $x_k\to x$ a.e. on $[0,\alpha)$ it follows that $\|x\|_X \le \liminf_{n\to\infty}\|x_n\|_X$;
(3) the *Fatou property* if from $\{x_k\}_{k=1}^\infty \subseteq X$, $x_k\to x$ a.e. on $[0,\alpha)$ and $\sup_k\|x_k\|_X<\infty$ it follows that $x\in X$ and $\|x\|_X\le\liminf_{k\to\infty}\|x_k\|_X$.

For a concise summary of how these properties relate to the properties of X see [10, Section 2].

Throughout the paper, we will denote by $I(A,B)$ the set of all 1-*interpolation spaces* (see e.g. [11, 23, 25]) between symmetric spaces A and B. For a sequence of functions $\{f_k\}_{k=1}^\infty \subseteq X(I)$, consider their disjoint translates

$$\bar{f}_k(t) := f_k(t-k+1)\chi_{[k-1,k)}(t), \qquad t>0.$$

Analogously, if $\{f_k(t)x_k(s)\}_{k=1}^\infty \subseteq X(I\times I)$, then

$$\overline{f_k(t)x_k(s)} := (f_kx_k)^*(t-k+1)\chi_{[k-1,k)}(t), \qquad t>0.$$

Finally, if X is a symmetric function space on $[0,1]$ or $[0,\infty)$, the *Boyd indices* p_X and q_X are defined by

$$p_X = \lim_{s\to\infty}\frac{\log s}{\log\|\sigma_s\|} = \sup_{s>1}\frac{\log s}{\log\|\sigma_s\|},$$
$$q_X = \lim_{s\to 0^+}\frac{\log s}{\log\|\sigma_s\|} = \sup_{0<s<1}\frac{\log s}{\log\|\sigma_s\|},$$

where $\sigma_a : X\to X$, $a>0$, is the dilation operator defined by $\sigma_a x(t) := x(t/a)$ if X is a symmetric space on $[0,\infty)$, while for $[0,1]$ we set

$$\sigma_a x(t) = \begin{cases} x(t/a) & \text{for } 0\le t\le \min\{1,a\},\\ 0 & \text{for } \min\{1,a\}<t\le 1.\end{cases}$$

2.2. The Kruglov property and the operator $\mathcal{K}$. In this section, let X denote a symmetric space on $[0,1]$.

Let f be a measurable function (random variable) on $[0,1]$. By $\pi(f)$ we denote a random variable $\sum_{i=1}^N f_i$ where f_i's are independent copies of f and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$. The characteristic function of $\pi(f)$ is given by

$$\varphi_{\pi(f)}(t) = \exp\Big(\int_{-\infty}^{\infty} (e^{itx} - 1)\, d\mathcal{F}_f(x)\Big)$$

where $\mathcal{F}_f$ is the distribution function of f [12, p. 7].

The following definition is due to Braverman [12, Definition 1.4, p. 11].

Definition 2.3. A symmetric space is said to have the *Kruglov property* ($X \in \mathbb{K}$) if

$$f \in X \;\Leftrightarrow\; \pi(f) \in X.$$

It was noted in [12, p. 11] that only the implication $f \in X \Rightarrow \pi(f) \in X$ is non-trivial. It is known that a symmetric space X has the Kruglov property if $X \supset L_p$ for some $p < \infty$ [12, p. 16]. In particular, this is satisfied by symmetric spaces with non-trivial upper Boyd index, i.e. $q_X < \infty$. However, some exponential Orlicz spaces which do not contain L_p for any $p < \infty$ also have this property. For a detailed discussion of this property and its relation to various geometric properties of Banach spaces, we refer to [10].

There is an operator $\mathcal{K}$ defined on $S([0,1],\lambda)$ which is closely linked to the Kruglov property (see [7, 8, 9]). Throughout this paper we write

$$(G,\nu) := \prod_{k=0}^{\infty}([0,1],\lambda_k)$$

where λ_k is the Lebesgue measure on $[0,1]$ for all $k \geq 0$.

Definition 2.4. Let $\{A_m\}$ be a sequence of pairwise disjoint subsets of [0,1] with $\lambda(A_m) = 1/(e\cdot m!)$, $m \in \mathbb{N}$. Given an $f \in S([0,1],\lambda)$, we set

$$\mathcal{K}f(\omega_0,\omega_1,\dots) := \sum_{m=1}^{\infty}\sum_{j=1}^{m} f(\omega_j)\chi_{A_m}(\omega_0).$$

Since we work only with symmetric spaces, the main object of interest is the distribution of the function $\mathcal{K}f$. Hence we can also consider the following equivalent form of the operator $\mathcal{K}$.

If $f \in S([0,1],\lambda)$ and $\{f_{m,j}\}_{j=1}^{m}$, $m \in \mathbb{N}$, is a sequence of measurable functions on $[0,1]$ such that for every $m \in \mathbb{N}$, $f_{m,1}, f_{m,2}, \dots, f_{m,m}, \chi_{A_m}$ are independent random variables with $\mathcal{F}_{f_{m,j}} = \mathcal{F}_f$, $j = 1,\dots,m$, then we write

$$\mathcal{K}'f(x) := \sum_{m=1}^{\infty}\sum_{j=1}^{m} f_{m,j}(x)\chi_{A_m}(x), \qquad x \in [0,1]. \tag{2.1}$$

Since for all $f \in S([0,1],\lambda)$ and $t \in \mathbb{R}$,

$$\nu\{(\omega_0,\omega_1,\dots) \in G : \mathcal{K}f(\omega_0,\omega_1,\dots) > t\} = \lambda\{s \in [0,1] : \mathcal{K}'f(s) > t\},$$

we can regard $\mathcal{K}$ as an operator acting from $S([0,1],\lambda)$ to $S([0,1],\lambda)$.

It was shown in [7] that if X is a symmetric space on $[0,1]$, then the operator $\mathcal{K}$ maps X boundedly into itself if and only if $X \in \mathbb{K}$. In fact, in [7–10], the operator $\mathcal{K}$ is shown to play an important role in estimating the norms of sums of independent random variables by the norms of the sums of their disjoint translates. In Section 4, we show how an extended version of the operator $\mathcal{K}$ plays a similar role for conditionally independent random variables.

To do so, we will need to move into Banach function spaces with mixed norm (see, for example, [22, §11.1, p. 400]). Let X be a Banach lattice and Y be a Banach lattice with order semicontinuous norm on $I = [0,1]$. The *space with mixed norm* $X[Y]$ consists of all functions $x(s,t)$ measurable on the square $I \times I$ and satisfying the conditions: (1) $x(s,\cdot) \in Y$ for a.e. $s \in I$; (2) $\varphi_x(s) = \|x(s,\cdot)\|_Y \in X$. Then $X[Y]$ endowed with the norm $\|x\|_{X[Y]} = \|\varphi_x\|_X$ is a Banach lattice on $I \times I$.

3. Disjointification of martingale differences. In this section we establish one-sided disjointification inequalities for martingale difference sequences (mds) in symmetric spaces. The main results, Theorems 3.1 and 3.5, give necessary and sufficient conditions for the right-hand side and left-hand side disjointification inequalities respectively. We will also show that the form of these inequalities essentially depends on which side of L_2 the space X lies on.

Let X be a symmetric space on $[0,1]$. Denote by $\widetilde{X(l_2)}$ the set of all sequences $\{x_k\}_{k=1}^\infty$ of functions from X such that

$$\|\{x_k\}\|_{\widetilde{X(l_2)}} := \sup_{n=1,2,\dots} \Big\|\Big(\sum_{k=1}^n x_k^2\Big)^{1/2}\Big\|_X < \infty.$$

The closed subspace of $\widetilde{X(l_2)}$ generated by the set of all eventually vanishing sequences $\{x_k\}$ will be denoted by $X(l_2)$.

Before stating the main result for the right-hand side disjointification inequality for mds (see (3.2)), we need to define a linear operator which acts on the space $S(0,\infty)$ into the space of sequences of functions from $S(0,1)$ by

$$Bx(t) := \{x(t+k-1)\}_{k=1}^\infty \quad (0 \le t \le 1). \tag{3.1}$$

It turns out that the boundedness of the operator B gives a necessary and sufficient condition for the right-hand disjointification inequality.

THEOREM 3.1. *Let X be a symmetric space on $[0,1]$ such that $q_X < \infty$. The following conditions are equivalent:*

(i) *B is bounded from Z_X^2 into $\widetilde{X(l_2)}$;*
(ii) *there exists $C > 0$ such that for any mds $\{d_k\}_{k=1}^n \subset X$ we have*

$$\Big\| \max_{k=1,\dots,n} \Big| \sum_{i=1}^{k} d_i \Big| \Big\|_X \le C \Big\| \sum_{i=1}^{n} \bar{d}_i \Big\|_{Z_X^2}; \tag{3.2}$$

(iii) *there exists $C > 0$ such that for any sequence $\{x_k\}_{k=1}^n \subset X$ we have*

$$\Big\| \sum_{i=1}^{n} r_i(t) x_i(s) \Big\|_{X(I\times I)} \le C \Big\| \sum_{i=1}^{n} \bar{x}_i \Big\|_{Z_X^2}, \tag{3.3}$$

where $\{r_i\}$ are the Rademacher functions, i.e., $r_i(t) = \operatorname{sign}(\sin 2^i \pi t)$ $(i = 1, 2, \dots)$ for $t \in [0, 1]$.

Proof. If (i) holds, then

$$\Big\| \Big(\sum_{i=1}^{n} d_i^2 \Big)^{1/2} \Big\|_X = \Big\| B\Big(\sum_{i=1}^{n} \bar{d}_i \Big) \Big\|_{\widetilde{X(l_2)}} \le \|B\|_{Z_X^2 \to \widetilde{X(l_2)}} \Big\| \sum_{i=1}^{n} \bar{d}_i \Big\|_{Z_X^2} \tag{3.4}$$

for any mds $\{d_k\}_{k=1}^n \subset X$. The assumption $q_X < \infty$ then allows us to apply [20, Theorem 3] to get

$$\Big\| \max_{k=1,\dots,n} \Big| \sum_{i=1}^{k} d_i \Big| \Big\|_X \le C_1 \Big\| \Big(\sum_{i=1}^{n} d_i^2 \Big)^{1/2} \Big\|_X$$

for any mds $\{d_k\}_{k=1}^n \subset X$. Combining this with (3.4) we obtain (3.2).

