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Some weighted norm inequalities for a one-sided version of g_λ^*

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Abstract. We study the boundedness of the one-sided operator $g^+_{\lambda,\varphi}$ between the weighted spaces $L^p(M^-w)$ and $L^p(w)$ for every weight w. If $\lambda = 2/p$ whenever $1 < p < 2$, and in the case $p = 1$ for $\lambda > 2$, we prove the weak type of $g^+_{\lambda,\varphi}$. For every $\lambda > 1$ and $p = 2$, or $\lambda > 2/p$ and $1 < p < 2$, the boundedness of this operator is obtained. For $p > 2$ and $\lambda > 1$, we obtain the boundedness of $g^+_{\lambda,\varphi}$ from $L^p((M^-)^{[p/2]+1}w)$ to $L^p(w)$, where $(M^-)^k$ denotes the operator M^- iterated k times.

1. Notations and definitions. As usual, $\mathcal{S}$ denotes the class of all those C^∞-functions defined on $\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} |x^m(D^n\varphi)(x)| < \infty$$

for all non-negative integers m and n. We also consider the space C_0^∞ of all C^∞-functions defined on $\mathbb{R}$ with compact support.

If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its Lebesgue measure by $|E|$, and the characteristic function of E by $\chi_E(x)$.

Let f be a measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions M^-f and M^+f are given by

$$M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|\,dt, \qquad M^+f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)|\,dt.$$

A *weight* w is a measurable and non-negative function defined on $\mathbb{R}$. If $E \subset \mathbb{R}$ is a measurable set, we denote its w-measure by $w(E) = \int_E w(t)\,dt$. Given $p \ge 1$, $L^p(w)$ is the space of all measurable functions f such that

$$\|f\|_{L^p(w)} = \Big(\int_{-\infty}^{\infty} |f(x)|^p w(x)\,dx\Big)^{1/p} < \infty.$$

If $w = 1$, we simply write L^p and $\|f\|_{L^p}$.

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We shall say that a function $B : [0,\infty) \to [0,\infty)$ is a *Young function* if it is continuous, convex, increasing and satisfies $\lim_{t\to\infty} B(t) = \infty$. The *Luxemburg norm* of a function f is given by

$$\|f\|_B = \inf\Big\{\lambda > 0 : \int B(|f|/\lambda) \le 1\Big\},$$

and the *average* over an interval I is:

$$\|f\|_{B,I} = \inf\Big\{\lambda > 0 : \frac{1}{|I|}\int_I B(|f|/\lambda) \le 1\Big\}.$$

The one-sided maximal operators associated to B are defined as

$$M_B^+(f)(x) = \sup_{h>0} \|f\|_{B,[x,x+h]}, \qquad M_B^-(f)(x) = \sup_{h>0} \|f\|_{B,[x-h,x]}.$$

Let φ belong to $\mathcal{S}$ and be supported on $(-\infty,0]$ with $\int \varphi(x)\,dx = 0$. For every $\lambda > 1$, the one-sided operator $g_{\lambda,\varphi}^+$ was defined in [RoSe] as

$$g_{\lambda,\varphi}^+(f)(x) = \bigg(\int_0^\infty \int_x^\infty \Big(\frac{t}{t+y-x}\Big)^\lambda |f*\varphi_t(y)|^2 \frac{dy\,dt}{t^2}\bigg)^{1/2}.$$

Throughout this paper the letter C will always mean a positive constant not necessarily the same at each occurrence. If $1 < p < \infty$ then p' denotes its conjugate exponent: $p + p' = pp'$.

2. Statement of the results. In [CW], S. Chanillo and R. Wheeden obtained the boundedness of the area integral between the spaces $L^p(Mw)$ and $L^p(w)$ when $1 < p \le 2$. For $p = 2$ and $\lambda > 1$, if the support of φ is compact, they showed in [CW, Lemma (1.1)] that the operator $g_{\lambda,\varphi}^*$ maps $L^2(Mw)$ into $L^2(w)$. We shall give, in Theorem A, a one sided-version of this result without the restriction on the support of φ. For $1 < p < 2$ and $\lambda = 2/p$, in order to prove Theorem B below, we use some arguments due to C. Fefferman (see [F]). As a consequence of Theorems A and B, for $1 < p \le 2$ and $\lambda > 2/p$, we obtain, in Theorem C, the boundedness of $g_{\lambda,\varphi}^+$ between $L^p(M^-w)$ and $L^p(w)$. For $p > 2$, the known techniques (see [P]) allow us to prove Theorem D.