Since the Rademacher functions $\{r_i\}$ are independent, the implication (ii)⇒(iii) is trivial. So it remains to prove (iii)⇒(i). By a well-known consequence of the Khintchine inequality [25, 2.d.1], for every symmetric space X there is a constant $c > 0$ such that

$$\Big\| \sum_{i=1}^{n} r_i(t) x_i(s) \Big\|_{X(I\times I)} \ge c \Big\| \Big(\sum_{i=1}^{n} x_i^2 \Big)^{1/2} \Big\|_X.$$

Therefore, from (3.3), it follows that

$$\Big\| \Big(\sum_{i=1}^{n} x_i^2 \Big)^{1/2} \Big\|_X \le C \Big\| \sum_{i=1}^{n} \bar{x}_i \Big\|_{Z_X^2}$$

for every sequence $\{x_k\}_{k=1}^n \subset X$. It is clear that this is equivalent to (i). ∎

The following result is an easy consequence of Theorem 3.1 and [5, Lemma 3.5] which asserts that if X is a symmetric space on $[0, 1]$, $X \in I(L_1, L_2)$, then the operator B is bounded from Z_X^2 into $\widetilde{X(l_2)}$.

Corollary 3.2. *Let X be a symmetric space on $[0, 1]$, $X \in I(L_1, L_2)$. Then there exists a constant $C > 0$ such that (3.2) holds for every mds $\{d_k\}_{k=1}^n \subset X$.*

In the following corollary, we consider a special case of mds, which is related to the conditionally independent random variables we study in Section 5.

COROLLARY 3.3. *Let X be a symmetric space on $[0,1]$, $X \in I(L_1, L_2)$. Then there exists a constant $C > 0$ such that for every sequence $\{f_k\}_{k=1}^n \subset X$ of mean zero independent functions and arbitrary sequence $\{x_k\}_{k=1}^n \subset X$ we have*

$$\Big\| \max_{k=1,\dots,n} \Big| \sum_{i=1}^{k} f_i(s)x_i(t) \Big| \Big\|_{X(I\times I)} \le C \Big\| \sum_{i=1}^{n} \overline{f_i(s)x_i(t)} \Big\|_{Z_X^2}. \tag{3.5}$$

Proof. It is sufficient to note that the sequence $\{f_i(s)x_i(t)\}_{i=1}^n$ is an mds with respect to the increasing sequence of σ-algebras, $\{\Sigma_k\}_{k=1}^n$, of subsets of the square $I \times I$ where Σ_k is generated by the functions $\{f_i(s)x_i(t)\}_{i=1}^k$ $(k = 1,\dots,n)$, and apply the previous corollary. ∎

Let us show that any of the conditions of Theorem 3.1 implies that $X \supset L_2$.

THEOREM 3.4. *Let X be a symmetric space on $[0,1]$ that is separable or has the Fatou property. If the operator B is bounded from Z_X^2 into $\widetilde{X(l_2)}$, then $X \supset L_2$.*

Proof. Without loss of generality, we may (and will) assume that $\|\chi_{[0,1]}\|_X = 1$. Suppose, on the contrary, that $X \not\supset L_2$. Since X either has the Fatou property or is separable, for every $M > 0$ there is a step function $x(t) = \sum_{j=1}^m a_j \chi_{E_j}(t)$ such that the sets $E_j \subset [0,1]$ are pairwise disjoint, $a_j \ge 0$ $(j = 1,\dots,m)$, $\|x\|_{L_2} = 1$, and $\|x\|_X \ge M + 1$. Denote $y(t) = \sum_{j=1}^m [a_j]\chi_{E_j}(t)$, where $[a]$ is the integer part of a real number a. Then $x = y + z$, where $|z(t)| \le 1$. Therefore, by the previous inequality,

$$\|y\|_X \ge M. \tag{3.6}$$

Setting $y_i(t) := \min(y(t)^2, i)$ and $f_i = y_i - y_{i-1}$ $(i = 1,2,\dots)$, where $y_0 = 0$, we deduce that $y(t)^2 = \sum_{i=1}^l f_i(t)$ for some $l \ge 1$. Since f_i's are indicator functions of some measurable subsets of $[0,1]$, we see that $f := \sum_{i=1}^l \bar{f}_i$ is the indicator function of a measurable subset of $(0,\infty)$ whose Lebesgue measure equals $\sum_{j=1}^m [a_j]^2\lambda(E_j)$. Hence,

$$\begin{aligned}\|f\|_{Z_X^2} &= \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_2} \\ &\le 1 + \Big(\sum_{j=1}^m [a_j]^2\lambda(E_j)\Big)^{1/2} \le 1 + \|x\|_{L_2} \le 2.\end{aligned}$$

On the other hand, since $\sum_{i=1}^l f_i(t) = \sum_{i=1}^l f_i(t)^2, t > 0$, we have

$$y(t) = \Big(\sum_{j=1}^m [a_j]^2\chi_{E_j}(t)\Big)^{1/2} = \Big(\sum_{i=1}^l f_i(t)^2\Big)^{1/2}.$$

Thus, by (3.6),

$$\|Bf\|_{\widetilde{X(l_2)}} = \Big\|\Big(\sum_{i=1}^{l} f_i(t)^2\Big)^{1/2}\Big\|_X = \|y\|_X \geq M.$$

Since M can be arbitrarily large and $\|f\|_{Z_X^2} \leq 2$ independently of M, this contradicts our assumption on the boundedness of B and the proof is complete. ∎

Now consider the converse of inequalities (3.2) and (3.5). In this case, the boundedness of the operator A defined below provides a necessary and sufficient condition for the inequalities to hold.

Let X be a symmetric space on $[0,1]$. Define on $X(l_2)$ the following linear operator acting into the set $S(0,\infty)$:

$$A(\{x_k\})(u) := \sum_{k=1}^{\infty} x_k(u-k+1)\chi_{[k-1,k)}(u).$$

The proof of Theorem 3.5 is very similar to the proof of Theorem 3.1 and is therefore omitted.

THEOREM 3.5. *Let X be a symmetric space on $[0,1]$ such that $q_X < \infty$. The following conditions are equivalent:*

(i) *the operator A is bounded from $X(l_2)$ into Z_X^2;*

(ii) *there exists $C > 0$ such that for any mds $\{d_k\}_{k=1}^n \subset X$ we have*

$$\Big\|\sum_{i=1}^{n} \bar{d}_i\Big\|_{Z_X^2} \leq C\Big\|\max_{k=1,\dots,n}\Big|\sum_{i=1}^{k} d_i\Big|\Big\|_X; \tag{3.7}$$

(iii) *there exists $C > 0$ such that for any sequence $\{x_k\}_{k=1}^n \subset X$ we have*

$$\Big\|\sum_{i=1}^{n} \bar{x}_i\Big\|_{Z_X^2} \leq C\Big\|\sum_{i=1}^{n} r_i(t)x_i(s)\Big\|_{X(I\times I)}. \tag{3.8}$$

The following corollary is the analogue of Corollaries 3.2 and 3.3 for the left-hand side inequality.

COROLLARY 3.6. *Let X be a symmetric space on $[0,1]$ with order semicontinuous norm such that $q_X < \infty$ and $X \in I(L_2, L_\infty)$. Then there exists a constant $C > 0$ such that (3.7) holds for every mds $\{d_k\}_{k=1}^n \subset X$.*

In particular, for every sequence $\{f_k\}_{k=1}^n \subset X$ of mean zero independent functions and arbitrary sequence $\{x_k\}_{k=1}^n \subset X$ we have

$$\Big\|\sum_{i=1}^{n} \overline{f_i(t)x_i(s)}\Big\|_{Z_X^2} \leq C\Big\|\max_{k=1,\dots,n}\Big|\sum_{i=1}^{k} f_i(t)x_i(s)\Big|\Big\|_{X(I\times I)}. \tag{3.9}$$

Proof. By Theorem 3.5, it suffices to show that the operator A is bounded from $X(l_2)$ into Z_X^2 for spaces X satisfying the conditions above.

Firstly, it is easy to check that the space Z_X^2 has order semicontinuous norm as X does. Therefore, from the equality $(Z_X^2)' = Z_{X'}^2$ (see for instance

[5, Lemma 3.3]), it follows that

$$\|A(\{x_k\})\|_{Z_X^2} = \sup_{\|y\|_{Z^2_{X'}}\le 1} \Big|\int_0^\infty A(\{x_k\})(u)y(u)\,du\Big| \tag{3.10}$$

for every $\{x_k\}_{k=1}^\infty \in X(l_2)$.

Moreover, for any $\{x_k\}_{k=1}^\infty \in X(l_2)$ and any $y \in Z^2_{X'}$ we have

$$\int_0^\infty A(\{x_k\})(u)y(u)\,du = \sum_{k=1}^\infty \int_0^1 (By)_k(u)x_k(u)\,du, \tag{3.11}$$

which follows from

$$\begin{aligned}\int_0^\infty A(\{x_k\})(u)y(u)\,du &= \int_0^\infty \sum_{k=1}^\infty x_k(u-k+1)\chi_{[k-1,k)}(u)y(u)\,du\\ &= \sum_{k=1}^\infty \int_{k-1}^k x_k(u-k+1)y(u)\,du\\ &= \sum_{k=1}^\infty \int_0^1 y(u+k-1)x_k(u)\,du\\ &= \sum_{k=1}^\infty \int_0^1 (By)_k(u)x_k(u)\,du.\end{aligned}$$

Since (L_2, L_∞) is a K-monotone couple [13, 4.4.38] and $L_2'+L_\infty' = L_2+L_1$ is separable, it follows from [27, Theorem 4.1] that $X' \in I(L_1, L_2)$. Hence, by [5, Lemma 3.5], the operator B is bounded from $Z^2_{X'}$ into $\widetilde{X'(l_2)}$. Then, using (3.10) and (3.11), we obtain

$$\begin{aligned}\|A(\{x_k\})\|_{Z_X^2} &\le \sup_{\|y\|_{Z^2_{X'}}\le 1} \Big|\sum_{k=1}^\infty \int_0^1 (By)_k(u)x_k(u)\,du\Big|\\ &\le \sup_{\|y\|_{Z^2_{X'}}\le 1} \sup_{n=1,2,\ldots} \bigg(\Big\|\Big(\sum_{k=1}^n (By)_k^2\Big)^{1/2}\Big\|_{X'} \cdot \Big\|\Big(\sum_{k=1}^n x_k^2\Big)^{1/2}\Big\|_X\bigg)\\ &\le \sup_{\|y\|_{Z^2_{X'}}\le 1} \|By\|_{\widetilde{X'(l_2)}} \|\{x_k\}\|_{X(l_2)} = \|B\|\,\|\{x_k\}\|_{X(l_2)}.\end{aligned}$$

Therefore, A is bounded from $X(l_2)$ into Z_X^2 and the proof is complete. ∎

Next, we show that any of the conditions of Theorem 3.5 implies that $X \subset L_2$.

THEOREM 3.7. *Let X be a symmetric space on $[0,1]$ that is separable or has the Fatou property. If the operator A is bounded from $X(l_2)$ into Z_X^2, then $X \subset L_2$.*

Proof. We claim that the operator B is bounded from $Z^2_{X'}$ into $\widetilde{X'(l_2)}$. In fact, for any $y \in Z^2_{X'}$ and $n \in \mathbb{N}$,

$$\Big\|\Big(\sum_{k=1}^{n}(By)_k^2\Big)^{1/2}\Big\|_{X'} = \sup_{\|x\|_X \le 1}\int_0^1 \Big(\sum_{k=1}^{n}(By)_k(u)^2\Big)^{1/2} x(u)\,du.$$

It is clear that there are measurable functions $z_k(t)$ $(k = 1, \ldots, n)$ such that $\sum_{k=1}^{n} z_k^2(u) = 1$ $(0 \le u \le 1)$ and

$$\Big(\sum_{k=1}^{n}(By)_k(u)^2\Big)^{1/2} = \sum_{k=1}^{n}(By)_k(u)z_k(u) \qquad (0 \le u \le 1).$$

If $x \in X$ and $x_k = xz_k$ $(k = 1, \ldots, n)$ and $x_k = 0$ $(k > n)$, then $\{x_k\} \subset X(l_2)$ and $\|\{x_k\}\|_{X(l_2)} = \|x\|_X$. Therefore,

$$\Big\|\Big(\sum_{k=1}^{n}(By)_k^2\Big)^{1/2}\Big\|_{X'} \le \sup_{\|\{x_k\}\|_{X(l_2)} \le 1} \sum_{k=1}^{n}\int_0^1 (By)_k(u)x_k(u)\,du.$$

Hence, by hypothesis and equality (3.11),

$$\|By\|_{\widetilde{X'(l_2)}} = \sup_{n=1,2,\ldots}\Big\|\Big(\sum_{k=1}^{n}(By)_k^2\Big)^{1/2}\Big\|_{X'} \le \|A\| \cdot \|y\|_{Z^2_{X'}},$$

and our claim is proved.

Therefore, we can apply Theorem 3.4 and conclude that $X' \supset L_2$, or equivalently, $X'' \subset L_2$. This implies $X \subset L_2$ and the proof is complete. ∎

The following corollary is an immediate consequence of Theorems 3.1, 3.4, 3.5 and 3.7. It shows, in contrast to the case of sequences of independent random variables ([8, Theorem 3.1] and [21, Theorem 1]), that two-sided disjointification inequalities hold only in L_2.

COROLLARY 3.8. *For a given symmetric space X on $[0, 1]$ the following conditions are equivalent:*

(1) *for every mds $\{d_k\}_{k=1}^n \subset X$,*

$$\Big\|\max_{k=1,\ldots,n}\Big|\sum_{i=1}^{k} d_i\Big|\Big\|_X \asymp \Big\|\sum_{i=1}^{n} \bar{d}_i\Big\|_{Z^2_X};$$

(2) *for every sequence $\{f_k\}_{k=1}^n \subset X$ of mean zero independent functions and arbitrary sequence $\{x_k\}_{k=1}^n \subset X$,*

$$\Big\|\max_{k=1,\ldots,n}\Big|\sum_{i=1}^{k} f_i(t)x_i(s)\Big|\Big\|_{X(I\times I)} \asymp \Big\|\sum_{i=1}^{n} \overline{f_i(t)x_i(s)}\Big\|_{Z^2_X};$$

(3) *for any sequence* $\{x_k\}_{k=1}^n \subset X$,

$$\Big\| \sum_{i=1}^n r_i(t) x_i(s) \Big\|_{X(I\times I)} \asymp \Big\| \sum_{i=1}^n \bar{x}_i \Big\|_{Z_X^2};$$

(4) $X = L_2$ (*with equivalence of norms*).

Remark 3.9. Let X be the Lorentz space $L_{2,q}$. It is shown in [16, pp. 154–155] that in the case when $q \neq 2$ neither is B bounded from Z_X^2 into $\widetilde{X(l_2)}$ nor is A bounded from $X(l_2)$ into Z_X^2. Therefore, for $L_{2,q}$ ($q \neq 2$) neither (3.2) nor (3.7) holds.

Later, we will see that better disjointification results can be proved for conditionally independent random variables of a special form. To this end, we first introduce a modification of the operator $\mathcal{K}$.

4. On the operator $\mathcal{K} \otimes \mathbf{1}_X$

Definition 4.1. Let E, F and X be Banach function lattices on $[0,1]$. Given a linear operator $T : E \to F$ and the identity operator $\mathbf{1}_X$ on X we define the operator $T \otimes \mathbf{1}_X : E \otimes X \to F \otimes X$ as acting on all finite combinations of the form

$$\sum_{k=1}^n f_k \otimes x_k := \sum_{k=1}^n f_k(s)\cdot x_k(t), \qquad f_k \in E \text{ and } x_k \in X \ (k = 1,\dots,n)$$

as follows:

$$(T\otimes \mathbf{1}_X)\Big(\sum_{k=1}^n f_k \otimes x_k\Big) = \sum_{k=1}^n T(f_k)\otimes x_k.$$

Our aim is to find conditions under which the operator $T \otimes \mathbf{1}_X$ can be extended to the space $E[X]$ with mixed norm.

The following result is well known (see e.g. [1, Theorem 12.3]).

Proposition 4.2. *Let E and F be Banach function lattices. If T is a positive linear operator such that $T(E) \subseteq F$, then T is bounded from E into F.*

Theorem 4.3. *Let E and X be symmetric spaces on $[0,1]$ that either have the Fatou property or are separable and $E \neq L_\infty$, $X \neq L_\infty$. If T is a positive linear operator such that $T(E) \subseteq E$ and $\mathbf{1}_X$ is the identity operator on X, then $T \otimes \mathbf{1}_X$ can be extended to a bounded operator defined on the whole space $E[X]$ (which will also be denoted by $T \otimes \mathbf{1}_X$) and*

$$\|T\otimes \mathbf{1}_X\|_{E[X]\to E[X]} \le \|T\|_{E\to E}.$$

Proof. Firstly, let $u \in E\otimes X$, i.e. $u(s,t) = \sum_{i=1}^n f_i(s)x_i(t)$, where $n \in \mathbb{N}$, $f_i \in E$, $x_i \in X$. If $y \in X'$, then by the linearity of the integral, it follows that

$$(4.1)\qquad (T\otimes \mathbf{1}_X)\int_0^1 u(s,t)y(t)\,dt = \int_0^1 (T\otimes \mathbf{1}_X)u(s,t)y(t)\,dt.$$

Moreover, for every $y\in X'$ with $\|y\|_{X'}=1$, we have

$$\Big|\int_0^1 u(s,t)y(t)\,dt\Big| \le \|u(s,\cdot)\|_X.$$

Hence,

$$\begin{aligned} T(\|u(s,\cdot)\|_X) &= (T\otimes \mathbf{1}_X)(\|u(s,\cdot)\|_X)\\ &\ge (T\otimes \mathbf{1}_X)\Big|\int_0^1 u(s,t)y(t)\,dt\Big|\\ &\ge \Big|(T\otimes \mathbf{1}_X)\int_0^1 u(s,t)y(t)\,dt\Big| \qquad \text{(since } T\otimes \mathbf{1}_X \text{ is positive)}\\ &= \Big|\int_0^1 (T\otimes \mathbf{1}_X)u(s,t)y(t)\,dt\Big| \qquad \text{(from (4.1))}. \end{aligned}$$

Therefore,

$$\begin{aligned} T(\|u(s,\cdot)\|_X) &\ge \sup_{\substack{y\in X'\\ \|y\|=1}} \Big|\int_0^1 (T\otimes \mathbf{1}_X)u(s,t)y(t)\,dt\Big|\\ &= \|(T\otimes \mathbf{1}_X)u(s,\cdot)\|_X \end{aligned}$$

(since X has the Fatou property or is separable).