Next, we state the already mentioned Theorems A–D.

THEOREM A. *Let $\varphi \in \mathcal{S}$ with* $\operatorname{supp}(\varphi) \subset (-\infty,0]$ *and* $\int \varphi(x)\,dx = 0$. *Then, for every $\lambda > 1$,*

$$\Big(\int_{-\infty}^\infty g_{\lambda,\varphi}^+(f)(x)^2 w(x)\,dx\Big)^{1/2} \le C_{\lambda,\varphi}\Big(\int_{-\infty}^\infty |f(x)|^2 M^-w(x)\,dx\Big)^{1/2},$$

with a constant $C_{\lambda,\varphi}$ not depending on f.

THEOREM B. *Let $\varphi \in \mathcal{S}$ with* $\operatorname{supp}(\varphi) \subset (-\infty,0]$ *and* $\int \varphi(x)\,dx = 0$. *Let $\lambda > 2$ if $p = 1$, and $\lambda = 2/p$ whenever $1 < p < 2$. Then there exists a* *constant* $C_{p,\lambda,w,\varphi}$ *such that*

$$w(\{x : g^+_{\lambda,\varphi}(f)(x) > \mu\}) \le \frac{C_{p,\lambda,w,\varphi}}{\mu^p} \int_{-\infty}^{\infty} |f(x)|^p M^- w(x)\, dx$$

for every function f *and* $\mu > 0$.

THEOREM C. *Let* $\varphi \in \mathcal{S}$ *with* $\operatorname{supp}(\varphi) \subset (-\infty, 0]$ *and* $\int \varphi(x)\, dx = 0$. *Let* $1 < p \le 2$. *If* $\lambda > 2/p$, *then there exists a constant* $C_{p,\lambda,w,\varphi}$ *such that*

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^p w(x)\, dx \le C_{p,\lambda,w,\varphi} \int_{-\infty}^{\infty} |f(x)|^p M^- w(x)\, dx$$

for every function f.

THEOREM D. *Let* $\varphi \in \mathcal{S}$ *with* $\operatorname{supp}(\varphi) \subset (-\infty, 0]$ *and* $\int \varphi(x)\, dx = 0$. *Let* $\lambda > 1$ *and* $p > 2$. *Then there exists a constant* $C_{p,\lambda,w,\varphi}$ *such that*

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^p w(x)\, dx \le C_{p,\lambda,w,\varphi} \int_{-\infty}^{\infty} |f(x)|^p (M^-)^{[p/2]+1}(w)(x)\, dx.$$

3. Proof of the results. The following lemma and remark will be used in the proof of Theorem A.

LEMMA 1. *Let* $\varphi \in C_0^\infty$ *with* $\operatorname{supp}(\varphi) \subset [-2^s, 0]$, $s \ge 0$, *and* $\int \varphi(x)\, dx = 0$. *Then*

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)\, dx \le C_\lambda 2^{s\lambda} \Big(\int_{-\infty}^{\infty} |\widehat{\varphi}(t)|^2 \frac{dt}{|t|} \Big) \int_{-\infty}^{\infty} |f(x)|^2 M^- w(x)\, dx,$$

with a constant C_λ *depending neither on* f *nor on* φ.

Proof. By Fubini's theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)\, dx \\
&\qquad = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{x}^{\infty} \Big(\frac{t}{t+y-x} \Big)^{\lambda} |f * \varphi_t(y)|^2 \frac{dy\, dt}{t^2}\, w(x)\, dx \\
&\qquad = \int_{0}^{\infty} \int_{-\infty}^{\infty} |f * \varphi_t(y)|^2 \Big(\frac{1}{t} \int_{-\infty}^{y} \Big(\frac{t}{t+y-x} \Big)^{\lambda} w(x)\, dx \Big) \frac{dy\, dt}{t}.
\end{aligned}$$

For each integer k, we consider the set

$$A_k = \Big\{ (y,t) : 2^{k-1} < \frac{1}{t} \int_{-\infty}^{y} \Big(\frac{t}{t+y-x} \Big)^{\lambda} w(x)\, dx \le 2^k \Big\}.$$