Now, applying the monotonicity of the norm of E and Proposition 4.2, we get

$$\big\|\|(T\otimes \mathbf{1}_X)u\|_X\big\|_E \le \big\|T(\|u\|_X)\big\|_E \le \|T\|_{E\to E}\cdot\big\|\|u\|_X\big\|_E = \|T\|_{E\to E}\cdot\|u\|_{E[X]}.$$

Thus, the operator $T\otimes \mathbf{1}_X$ acts boundedly from the space $E\otimes X$ endowed with the norm $\|\cdot\|_{E[X]}$ into $E[X]$ and

$$(4.2)\qquad \|T\otimes \mathbf{1}_X\|_{E\otimes X\to E[X]} \le \|T\|_{E\to E}.$$

Now we want to extend the operator $T\otimes \mathbf{1}_X$ to the whole space $E[X]$. If E and X are separable symmetric spaces, then the set of functions of the form

$$(4.3)\qquad \sum_{i=1}^n \chi_{A_i}(t)f_i(s),$$

where $n \in \mathbb{N}$, $A_i \subset [0,1]$ are pairwise disjoint, and $f_i \in L_\infty$, is dense in the space $E[X]$ (see [22, §11.1, Lemma 2]). Therefore, by (4.2), $T \otimes \mathbf{1}_X$ can be extended to $E[X]$ so that

$$\|T \otimes \mathbf{1}_X\|_{E[X]\to E[X]} \le \|T\|_{E\to E}.$$

Let symmetric spaces E and X have the Fatou property. Then, as before, we extend $T \otimes \mathbf{1}_X$ to a bounded operator from $E_0[X_0]$ into $E[X]$, where E_0 and X_0 are subspaces of E and X respectively, with order continuous norm (see also [11, Theorem 3.8]). Thereby, $T \otimes \mathbf{1}_X$ is defined on $L_\infty(I \times I) \subset E_0[X_0]$. Let $u \in E[X]$, $u \ge 0$. Setting $u_m(s,t) := \min(u(s,t), m)$ $(m \in \mathbb{N})$, we see that $u_m \uparrow u$ a.e. on $I \times I$. Since T is positive, the sequence $(T \otimes \mathbf{1}_X)u_m$ increases a.e. Moreover, we have proved that for all $m \in \mathbb{N}$,

$$\|(T \otimes \mathbf{1}_X)u_m\|_{E[X]} \le \|T\|_{E\to E}\|u_m\|_{E[X]} \le \|T\|_{E\to E}\|u\|_{E[X]}.$$

Using the definition of a space with mixed norm it is not hard to check that together with E and X, the space $E[X]$ has the Fatou property. Therefore,

$$(T \otimes \mathbf{1}_X)u := \lim_{m\to\infty}(T \otimes \mathbf{1}_X)u_m \in E[X]$$

and

$$\|(T \otimes \mathbf{1}_X)u\|_{E[X]} \le \|T\|_{E\to E}\|u\|_{E[X]}. \tag{4.4}$$

For arbitrary $u \in E[X]$, we let $u = u_+ - u_-$, where $u_+ := \max(u,0)$ and $u_- := \max(-u,0)$, and set $(T \otimes \mathbf{1}_X)u := (T \otimes \mathbf{1}_X)u_+ - (T \otimes \mathbf{1}_X)u_-$. Then, by (4.4),

$$\|(T \otimes \mathbf{1}_X)u\|_{E[X]} \le \|(T \otimes \mathbf{1}_X)(|u|)\|_{E[X]} \le \|T\|_{E\to E}\|u\|_{E[X]},$$

and the proof is complete. ■

The following result is an immediate consequence of Theorem 4.3 and the fact that the Kruglov operator $\mathcal{K}$ is a positive linear operator.

Theorem 4.4. *Let E and X be symmetric spaces on $[0,1]$ that either have the Fatou property or are separable. If $\mathcal{K}(E) \subseteq E$ and $\mathbf{1}_X$ is the identity operator on X, then $\mathcal{K} \otimes \mathbf{1}_X$ is bounded in $E[X]$ and*

$$\|\mathcal{K} \otimes \mathbf{1}_X\|_{E[X]\to E[X]} \le \|\mathcal{K}\|_{E\to E}.$$

Corollary 4.5. *If a symmetric space X on $I = [0,1]$ is such that $q_X < \infty$ and either has the Fatou property or is separable, then $\mathcal{K} \otimes \mathbf{1}_{X[0,1]} : X(I \times I) \to X(I \times I)$.*

Proof. Since the Kruglov operator is bounded in $L_p := L_p(I)$ if $1 \le p < \infty$, and $L_p[L_p]$ is isometrically isomorphic to $L_p(I \times I)$, from Theorem 4.4

it follows that $\mathcal{K}\otimes \mathbf{1}_{L_p[0,1]} : L_p(I\times I)\to L_p(I\times I)$ $(1\le p<\infty)$ and

$$\|\mathcal{K}\otimes \mathbf{1}_{L_p[0,1]}\|_{L_p(I\times I)\to L_p(I\times I)} \le \|\mathcal{K}\|_{L_p\to L_p}.$$

Let now $q_X < r < \infty$. Since X either has the Fatou property or is separable, it is an interpolation space between L_1 and L_∞ [23, Theorems 2.4.9 and 2.4.10]. More specifically, from a one-sided version of the Boyd interpolation theorem [4, Theorem 1], it follows that X is an interpolation space between L_1 and L_r. Therefore, $\mathcal{K}\otimes \mathbf{1}_{X[0,1]}$ is bounded in $X(I\times I)$, and the proof is complete. ∎

5. Disjointification of conditionally independent random variables. By applying the results of the previous section, we are able to prove better results for conditionally independent random variables of the form $\{f_k(s)x_k(t)\}_{k=1}^n$, where f_k's are independent and x_k's are arbitrary, in the case when

$$\sum_{k=1}^{n}\lambda(\operatorname{supp} f_k)\le 1. \tag{5.1}$$

The idea of the following theorem comes from [7] and [8] but was not presented explicitly. We state it here as it plays a key role in the proof of the main theorem, Theorem 5.3. Recalling that we use φ_f to denote the characteristic function of a random variable f, we have

THEOREM 5.1. *If a sequence $\{f_k\}_{k=1}^n\subseteq S([0,1],\lambda)$ consists of pairwise disjointly supported functions satisfying condition* (5.1), *then $\{\mathcal{K}f_k\}_{k=1}^n$ is a sequence of independent random variables.*

Proof. We find the characteristic function of the random vector $(\mathcal{K}f_1,\dots$ $\dots,\mathcal{K}f_n)$. Note first that

$$\sum_{k=1}^{n}(\exp(it_kf_k)-1)=\exp\Big(i\sum_{k=1}^{n}t_kf_k\Big)-1 \qquad (t_k\in\mathbb{R}), \tag{5.2}$$

which follows from

$$\begin{aligned}\sum_{k=1}^{n}(\exp(it_kf_k)-1) &= \sum_{k=1}^{n}\exp(it_kf_k)-n\\ &= \exp\Big(i\sum_{k=1}^{n}t_kf_k\Big)+n-1-n\\ &\qquad \text{(since } f_k\text{'s are disjointly supported)}\\ &= \exp\Big(i\sum_{k=1}^{n}t_kf_k\Big)-1.\end{aligned}$$

Therefore,

$$\begin{aligned}(5.3)\qquad \varphi_{f_1,\dots,f_n}(t_1,\dots,t_n)-1 &= \int_0^1\Big(\exp\Big(i\sum_{k=1}^n t_kf_k(x)\Big)-1\Big)\,dx\\ &= \int_0^1\sum_{k=1}^n(\exp(it_kf_k(x))-1)\,dx \qquad \text{(by (5.2))}\\ &= \sum_{k=1}^n(\varphi_{f_k}(t_k)-1).\end{aligned}$$

Finally, this leads to the equality

$$\begin{aligned}\varphi_{\mathcal{K}f_1,\dots,\mathcal{K}f_n}(t_1,\dots,t_n) &= \varphi_{\sum_{k=1}^n t_k\mathcal{K}f_k}(1) = \varphi_{\mathcal{K}(\sum_{k=1}^n t_kf_k)}(1)\\ &= \exp(\varphi_{\sum_{k=1}^n t_kf_k}(1)-1)\\ &= \exp(\varphi_{f_1,\dots,f_n}(t_1,\dots,t_n)-1)\\ &= \exp\Big(\sum_{k=1}^n(\varphi_{f_k}(t_k)-1)\Big) \qquad \text{(by (5.3))}\\ &= \prod_{k=1}^n\varphi_{\mathcal{K}f_k}(t_k).\end{aligned}$$

Applying the fact that random variables $\xi_1,\dots,\xi_n$ are independent iff $\varphi_{\xi_1,\dots,\xi_n}(t_1,\dots,t_n)=\varphi_{\xi_1}(t_1)\dots\varphi_{\xi_n}(t_n)$ for all real $t_1,\dots,t_n$ (see e.g. [30, p. 284]), we conclude that the random variables $\{\mathcal{K}f_k\}_{k=1}^n$ are independent. ■

Next, we will need an auxiliary statement. In what follows, we consider the cube $\prod_{k=1}^n[0,1]$ with the usual n-dimensional Lebesgue measure λ_n.

LEMMA 5.2. *Let $\{f_k\}_{k=1}^n\subseteq S([0,1],\lambda)$ be a sequence of non-negative independent functions and $\{x_k\}_{k=1}^n\subseteq S([0,1],\lambda)$ be a sequence of arbitrary non-negative measurable functions. Suppose that $\{h_k\}_{k=1}^n$ is a sequence of independent functions on $[0,1]$ such that h_k and $\mathcal{K}f_k$ are equimeasurable for every $k=1,\dots,n$. Then, for every $\tau>0$,*

$$\begin{aligned}(5.4)\qquad \lambda_2\Big\{(s,t)\in I\times I:\sum_{k=1}^n f_k(s)x_k(t)>\tau\Big\}&\\ \le 2\lambda_2\Big\{(s,t)\in I\times I:\sum_{k=1}^n h_k(s)x_k(t)>\tau/2\Big\}&.\end{aligned}$$

Proof. Since $f_i\ge 0$, by the definition of the Kruglov operator, it follows that $\mathcal{K}f_i\ge g_i$, where g_i is equimeasurable with the function $\sigma_{1/2}f_i$ ($i=1,\dots,n$). In particular,

$$(5.5)\qquad \sum_{i=1}^n\mathcal{K}f_i(t_i)x_i(t_0)\ge\sum_{i=1}^n g_i(t_i)x_i(t_0).$$

It is clear that, for every $i = 1, \ldots, n$, the function f_i is equimeasurable with the sum $g_i' + g_i''$ of two disjointly supported functions g_i' and g_i'', each of which is equimeasurable with the function g_i. Therefore, by (5.5), we have

$$\begin{aligned}\lambda_{n+1}\Big\{(t_k)_{k=0}^n : \sum_{i=1}^n (g_i'(t_i) + g_i''(t_i))x_i(t_0) > \tau\Big\} \\ \leq 2\lambda_{n+1}\Big\{(t_k)_{k=0}^n : \sum_{i=1}^n g_i(t_i)x_i(t_0) > \tau/2\Big\} \\ \leq 2\lambda_{n+1}\Big\{(t_k)_{k=0}^n : \sum_{i=1}^n \mathcal{K}f_i(t_i)x_i(t_0) > \tau/2\Big\}.\end{aligned}$$

On the other hand, from Fubini's theorem and the assumptions on f_i, g_i', g_i'' as well as h_i and Kf_i ($i \in \mathbb{N}$), it follows that, for all $\tau > 0$,

$$\begin{aligned}\lambda_2\Big\{(s,t) \in I \times I : \sum_{i=1}^n f_i(s)x_i(t) > \tau\Big\} \\ = \lambda_{n+1}\Big\{(t_k)_{k=0}^n \in \prod_{k=0}^n [0,1] : \sum_{i=1}^n f_i(t_i)x_i(t_0) > \tau\Big\} \\ = \lambda_{n+1}\Big\{(t_k)_{k=0}^n \in \prod_{k=0}^n [0,1] : \sum_{i=1}^n (g_i'(t_i) + g_i''(t_i))x_i(t_0) > \tau\Big\}\end{aligned}$$

and

$$\begin{aligned}\lambda_2\Big\{(s,t) \in I \times I : \sum_{i=1}^n h_i(s)x_i(t) > \tau\Big\} \\ = \lambda_{n+1}\Big\{(t_k)_{k=0}^n \in \prod_{k=0}^n [0,1] : \sum_{i=1}^n h_i(t_i)x_i(t_0) > \tau\Big\} \\ = \lambda_{n+1}\Big\{(t_k)_{k=0}^n \in \prod_{k=0}^n [0,1] : \sum_{i=1}^n \mathcal{K}f_i(t_i)x_i(t_0) > \tau\Big\}.\end{aligned}$$

Combining these with the previous inequality yields the conclusion. ■

The main result of this section is the following.

THEOREM 5.3. *Let X be a symmetric space on $[0,1]$ that either has the Fatou property or is separable. Suppose $q_X < \infty$. Then there exists a constant $C > 0$ such that for every sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent functions satisfying condition* (5.1) *and for every sequence $\{x_k\}_{k=1}^n \subseteq X$ of arbitrary measurable functions we have*

$$\Big\|\sum_{k=1}^n f_k(s)x_k(t)\Big\|_{X(I\times I)} \leq C\Big\|\sum_{k=1}^n \overline{f_k(s)x_k(t)}\Big\|_X. \tag{5.6}$$

Proof. Without loss of generality, we may (and will) assume that $f_k \geq 0$ and $x_k \geq 0$.

Firstly, from Theorem 5.1, it follows that the functions $\mathcal{K}\bar{f}_k$ are independent. Moreover, it is clear that $\mathcal{K}\bar{f}_k$ and $\mathcal{K}f_k$ are equimeasurable for each $k = 1, \dots, n$. Therefore, by [23, Corollary 2.4.2], Lemma 5.2 and Corollary 4.5, we obtain

$$\Big\|\sum_{k=1}^{n} f_k(s)x_k(t)\Big\|_{X(I\times I)} \leq 4\Big\|\sum_{k=1}^{n} \mathcal{K}\bar{f}_k(s)x_k(t)\Big\|_{X(I\times I)}$$
$$\leq 4C'\Big\|\sum_{k=1}^{n} \bar{f}_k(s)x_k(t)\Big\|_{X(I\times I)} = 4C'\Big\|\sum_{k=1}^{n} \overline{f_k(s)x_k(t)}\Big\|_{X},$$

where C' is the norm of the operator $\mathcal{K}\otimes\mathbf{1}$ in the space $X(I\times I)$. ∎

REMARK 5.4. The converse of inequality (5.6) holds in every symmetric space X that either has the Fatou property or is separable (see the proof of Theorem 6.5 in the next section).

The following result shows the necessity of the condition $q_X < \infty$ in the last theorem.

THEOREM 5.5. *Suppose that a symmetric space X on $[0,1]$ has the following property: there exists a constant $C > 0$ such that for every sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent functions satisfying condition* (5.1) *and for every sequence $\{x_k\}_{k=1}^n \subseteq X$ of arbitrary measurable functions, inequality* (5.6) *holds. Then $q_X < \infty$.*

Proof. If $q_X = \infty$, then from Krivine's theorem for symmetric spaces (see e.g. [25, Theorem 2.b.6] or [3, Theorem 4]), it follows that for every integer m, X contains m pairwise disjointly supported functions $\{g_i\}_{i=1}^m$ all having the same distribution so that

$$\frac{1}{2}\max_{i=1,\dots,m}|a_i| \leq \Big\|\sum_{i=1}^{m} a_i g_i\Big\| \leq 2\max_{i=1,\dots,m}|a_i| \tag{5.7}$$

for every choice of scalars $\{a_i\}_{i=1}^m$. Obviously, we may assume that $g_i \geq 0$.

Now fix n and consider the aforementioned sequence $\{g_i\}_{i=1}^m$ with $m = n^{2n}\cdot 2^n$. Consider the sequence of functions $\{x_k\}_{k=1}^{2^n}$, where $x_k := \sum_{i=1}^{k\cdot n^{2n}} g_i$. Note that $0 \leq x_k \leq x_{k+1}$ for $k = 1, \dots, 2^n - 1$. Moreover, let $\{f_k\}_{k=1}^{2^n}$ be a sequence of independent copies of the function $\chi_{[0,2^{-n}]}$. We show that, for the sequences $\{f_k(s)x_k(t)\}_{k=1}^{2^n}$, the norms on the right-hand side of (5.6) are bounded but the ones on the left-hand side are not.

Recalling that σ_a denotes the dilation operator (see Section 2 for definition), we first estimate from above the right-hand side of (5.6):

$$\begin{aligned}
\Big\| \sum_{k=1}^{2^n} \overline{f_k(s)x_k(t)} \Big\|_X &= \Big\| \sum_{k=1}^{2^n} \overline{f_k(t_k)x_k(t_0)} \Big\|_X \\
&= \Big\| \sum_{k=1}^{2^n} \overline{\chi_{[0,2^{-n}]}(t_k)x_k(t_0)} \Big\|_X = \Big\| \sum_{k=1}^{2^n} \overline{\sigma_{2^{-n}}(x_k)} \Big\|_X \\
&\le \Big\| \sum_{k=1}^{2^n} \overline{\sigma_{2^{-n}}(x_{2^n})} \Big\|_X \qquad \text{(since } x_k \le x_{k+1}) \\
&= \|x_{2^n}\|_X \qquad \text{(since the } \overline{\sigma_{2^{-n}}(x_{2^n})}\text{'s are disjointly supported)} \\
&= \Big\| \sum_{i=1}^{2^n \cdot n^{2n}} g_i \Big\|_X \qquad \text{(from the definition of } x_{2^n}) \\
&\le 2 \qquad \text{(from (5.7)).}
\end{aligned}$$

Consider the left-hand side of (5.6). We show that $\|\sum_{k=1}^{2^n} f_k(s)x_k(t)\|_{X(I\times I)}$ is unbounded as $n \to \infty$. Since $x_k \le x_{k+1}$ for $k = 1, \dots, 2^n - 1$, we have

$$\Big\| \sum_{k=1}^{2^n} f_k(s)x_k(t) \Big\|_{X(I\times I)} \ge \Big\| \sum_{k=1}^{2^n} f_k(s)x_1(t) \Big\|_{X(I\times I)}. \tag{5.8}$$

Hence, we first estimate $\sum_{k=1}^{2^n} f_k(s)$ from below.

Since

$$\frac{2^n!}{(2^n-n)!} \ge (2^n-n)^n \quad \text{and} \quad 1 - \frac{1}{2^n} \ge 1 - \frac{n}{2^n},$$

for sufficiently large n we have

$$\begin{aligned}
\lambda\Big\{ s \in [0,1] : \sum_{k=1}^{2^n} f_k(s) \ge n \Big\} &\ge \frac{2^n!}{n!(2^n-n)!} \Big(\frac{1}{2^n}\Big)^n \Big(1 - \frac{1}{2^n}\Big)^{2^n-n} \\
&\ge \frac{(2^n-n)^n}{n!(2^n)^n} \cdot \frac{(2^n-n)^{2^n-n}}{(2^n)^{2^n-n}} \\
&= \frac{1}{n!}\Big(1 - \frac{n}{2^n}\Big)^{2^n} \ge \frac{1}{n^{2n}}.
\end{aligned}$$

Therefore, if $A := \{s \in [0,1] : \sum_{k=1}^{2^n} f_k(s) \ge n\}$, then by (5.8),

$$\Big\| \sum_{k=1}^{2^n} f_k(s)x_k(t) \Big\|_{X(I\times I)} \ge \|n\chi_A(s)x_1(t)\|_{X(I\times I)} = n\|\sigma_{n^{-2n}}(x_1)\|_X.$$

But by the definition of x_1 and (5.7), $\|\sigma_{n^{-2n}}(x_1)\|_X = \|g_1\|_X \ge 1/2$. Hence, taking into account the previous inequality, we obtain

$$\Big\|\sum_{k=1}^{2^n} f_k(s)x_k(t)\Big\|_{X(I\times I)} \to \infty \quad \text{as } n\to\infty.$$

Thus, the left-hand side of (5.6) is unbounded as $n\to\infty$ and the result follows. ■

REMARK 5.6. Assumption (5.1) for a sequence $\{f_k\}_{k=1}^n \subseteq X$ of independent functions is essential. In fact, Theorems 3.1 and 3.4 show that the inequality

$$\Big\|\sum_{i=1}^{n} r_i(t)x_i(s)\Big\|_{X(I\times I)} \le C\Big\|\sum_{i=1}^{n} \bar{x}_i\Big\|_{Z_X^2},$$

where $\{r_i(t)\}$ are the Rademacher functions, holds for any sequence $\{x_i\}_{i=1}^n \subset X$ only if $X\supset L_2$.