Then

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)\,dx \le \sum_{k\in\mathbb{Z}} 2^k \int_0^{\infty}\int_{-\infty}^{\infty} |f*\varphi_t(y)|^2 \chi_{A_k}(y,t)\,\frac{dy\,dt}{t}. \tag{2}$$

For every (y,t) belonging to A_k and $y \le z \le y+2^s t$, we have

$$\begin{aligned}\frac{1}{t}\int_{-\infty}^{z}\left(\frac{t}{t+z-x}\right)^{\lambda} w(x)\,dx &\ge \frac{1}{2^{(s+1)\lambda}}\frac{1}{t}\int_{-\infty}^{y}\left(\frac{t}{t+y-x}\right)^{\lambda} w(x)\,dx\\ &> \frac{2^{k-1}}{2^{(s+1)\lambda}}.\end{aligned}$$

On the other hand, since $\lambda > 1$, there exists a constant C_λ such that for every z,

$$\frac{1}{t}\int_{-\infty}^{z}\left(\frac{t}{t+z-x}\right)^{\lambda} w(x)\,dx \le C_\lambda M^- w(z).$$

Therefore, if $(y,t)\in A_k$ and $y\le z\le y+2^s t$ then z belongs to $E_k = \{z : M^-w(z) \ge (C_\lambda/2^{(s+1)\lambda})2^{k-1}\}$. Taking into account that $\operatorname{supp}(\varphi)\subset[-2^s,0]$, we get

$$f*\varphi_t(y) = \int f(z)\chi_{E_k}(z)\varphi_t(y-z)\,dz = (f\chi_{E_k}*\varphi_t)(y).$$

Then, by Plancherel's and Fubini's theorems, (2) is majorized by

$$\sum_{k\in\mathbb{Z}} 2^k \int_0^{\infty}\int_{-\infty}^{\infty} |f\chi_{E_k}*\varphi_t(y)|^2\,\frac{dy\,dt}{t} = \sum_{k\in\mathbb{Z}} 2^k \int_{-\infty}^{\infty} |\widehat{f\chi_{E_k}}(y)|^2 \int_0^{\infty} |\widehat{\varphi}(ty)|^2\,\frac{dt}{t}\,dy.$$

The inner integral is bounded by $C_\varphi = \int_{-\infty}^{\infty}(|\widehat{\varphi}(t)|^2/|t|)\,dt$. Thus, applying Plancherel's theorem again, we get

$$\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)\,dx \le C_\varphi \int_{-\infty}^{\infty} |f(y)|^2 \sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}(y)\,dy.$$

Finally, we observe that by the definition of E_k,

$$\sum_{k\in\mathbb{Z}} 2^k\chi_{E_k}(y) \le C_\lambda 2^{s\lambda} M^- w(y)$$

for almost every y, ending the proof of the lemma. ∎

Remark. We observe that if $\varphi\in\mathcal{S}$ and $\int\varphi(x)\,dx = 0$, then

$$\int_{-\infty}^{\infty} |\widehat{\varphi}(s)|^2\,\frac{ds}{|s|} \le 4\pi^2\Big(\int_{-\infty}^{\infty} |s|\,|\varphi(s)|\,ds\Big)^2 + \int_{-\infty}^{\infty} |\varphi(s)|^2\,ds. \tag{3}$$

In fact, since $\int\varphi(x)\,dx = 0$, we have

$$|\widehat{\varphi}(s)| = \Big|\int_{-\infty}^{\infty}\varphi(t)(e^{-2\pi ist}-1)\,dt\Big| \le 2\pi|s|\int_{-\infty}^{\infty}|t|\,|\varphi(t)|\,dt.$$

Consequently,

$$\int_{|s|\le 1} |\widehat{\varphi}(s)|^2 \frac{ds}{|s|} \le 4\pi^2 \Big(\int_{-\infty}^{\infty} |s|\, |\varphi(s)|\, ds \Big)^2.$$

On the other hand, in view of Plancherel's theorem

$$\int_{|s|\ge 1} |\widehat{\varphi}(s)|^2 \frac{ds}{|s|} \le \int_{-\infty}^{\infty} |\widehat{\varphi}(s)|^2\, ds \le \int_{-\infty}^{\infty} |\varphi(s)|^2\, ds,$$

which shows that (3) holds.