REMARK 5.7. Recall that some disjointification relations using the space Z_X^1 instead of Z_X^2 were obtained earlier for non-negative independent random variables as well (see [21, 7, 9]). However, in the case of mds and even conditionally independent functions, analogous formulas do not hold for any other symmetric spaces apart from L_1.

In fact, suppose that a symmetric space X satisfies the following condition: there is a constant $C>0$ such that for any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions and any sequence $\{x_k\}_{k=1}^n \subset X$ of measurable functions,

$$\Big\|\sum_{i=1}^{n} f_i(s)x_i(t)\Big\|_{X(I\times I)} \le C\Big\|\sum_{i=1}^{n} \overline{f_i(s)x_i(t)}\Big\|_{Z_X^1}. \tag{5.9}$$

Take $f_k(s)=1$ and $x_k(t)=\chi_{[0,1/n]}(t)$ $(k=1,\ldots,n)$. Then $\sum_{k=1}^n f_k(s)x_k(t) = n\chi_{[0,1/n]}(t)$ and the sum $\sum_{k=1}^n \overline{f_k(s)x_k(t)}$ is equimeasurable with $\chi_{[0,1]}$. Then inequality (5.9) gives

$$n\|\chi_{[0,1/n]}\|_X \le C\|\chi_{[0,1]}\|_X.$$

At the same time,

$$\|\chi_{[0,1]}\|_X = \Big\|\sum_{k=1}^{n} \overline{\chi_{[0,1/n]}}\Big\|_X \le \sum_{k=1}^{n}\|\overline{\chi_{[0,1/n]}}\|_X = n\|\chi_{[0,1/n]}\|_X.$$

Hence,

$$\|\chi_{[0,1/n]}\|_X \asymp \frac{1}{n}\|\chi_{[0,1]}\|_X \quad (n\in\mathbb{N}). \tag{5.10}$$

Next, recall that given the symmetric space X, the function $\phi_X(t) := \|\chi_A\|_X$, where $\lambda(A)=t$, is the *fundamental function* of X. Recall also that $\Lambda_{\phi_X}\subseteq X$, where Λ_{ϕ_X} is the corresponding Lorentz space (see Section 2.1).

It is not hard to check that (5.10) implies $\phi_X(t) \asymp t$. Therefore, $\Lambda_{\phi_X} = L_1$, whence $L_1 \subseteq X$. But for all symmetric spaces X on $[0,1]$ we have $X \subseteq L_1$. Thus, inequality (5.9) holds only in the case $X = L_1$.

6. Comparison of martingale differences and non-negative random variables with their independent copies. The results on the disjointification of martingale differences proven in the previous sections allow us to compare norms of sums of martingale differences with the norms of sums of their independent copies.

THEOREM 6.1. *Let X be a symmetric space on $[0,1]$, $X \in I(L_1, L_2)$. Then there exists a constant $C > 0$ such that for every mds $\{d_k\}_{k=1}^n \subset X$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions such that f_k is equimeasurable with d_k for every $k = 1, \dots, n$ we have*

$$\Big\| \max_{k=1,\dots,n} \Big| \sum_{i=1}^k d_i \Big| \Big\|_X \le C \Big\| \sum_{k=1}^n f_k \Big\|_X. \tag{6.1}$$

Proof. Since $\int_0^1 f_k(t)\,dt = \int_0^1 d_k(t)\,dt = 0$ $(k = 1, \dots, n)$, it follows that $\{f_k\}_{k=1}^n$ is a sequence of mean zero independent functions. Therefore, by [21, Theorem 1], there is a constant $C' > 0$, depending only on X, such that

$$\Big\| \sum_{k=1}^n \bar{f}_k \Big\|_{Z_X^2} \le C' \Big\| \sum_{k=1}^n f_k \Big\|_X.$$

Combining this inequality with inequality (3.2) proved in Corollary 3.2 and applying the fact that the functions $\sum_{k=1}^n \bar{d}_k$ and $\sum_{k=1}^n \bar{f}_k$ are equimeasurable, we get the result. ∎

THEOREM 6.2. *Suppose that a symmetric space $X \in I(L_2, L_\infty)$ has order semicontinuous norm and $q_X < \infty$. Then there exists a constant $C > 0$ such that for every mds $\{d_k\}_{k=1}^n \subset X$ and any sequence $\{f_k\}_{k=1}^n \subset X$ of independent functions such that f_k is equimeasurable with d_k for every $k = 1, \dots, n$ we have*

$$\Big\| \sum_{k=1}^n f_k \Big\|_X \le C \Big\| \max_{k=1,\dots,n} \Big| \sum_{i=1}^k d_i \Big| \Big\|_X. \tag{6.2}$$

Proof. Since $q_X < \infty$, the operator $\mathcal{K}$ is bounded in X. So we may apply [8, Theorem 3.1] to find a constant $C' > 0$, depending only on X, such that

$$\Big\| \sum_{k=1}^n f_k \Big\|_X \le C' \Big\| \sum_{k=1}^n \bar{f}_k \Big\|_{Z_X^2}.$$

Inequality (6.2) follows from this and inequality (3.7) from Corollary 3.6. ∎

For sequences $\{f_k(s)x_k(t)\}_{k=1}^n$ of conditionally independent random variables satisfying condition (5.1) a rather stronger result holds. Theorem 5.3 and [21, Theorem 1] imply

THEOREM 6.3. *Let X be a symmetric space on $[0,1]$ that either has the Fatou property or is separable. Suppose $q_X < \infty$. Then there exists a constant $C > 0$ such that for arbitrary sequences $\{f_k\}_{k=1}^n \subseteq X$ of independent functions satisfying condition* (5.1)*, measurable functions $\{x_k\}_{k=1}^n \subseteq X$ and for any sequence $\{g_k\}_{k=1}^n \subseteq X$ of independent functions such that g_k is equimeasurable with the function $f_k(s)x_k(t)$ $(k = 1, \dots, n)$, we have*

$$\Big\|\sum_{k=1}^n f_k(s)x_k(t)\Big\|_{X(I\times I)} \le C\Big\|\sum_{k=1}^n g_k\Big\|_X. \tag{6.3}$$

REMARK 6.4. Let X be a symmetric space on $[0,1]$ that either has the Fatou property or is separable. Suppose also that $\mathcal{K}: X \to X$. Then, by [8, Theorem 3.1] and Remark 5.4, we obtain the converse of inequality (6.3), i.e., if sequences $\{f_k\}_{k=1}^n \subseteq X$, $\{x_k\}_{k=1}^n \subseteq X$, and $\{g_k\}_{k=1}^n \subseteq X$ satisfy the same conditions as in Theorem 6.3, then

$$\Big\|\sum_{k=1}^n g_k\Big\|_X \le C\Big\|\sum_{k=1}^n f_k(s)x_k(t)\Big\|_{X(I\times I)} \tag{6.4}$$

for some constant $C > 0$.

Now, we prove some inequalities comparing sums of non-negative random variables to sums of their independent copies in the setting of symmetric spaces. These inequalities are strongly related to the results of Theorems 6.1 and 6.2. Furthermore, the results on non-negative random variables and their independent copies provide a simpler method of obtaining the modular comparison inequalities (see [28, p. 82] for terminology) proved in [17] (see also [18]).

THEOREM 6.5. *Suppose the operator $\mathcal{K}$ is bounded on a symmetric space X on $[0,1]$ that either has the Fatou property or is separable. Then for arbitrary sequences $\{g_k\}_{k=1}^n$ of non-negative functions and $\{f_k\}_{k=1}^n$ of independent functions from X such that f_k is equimeasurable with g_k for every $k = 1, \dots, n$ we have*

$$\Big\|\sum_{k=1}^n f_k\Big\|_X \le \alpha\|\mathcal{K}\|_{X\to X}\Big\|\sum_{k=1}^n g_k\Big\|_X, \tag{6.5}$$

with a universal constant $\alpha > 0$.

Proof. First, let $Z[0,\infty)$ be a symmetric space on the semiaxis $[0,\infty)$ and $\{h_k\}_{k=1}^n \subset Z[0,\infty)$ $(n \in \mathbb{N})$ be an arbitrary sequence of non-negative measurable functions on $[0,\infty)$ whose supports have finite measure. Obviously, for every $\tau > 0$,

$$\int_0^\tau \Big(\sum_{k=1}^n \bar{h}_k\Big)^*(s)\,ds \le \int_0^\tau \Big(\sum_{k=1}^n h_k\Big)^*(s)\,ds.$$

Then if $Z[0,\infty) \in I(L_1[0,\infty), L_\infty[0,\infty))$, by [23, Theorem 2.4.3], we obtain

$$\Big\|\sum_{k=1}^n \bar{h}_k\Big\|_{Z[0,\infty)} \le C\Big\|\sum_{k=1}^n h_k\Big\|_{Z[0,\infty)}, \tag{6.6}$$

where C is the interpolation constant of the space $Z[0,\infty)$ in the couple $(L_1[0,\infty), L_\infty[0,\infty))$.

Note that the space Z_X^1 also either has the Fatou property or is separable (depending on X). Therefore, by assumption, $Z_X^1 \in I(L_1[0,\infty), L_\infty[0,\infty))$ with constant 1.

Now, let sequences $\{g_k\}_{k=1}^n$ and $\{f_k\}_{k=1}^n \subset X$ satisfy our assumptions. Then, applying arguments similar to that at the beginning of the proof together with inequality (6.6) to the space Z_X^1, we obtain

$$\Big\|\sum_{k=1}^n \bar{f}_k\Big\|_{Z_X^1} \le \Big\|\sum_{k=1}^n g_k\Big\|_X. \tag{6.7}$$

On the other hand, by [9, Theorem 1(ii)], there is a universal constant $\alpha > 0$ such that

$$\Big\|\sum_{k=1}^n f_k\Big\|_X \le \alpha\|\mathcal{K}\|_{X\to X}\Big\|\sum_{k=1}^n \bar{f}_k\Big\|_{Z_X^1}.$$

Combining this inequality with (6.7), we obtain (6.5). ∎

Remark 6.6. With a different (non-interpolation) method, inequality (6.6) in the case when a symmetric space $Z[0,\infty)$ is separable is proved in [19, Lemma 7.2].

Now we are able to prove the following modular inequality.

Corollary 6.7. *Let M be an Orlicz function on $[0,\infty)$ such that the operator $\mathcal{K}$ is bounded on the Orlicz space $L_M = L_M[0,1]$. Then there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for arbitrary sequences $\{g_k\}_{k=1}^n$ of non-negative measurable functions and $\{f_k\}_{k=1}^n$ of independent functions from X such that f_k is equimeasurable with g_k $(k = 1,\dots,n)$, there is a $\tau > 0$, depending only on $\|\sum_{k=1}^n g_k\|_{L_M}$, such that*

$$\int_0^1 M\Big(\frac{\sum_{k=1}^n f_k(s)}{\tau}\Big)\,ds \le \int_0^1 M\Big(\frac{C\sum_{k=1}^n g_k(s)}{\tau}\Big)\,ds. \tag{6.8}$$

In particular, if $\|\sum_{k=1}^n g_k\|_{L_M} = 1$*, we may take* $\tau = C$ *to obtain*

$$\int_0^1 M\Big(\frac{\sum_{k=1}^n f_k(s)}{C}\Big)\,ds \le \int_0^1 M\Big(\sum_{k=1}^n g_k(s)\Big)\,ds. \tag{6.9}$$

Proof. Since an arbitrary Orlicz space has the Fatou property, from Theorem 6.5, it follows that

$$\Big\|\sum_{k=1}^n f_k\Big\|_{L_M} \le C\Big\|\sum_{k=1}^n g_k\Big\|_{L_M},$$

where $C := \alpha\|\mathcal{K}\|_{L_M\to L_M}$. If $\tau := C\|\sum_{k=1}^n g_k\|_{L_M}$, then

$$\int_0^1 M\Big(\frac{C\sum_{k=1}^n g_k(s)}{\tau}\Big)\,ds = 1.$$

Hence, from the previous inequality and the definition of the Luxemburg norm in L_M we have

$$\int_0^1 M\Big(\frac{\sum_{k=1}^n f_k(s)}{\tau}\Big)\,ds \le 1,$$

which implies (6.8). Inequality (6.9) follows from (6.8) and the definition of τ. ■

Let us show how inequalities for non-negative random variables and their independent copies imply analogous inequalities for martingale differences and their independent copies. But first, we need an auxiliary result (see e.g. [6, Corollary 15]).

Recall that a Banach lattice E is said to be *p-convex* ($1 \le p < \infty$) with constant $K \ge 1$ if

$$\Big\|\Big(\sum_{k=1}^n |x_k|^p\Big)^{1/p}\Big\| \le K\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p}$$

for every choice of vectors $x_1, \dots, x_n$ from E. For a p-convex Banach function lattice E we may define its *p-concavification*, i.e., the Banach lattice $E_{(p)}$ with the norm $\|x\|_{E_{(p)}} := \|\,|x|^{1/p}\|_E^p$.

LEMMA 6.8. *Suppose that* $p > 1$ *and the Kruglov operator* $\mathcal{K}$ *is bounded on a p-convex symmetric space* X*. Then* $\mathcal{K}$ *is bounded on* $X_{(p)}$ *as well and* $\|\mathcal{K}\|_{X_{(p)}\to X_{(p)}} \le \|\mathcal{K}\|_{X\to X}^p$.

Proof. By definition, $\mathcal{K}f$ is equimeasurable with the function

$$\sum_{n=1}^\infty \sum_{k=1}^n f_{n,k}(x)\chi_{E_n}(x), \qquad x \in [0,1],$$

where $\{E_n\}$ is a sequence of measurable pairwise disjoint subsets of $[0,1]$, $\lambda(E_n) = 1/(en!)$ $(n \in \mathbb{N})$, and $f_{n,1}, \dots, f_{n,n}$ are copies of f such that the sequence $f_{n,1}, \dots, f_{n,n}, \chi_{E_n}$ consists of independent functions. Then, by the elementary inequality

$$\Big|\sum_{k=1}^{n} \alpha_k\Big|^{1/p} \le \sum_{k=1}^{n} |\alpha_k|^{1/p} \qquad (\alpha_k \in \mathbb{R}),$$

we see that $(\mathcal{K}f)^{*1/p} \le (\mathcal{K}(|f|^{1/p}))^*$. Therefore,

$$\|\mathcal{K}f\|_{X_{(p)}} = \big\|\,|\mathcal{K}f|^{1/p}\big\|_X^p \le \|\mathcal{K}(|f|^{1/p})\|_X^p \le \|\mathcal{K}\|_{X\to X}^p \|f\|_{X_{(p)}},$$

and we obtain the result. ∎

THEOREM 6.9. *Suppose that the operator $\mathcal{K}$ is bounded in a 2-convex symmetric space X on $[0,1]$ that either has the Fatou property or is separable. Then for an arbitrary mds $\{d_k\}_{k=1}^n \subseteq X$ and for any sequence $\{f_k\}_{k=1}^n$ of independent functions such that f_k is equimeasurable with d_k for every $k = 1, \dots, n$ we have*

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_X \le C_1 \Big\|\Big(\sum_{k=1}^{n} d_k^2\Big)^{1/2}\Big\|_X, \tag{6.10}$$

with a constant $C_1 > 0$.

In particular, if $q_X < \infty$, then there is a constant $C_2 > 0$ such that

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_X \le C_2 \Big\|\max_{k=1,\dots,n}\Big|\sum_{i=1}^{k} d_i\Big|\Big\|_X. \tag{6.11}$$

Proof. Note that the space $X_{(2)}$ has the Fatou property (respectively, is separable) if the space X has the Fatou property (respectively, is separable). Therefore, applying Lemma 6.8 and Theorem 6.5 for sequences $\{d_k^2\}_{k=1}^n$ and $\{f_k^2\}_{k=1}^n$, we obtain

$$\Big\|\sum_{k=1}^{n} f_k^2\Big\|_{X_{(2)}} \le \alpha \|\mathcal{K}\|_{X\to X}^2 \Big\|\sum_{k=1}^{n} d_k^2\Big\|_{X_{(2)}},$$

which implies

$$\Big\|\Big(\sum_{k=1}^{n} f_k^2\Big)^{1/2}\Big\|_X \le \alpha^{1/2}\|\mathcal{K}\|_{X\to X} \Big\|\Big(\sum_{k=1}^{n} d_k^2\Big)^{1/2}\Big\|_X.$$

Since $\{f_k\}_{k=1}^n$ is a sequence of mean zero independent functions and $\mathcal{K}$ is bounded in X, we see by [2, Theorem 1] that

$$\Big\|\Big(\sum_{k=1}^{n} f_k^2\Big)^{1/2}\Big\|_X \asymp \Big\|\sum_{k=1}^{n} f_k\Big\|_X.$$

This and the previous inequality imply (6.10). Assuming $q_X < \infty$, we get (6.11) as an immediate consequence of (6.10) and [20, Theorem 3]. ∎

Remark 6.10. In Theorem 6.2, inequality (6.11) was proved under the weaker assumption $X \in I(L_2, L_\infty)$.

Theorems 6.1, 6.5 and 6.9 can be viewed as extensions of the modular inequalities proved in [17] and [18] to the setting of symmetric spaces. On the other hand, using the results obtained here, we can prove those modular inequalities. Let us prove, for instance, the modular version of (6.5).

In the following, we will use the notation $\mathbb{E}F = \int_0^1 F(s)\,ds$. Recall that an Orlicz function M on $[0,\infty)$ satisfies *the Δ_2-condition at infinity* if there exist $C > 0$ and $u_0 > 0$ such that $M(2u) \le CM(u)$ for all $u \ge u_0$.

Theorem 6.11. *Let M be an Orlicz function on $[0,\infty)$ satisfying the Δ_2-condition at infinity. Then, possibly after changing the function M on the interval $[0,1]$, there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ and for arbitrary sequences $\{g_k\}_{k=1}^n$ of non-negative measurable functions and $\{f_k\}_{k=1}^n$ of independent functions from the Orlicz space L_M on $[0,1]$ such that f_k is equimeasurable with g_k $(k = 1,\dots,n)$, we have*

$$\mathbb{E}\Big(M\Big(\sum_{k=1}^n f_k\Big)\Big) \le C\mathbb{E}\Big(M\Big(\sum_{k=1}^n g_k\Big)\Big). \tag{6.12}$$

Proof. First of all, note that by assumption, the upper Boyd index q_{L_M} of L_M is finite (see e.g. [26, Theorem 11.7]). Therefore, since $q_{L_{M_t}} = q_{L_M} < \infty$, where $M_t := t^{-1}M$ $(t > 0)$, we have $\mathcal{K} : L_{M_t} \to L_{M_t}$. Moreover, for any $t > 0$, L_{M_t} has the Fatou property. Therefore, by Theorem 6.5,

$$\Big\|\sum_{k=1}^n f_k\Big\|_{L_{M_t}} \le \alpha\|\mathcal{K}\|_{L_{M_t}\to L_{M_t}}\Big\|\sum_{k=1}^n g_k\Big\|_{L_{M_t}}, \tag{6.13}$$

where $\alpha > 0$ is a universal constant. Let us show that

$$\sup_{t>0}\|\mathcal{K}\|_{L_{M_t}\to L_{M_t}} \le C_1\|\mathcal{K}\|_{L_M\to L_M} \tag{6.14}$$

with some $C_1 > 0$.