Let η be a non-negative and C_0^∞-function with support contained in $[-2,-1]$ and $\int \eta(x)\, dx = 1$. For every non-negative integer k, let $\eta_k(x) = 2^{-k}\eta(2^{-k}x)$. We define

$$\theta(x) = \int_{|x|/2\le |t|\le |x|} \eta(t)\, dt.$$

Then $\theta \in C_0^\infty$ and $\operatorname{supp}(\theta) \subset [-4,-1] \cup [1,4]$. For every positive integer k, let

$$\theta_k(x) = \theta(2^{-k+1}x),$$

and for $k = 0$, let

$$\theta_0(x) = 1 - \int_{|y|\le |x|} \eta(y)\, dy.$$

Then $\sum_{k=0}^{\infty} \theta_k(x) = 1$ for every x. Given $\varphi \in \mathcal{S}$ with $\operatorname{supp}(\varphi) \subset (-\infty, 0]$ and $\int \varphi(x)\, dx = 0$, we define

$$a_k = \int \sum_{h=0}^{k} \theta_h(y)\varphi(y)\, dy, \qquad k \ge 0, \qquad a_{-1} = 0.$$

For every non-negative integer k, let ϱ_k be given by

$$\varrho_k(x) = \theta_k(x)\varphi(x) + a_{k-1}\eta_{k-1}(x) - a_k\eta_k(x). \tag{4}$$

It is easy to check that $\operatorname{supp}(\varrho_k) \subset [-2^{k+1}, -2^{k-1}]$ for $k \ge 1$, and $\operatorname{supp}(\varrho_0) \subset [-2, 0]$. Moreover, $\int \varrho_k(x)\, dx = 0$ for every $k \ge 0$, and $\sum_{k=0}^{\infty} \varrho_k = \varphi$. We shall show that for every $N > 2$,

$$C_{\varrho_k} = \int_{-\infty}^{\infty} |\widehat{\varrho}_k(s)|^2 \frac{ds}{|s|} \le C_{N,\varphi} 2^{-2k(N-2)}. \tag{5}$$

By definition of ϱ_k,

$$\Big(\int_{-\infty}^{\infty} |\varrho_k(x)|^2\, dx \Big)^{1/2} \le \Big(\int_{-\infty}^{\infty} |\theta_k(x)\varphi(x)|^2\, dx \Big)^{1/2} + |a_{k-1}| \Big(\int_{-\infty}^{\infty} |\eta_{k-1}(x)|^2\, dx \Big)^{1/2} + |a_k| \Big(\int_{-\infty}^{\infty} |\eta_k(x)|^2\, dx \Big)^{1/2}. \tag{6}$$

Since $0 \leq \theta_k(x) \leq 1$ and $\operatorname{supp}(\theta_k\varphi) \subset [-2^{k+1}, -2^{k-1}]$ for $k \geq 1$, and $\operatorname{supp}(\theta_0\varphi) \subset [-2, 0]$, we have

$$\tag{7} \Big(\int_{-\infty}^{\infty} |\theta_k(x)\varphi(x)|^2\,dx\Big)^{1/2} \leq \Big(\int_{\operatorname{supp}(\theta_k\varphi)} \frac{C_{N,\varphi}}{(1+|x|)^{2N}}\,dx\Big)^{1/2} \leq C_{N,\varphi}2^{-k(N-1/2)}.$$

By definition of a_k, and taking into account that $\int \varphi(x)\,dx = 0$, we get

$$|a_k| = \Big|-\int \sum_{h=k+1}^{\infty} \theta_h(y)\varphi(y)\,dy\Big| \leq \int_{|y|\geq 2^k} |\varphi(y)|\,dy \leq C_{N,\varphi}\int_{|y|\geq 2^k} \frac{dy}{(1+|y|)^N} \leq C_{N,\varphi}2^{-k(N-1)}.$$

Thus,

$$\tag{8} |a_k|\Big(\int_{-\infty}^{\infty} |\eta_k(x)|^2\,dx\Big)^{1/2} = \frac{|a_k|}{2^{k/2}}\Big(\int_{-\infty}^{\infty} |\eta(x)|^2 dx\Big)^{1/2} \leq C_{N,\varphi}2^{-k(N-1/2)}.$$

Then, by (6)–(8),

$$\int_{-\infty}^{\infty} |\varrho_k(x)|^2\,dx \leq C_{N,\varphi}2^{-2k(N-1/2)}.$$

Simple calculations show that

$$\int_{-\infty}^{\infty} |x|\,|\varrho_k(x)|^2\,dx \leq C_{N,\varphi}2^{-2k(N-2)}.$$

Now, using (3) we obtain (5).