Without loss of generality, we may (and do) assume that

$$M(u+v) \le \gamma(M(u) + M(v)) \tag{6.15}$$

for some constant $\gamma > 0$ and all $u, v \ge 0$ (see for instance [14, formula (7.9)] or [25, Proposition 2.b.5]).

Let now $f \ge 0$. Suppose that $\mathbb{E}M(f) := A < \infty$. If $\{f_k\}_{k=1}^\infty$ is a sequence of independent functions equimeasurable with f, then, by (6.15), for every

$n \in \mathbb{N}$, we have

$$\mathbb{E}\Big(M\Big(\sum_{k=1}^{n} f_k\Big)\Big) \le \gamma\mathbb{E}\Big(M\Big(\sum_{k=1}^{n-1} f_k\Big) + M(f_n)\Big) = \gamma\Big(\mathbb{E}M\Big(\sum_{k=1}^{n-1} f_k\Big) + A\Big)$$
$$\le \gamma^2\Big(\mathbb{E}M\Big(\sum_{k=1}^{n-2} f_k\Big) + 2A\Big) \le \cdots \le \gamma^{n-1} nA.$$

By definition, $\mathcal{K}f$ equals $\sum_{k=1}^{n} f_k$ on a set E_n with Lebesgue measure $\lambda(E_n) = 1/(en!)$ $(n = 1, 2, \dots)$; moreover, the E_n are pairwise disjoint and the family $\{f_1, \dots, f_n, \chi_{E_n}\}$ consists of independent functions. Therefore,

$$\mathbb{E}M(\mathcal{K}f) = \sum_{n=1}^{\infty} \mathbb{E}M\Big(\sum_{k=1}^{n} f_k\chi_{E_n}\Big) = \sum_{n=1}^{\infty} \mathbb{E}M\Big(\sum_{k=1}^{n} f_k\Big)\lambda(E_n)$$
$$\le \sum_{n=1}^{\infty} \gamma^{n-1} nA\lambda(E_n) = \frac{A}{e}\sum_{n=1}^{\infty} \frac{\gamma^{n-1}}{(n-1)!} = e^{\gamma-1}A,$$

whence,

$$\mathbb{E}M_t(\mathcal{K}f) \le e^{\gamma-1}\mathbb{E}M_t(f) \quad (t > 0).$$

From this it follows that

$$\|\mathcal{K}f\|_{L_{M_t}} \le e^{\gamma-1}\|f\|_{L_{M_t}} \quad (t > 0),$$

i.e., (6.14) is proved. Therefore, by (6.13), there is a constant $C > 0$ such that

$$(6.16) \qquad \Big\|\sum_{k=1}^{n} f_k\Big\|_{L_{M_t}} \le C\Big\|\sum_{k=1}^{n} g_k\Big\|_{L_{M_t}} \quad (t > 0)$$

for every $n \in \mathbb{N}$, where $\{g_k\}_{k=1}^{n}$ is an arbitrary sequence of non-negative measurable functions from L_M and $\{f_k\}_{k=1}^{n}$ is any sequence of their independent copies.

Furthermore, we may find $t > 0$ such that

$$(6.17) \qquad \int_0^1 M\Big(C\sum_{k=1}^{n} g_k(s)\Big)\,ds = t,$$

or equivalently,

$$\Big\|\sum_{k=1}^{n} g_k\Big\|_{L_{M_t}} = \frac{1}{C}.$$

Then from (6.16) it follows that

$$\int_0^1 M\Big(C\sum_{k=1}^{n} f_k(s)\Big)\,ds \le t.$$

Combining this with equality (6.17), we obtain

$$\int_0^1 M\Big(\sum_{k=1}^n f_k(s)\Big)\,ds \le \int_0^1 M\Big(C\sum_{k=1}^n g_k(s)\Big)\,ds$$

and the proof is complete because M satisfies the Δ_2-condition at infinity. ∎

Using Theorem 6.9, arguments analogous to the ones used in the proof of the previous theorem and the Burkholder–Davis–Gundy square function inequality from [15], it is not hard to prove the following comparison assertion.

THEOREM 6.12. *Let M be an Orlicz function on $[0,\infty)$ satisfying the Δ_2-condition at infinity such that the function $M(\sqrt{t})$ is convex for $t>0$. Then, possibly after changing the function M on $[0,1]$, there exists a constant $C>0$ such that for an arbitrary mds $\{d_k\}_{k=1}^n$ on $[0,1]$ and for any sequence $\{f_k\}_{k=1}^n$ of independent functions such that f_k is equimeasurable with d_k $(k=1,\dots,n)$, we have*

$$\mathbb{E}\Big(M\Big(\sum_{k=1}^n f_k\Big)\Big) \le C\mathbb{E}\Big(M\Big(\max_{k=1,\dots,n}\Big|\sum_{i=1}^k d_i\Big|\Big)\Big). \tag{6.18}$$

REMARK 6.13. Similar disjointification methods allow us to prove the converses of the modular inequalities (6.12) and (6.18) (see [17] and [18]) as well. We do not pursue this subject here, noting only that the relation (6.1) proved in Theorem 6.1 is an extension of these inequalities to the setting of symmetric spaces.

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References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, FL, 1985.

[2] S. V. Astashkin, *Independent functions in symmetric spaces and the Kruglov property*, Mat. Sb. 199 (2008), no. 7, 3–20 (in Russian); English transl.: Sb. Math. 199 (2008), 945–963.

[3] —, *On finite representability of l_p-spaces in symmetric spaces*, Algebra i Analiz 23 (2011), no. 2, 77–101 (in Russian).

[4] S. V. Astashkin and L. Maligranda, *Interpolation between L_1 and L_p, $1<p<\infty$*, Proc. Amer. Math. Soc. 132 (2004), 2929–2938.

[5] S. V. Astashkin, E. M. Semenov and F. A. Sukochev, *The Banach–Saks p-property*, Math. Ann. 332 (2005), 879–900.

[6] S. V. Astashkin, E. M. Semenov, F. A. Sukochev and D. V. Zanin, *Kruglov operator and operators defined by random permutations*, Funktsional. Anal. i Prilozhen. 43 (2009), no. 2, 3–18 (in Russian); English transl.: Funct. Anal. Appl. 43 (2009), 83–95.

[7] S. V. Astashkin and F. A. Sukochev, *Series of independent random variables in rearrangement invariant spaces: an operator approach*, Israel J. Math. 145 (2005), 125–156.

[8] —, —, *Series of independent mean zero random variables in symmetric spaces with the Kruglov property*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 345 (2007), 25–50, 140 (in Russian); English transl.: J. Math. Sci. (N. Y.) 148 (2008), 795–809.

[9] —, —, *Best constants in Rosenthal-type inequalities and the Kruglov operator*, Ann. Probab. 38 (2010), 1986–2008.

[10] —, —, *Independent functions and Banach space geometry*, Uspekhi Mat. Nauk 65 (2010), no. 6, 3–86 (in Russian); English transl.: Russian Math. Surveys 65 (2010), 1003–1081.

[11] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Orlando, FL, 1988.

[12] M. Sh. Braverman, *Independent Random Variables and Rearrangement Invariant Spaces*, Cambridge Univ. Press, Melbourne, 1994.

[13] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces, Vol. I*, North-Holland, Amsterdam, 1991.

[14] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1 (1973), 19–42.

[15] D. L. Burkholder, B. J. Davis and R. F. Gundy, *Integral inequalities for convex functions of operators on martingales*, in: Proc. Sixth Berkeley Sympos. on Mathematical Statistics and Probability, Vol. II: Probability theory, Univ. California Press, Berkeley, CA, 1972, 223–240.

[16] N. L. Carothers and S. J. Dilworth, *Equidistributed random variables in* $L_{p,q}$, J. Funct. Anal. 84 (1989), 146–159.

[17] V. H. de la Peña, *Bounds on the expectation of functions of martingales and sums of positive rv's in terms of norms of sums of independent random variables*, Proc. Amer. Math. Soc. 108 (1990), 233–239.

[18] P. Hitczenko, *A remark on the paper "Martingale inequalities in rearrangement invariant function spaces" by W. B. Johnson and G. Schechtman*, in: Functional Analysis (Austin, TX, 1987/1989), Lecture Notes in Math. 1470, Springer, Berlin, 1991, 177–182.

[19] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. 19 (1979), no. 217.

[20] W. B. Johnson and G. Schechtman, *Martingale inequalities in rearrangement invariant function spaces*, Israel J. Math. 64 (1988), 267–275.

[21] —, —, *Sums of independent random variables in rearrangement invariant function spaces*, Ann. Probab. 17 (1989), 789–808.

[22] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Nauka, Moscow, 1977 (in Russian); English transl.: Pergamon Press, Oxford, 1982.

[23] S. G. Kreĭn, Yu. I. Petunin and E. M. Semënov, *Interpolation of Linear Operators*, Transl. Math. Monogr. 54, Amer. Math. Soc., 1982.

[24] V. M. Kruglov, *A remark on the theory of infinitely divisible distributions*, Teor. Veroyatnost. i Primenen. 15 (1970), 330–336 (in Russian); English transl.: Theor. Probab. Appl. 15 (1970), 319–324.

[25] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vol. II, Springer, Berlin, 1979.

[26] L. Maligranda, *Orlicz Spaces and Interpolation*, Sem. Mat. 5, Dep. Math., Univ. Estad. Campinas, Campinas SP, 1989.

[27] M. Mastyło, *Interpolation of linear operators in the Köthe dual spaces*, Ann. Mat. Pura Appl. (4) 154 (1989), 231–242.

[28] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Dekker, New York, 1991.

[29] H. P. Rosenthal, *On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273–303.

[30] A. N. Shiryaev, *Probability*, Springer, New York, 1984.

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