Proof of Theorem A. We consider the sequence of functions $\{\varrho_k,\, k \geq 0\}$ defined in (4). Since $\sum_{k=0}^{\infty} \varrho_k = \varphi$ and $\sum_{k=0}^{\infty} \chi_{\operatorname{supp}(\varrho_k)}(x) \leq 3$, we have

$$f * \varphi_t(y) = \sum_{k=0}^{\infty} f * (\varrho_k)_t(y)$$

for every y. Then

$$\tag{9} \begin{aligned} &\Big(\int_{-\infty}^{\infty} g^+_{\lambda,\varphi}(f)(x)^2 w(x)\,dx\Big)^{1/2} \\ &\quad\leq \sum_{k=0}^{\infty}\Big(\int_{-\infty}^{\infty}\int_0^{\infty}\int_x^{\infty}\Big(\frac{t}{t+y-x}\Big)^{\lambda}|f*(\varrho_k)_t(y)|^2\,\frac{dy\,dt}{t^2}\,w(x)\,dx\Big)^{1/2} \\ &\quad= \sum_{k=0}^{\infty}\Big(\int_{-\infty}^{\infty} g^+_{\lambda,\varrho_k}(f)(x)^2 w(x)\,dx\Big)^{1/2}. \end{aligned}$$

Keeping in mind that $\operatorname{supp}(\varrho_k) \subset [-2^{k+1}, 0]$ and $\int \varrho_k(x)\,dx = 0$, we can apply Lemma 1. Then, by the estimate (5) with $N > \lambda + 2$, we find that (9) is bounded by a constant times

$$\sum_{k=0}^{\infty} 2^{(k+1)\lambda/2}\Big(\int_{-\infty}^{\infty} |\widehat{\varrho}_k(t)|^2 \frac{dt}{|t|}\Big)^{1/2}\Big(\int_{-\infty}^{\infty} |f(x)|^2 M^- w(x)\,dx\Big)^{1/2}$$
$$\le C_{\lambda,\varphi}\Big(\int_{-\infty}^{\infty} |f(x)|^2 M^- w(x)\,dx\Big)^{1/2}. \ \blacksquare$$

In order to prove Theorem B, we shall need the following one-sided Fefferman–Stein type inequality and Lemma 11.

LEMMA 10. *There exists a positive constant C, such that*

$$w(\{x : M^+(f)(x) > \mu\}) \le \frac{C}{\mu}\int_{-\infty}^{\infty} |f(x)| M^- w(x)\,dx$$

for every function f, and $\mu > 0$.

Proof. The proof is similar to the proof of Theorem 1 in [M, p. 693], and it shall not be given. $\blacksquare$

LEMMA 11. *Let $I = (\alpha, \beta)$, a bounded interval, $1 < \lambda < 2$, and $k \ge 4$. Then there exists a constant $C_{\lambda,k}$ such that for every $x < \alpha - 2|I|$,*

$$\int_0^{\infty}\int_x^{\alpha-2|I|}\Big(\frac{t}{t+y-x}\Big)^{\lambda}\Big(\frac{t}{t+\alpha-y}\Big)^{k}\frac{dy\,dt}{t^4} \le C_{\lambda,k}\frac{|I|^{\lambda-2}}{(\alpha-x)^{\lambda}}.$$

Proof. Changing the variables (y, t) to

$$z = (\alpha - y)/t \quad\text{and}\quad u = (\alpha - x)/t,$$

we obtain

$$\int_0^{\infty}\int_{\alpha-x\ge\alpha-y\ge 2|I|}\Big(\frac{1}{1+\frac{y-x}{t}}\Big)^{\lambda}\Big(\frac{1}{1+\frac{\alpha-y}{t}}\Big)^{k}\frac{dy\,dt}{t^4}$$
$$= \frac{1}{(\alpha-x)^2}\int_0^{\infty}\int_{u\ge z\ge 2|I|u/(\alpha-x)}\frac{1}{(1+u-z)^{\lambda}}\,\frac{1}{(1+z)^k}\,u\,du\,dz.$$

We set $A = 2|I|/(\alpha - x)$. Applying Fubini's theorem, it is enough to show that

$$\int_0^{\infty}\frac{1}{(1+z)^k}\int_{z\le u\le z/A}\frac{u}{(1+u-z)^{\lambda}}\,du\,dz \le C_{\lambda,k}A^{\lambda-2}.$$

Recalling that $1<\lambda<2$, we have

$$\int_0^\infty \frac{1}{(1+z)^k} \int_{z\le u\le z/A,\, u-z>u/2} \frac{u}{(1+u-z)^\lambda}\,du\,dz$$
$$\le \int_0^\infty \frac{1}{(1+z)^k} \int_0^{z/A} \left(\frac{2}{u}\right)^\lambda u\,du\,dz = C_\lambda \int_0^\infty \frac{1}{(1+z)^k}\left(\frac{z}{A}\right)^{2-\lambda} dz = A^{\lambda-2}.$$

Since $k\ge 4, A<1$ and $\lambda<2$, it follows that

$$\int_0^\infty \frac{1}{(1+z)^k} \int_{z\le u\le z/A,\, u-z\le u/2} \frac{u}{(1+u-z)^\lambda}\,du\,dz$$
$$\le \int_0^\infty \frac{1}{(1+z)^k} \int_0^{2z} u\,du\,dz = 2\int_0^\infty \frac{z^2}{(1+z)^k}\,dz \le C_k A^{\lambda-2},$$

which ends the proof of the lemma. ∎

Proof of Theorem B. By a density argument it is enough to consider $f\in L^p(M^-w)\cap L^p$. It is well known that the set $\Omega=\{x: M^+(|f|^p)(x)^{1/p}>\mu\}$ is open. Let $\{I_j\}_{j\ge1}$ be its connected components. Since $f\in L^p$, each I_j is a bounded interval, and it is well known (see [HSt, pp. 421–424]) that

$$\frac{1}{|I_j|}\int_{I_j} |f(x)|^p\,dx = \mu^p. \tag{12}$$

Given $I_j=(\alpha_j,\beta_j)$, we write $I_j^-=(\alpha_j-4|I_j|,\alpha_j)$. By (12), we have

$$w(I_j^-)=\frac{1}{\mu^p}\int_{I_j}|f(x)|^p\,\frac{w(I_j^-)}{|I_j|}\,dx \le \frac{5}{\mu^p}\int_{I_j}|f(x)|^p M^-w(x)\,dx.$$

Therefore, if we define $\widetilde{\Omega}=\bigcup_{j\ge1} I_j\cup I_j^-$, applying Lemma 10 we obtain

$$\begin{aligned} w(\widetilde{\Omega}) &\le w(\Omega)+\sum_{j\ge1} w(I_j^-)\\ &\le \frac{C}{\mu^p}\int_{-\infty}^{\infty}|f(x)|^p M^-w(x)\,dx + \frac{5}{\mu^p}\sum_{j\ge1}\int_{I_j}|f(x)|^p M^-w(x)\,dx\\ &\le \frac{C}{\mu^p}\int_{-\infty}^{\infty}|f(x)|^p M^-w(x)\,dx. \end{aligned}$$

Consequently, it is enough to prove that

$$w(\{x\notin\widetilde{\Omega}: g^+_{\lambda,\varphi}(f)(x)>\mu\}) \le \frac{C}{\mu^p}\int_{-\infty}^{\infty}|f(x)|^p M^-w(x)\,dx. \tag{13}$$

We define

$$g(x) = f(x)\chi_{\Omega^c}(x) + \sum_{j\ge 1}\bigg(\frac{1}{|I_j|}\int_{I_j} f\bigg)\chi_{I_j}(x),$$

$$b_j(x) = \bigg(f(x) - \frac{1}{|I_j|}\int_{I_j} f\bigg)\chi_{I_j}(x), \qquad j \ge 1.$$

Then $f = g + b$ where $b = \sum_{j\ge 1} b_j$.

By Chebyshev's inequality and applying Theorem A, we get

$$\begin{aligned}
(14)\qquad w(\{x \notin \widetilde{\Omega} : g^+_{\lambda,\varphi}(f)(x) > \mu\}) &\le \frac{1}{\mu^2}\int_{\widetilde{\Omega}^c} g^+_{\lambda,\varphi}(g)(x)^2 w(x)\,dx \\
&\le \frac{C}{\mu^2}\int_{-\infty}^{\infty} |g(x)|^2 M^-(w\chi_{\widetilde{\Omega}^c})(x)\,dx \\
&= \frac{C}{\mu^2}\int_{-\infty}^{\infty} |g(x)|^{2-p}|g(x)|^p M^-(w\chi_{\widetilde{\Omega}^c})(x)\,dx.
\end{aligned}$$

We observe that $|g(x)| \le \mu$ almost everywhere. Then, by the definition of g and Hölder's inequality, (14) is bounded by

$$\frac{C}{\mu^p}\bigg[\int_{\Omega^c} |f(x)|^p M^-(w\chi_{\widetilde{\Omega}^c})(x)\,dx + \sum_{j\ge 1}\int_{I_j}\bigg(\frac{1}{|I_j|}\int_{I_j}|f(z)|^p\,dz\bigg)M^-(w\chi_{\widetilde{\Omega}^c})(x)dx\bigg].$$

It is easy to see that $M^-(w\chi_{\widetilde{\Omega}^c})(x) \le CM^-(w)(z)$ for every $x, z \in I_j$. Thus,

$$(15)\qquad w(\{x \notin \widetilde{\Omega} : g^+_{\lambda,\varphi}(g)(x) > \mu\}) \le \frac{C}{\mu^p}\int_{-\infty}^{\infty} |f(x)|^p M^- w(x)\,dx.$$

We define $I_j^* = (\alpha_j - 2|I_j|, \beta_j)$ for every $j \ge 1$. We can write

$$(16)\qquad g^+_{\lambda,\varphi}(b)(x) \le g^1(x) + g^2(x),$$

where

$$g^1(x) = \bigg(\int_0^\infty\int_x^\infty\bigg(\frac{t}{t+y-x}\bigg)^\lambda\bigg|\sum_{i\,:\,y\notin I_i^*} b_i * \varphi_t(y)\bigg|^2\frac{dy\,dt}{t^2}\bigg)^{1/2},$$

$$g^2(x) = \bigg(\int_0^\infty\int_x^\infty\bigg(\frac{t}{t+y-x}\bigg)^\lambda\bigg|\sum_{i\,:\,y\in I_i^*} b_i * \varphi_t(y)\bigg|^2\frac{dy\,dt}{t^2}\bigg)^{1/2}.$$

Let us consider $g^1(x)$. Taking into account that $b_i * \varphi_t(y) = 0$ if $y > \beta_i$, and $\int |b_i(z)|\,dz \le 2|I_i|\mu$, it follows that

$$\bigg|\sum_{i\,:\,y\notin I_i^*} b_i * \varphi_t(y)\bigg| \le \frac{2\mu}{t}\sum_{i\,:\,y\notin I_i^*,\,y<\beta_i} |I_i|\sup_{z\in I_i}\bigg|\varphi\bigg(\frac{y-z}{t}\bigg)\bigg|.$$

Since $\varphi \in \mathcal{S}$, and $\operatorname{supp}(\varphi) \subset (-\infty, 0]$, we deduce that

$$\left|\varphi\Big(\frac{y-z}{t}\Big)\right| \le \frac{C}{\big(1+\frac{w-y}{t}\big)^2} \qquad \text{for } y \notin I_i^* \text{ and } z, w \in I_i.$$

Then

$$\Big|\sum_{i\,:\,y\notin I_i^*} b_i * \varphi_t(y)\Big| \le \frac{C\mu}{t} \sum_{i\,:\,y\notin I_i^*,\, y<\beta_i} \int_{I_i} \frac{dw}{\big(1+\frac{w-y}{t}\big)^2} \le c\mu.$$

Therefore,

$$g^1(x)^2 \le C\mu \int_0^\infty \int_x^\infty \Big(\frac{t}{t+y-x}\Big)^\lambda \Big|\sum_{i\,:\,y\notin I_i^*} b_i * \varphi_t(y)\Big| \frac{dy\,dt}{t^2} = C\mu F(x),$$

and by Chebyshev's inequality we get

$$w(\{x \notin \widetilde{\Omega} : g^1(x) > \mu\}) \le \frac{C}{\mu} \int_{\widetilde{\Omega}^{\mathrm{c}}} F(x) w(x)\, dx. \tag{17}$$

Since $\int b_i(z)\, dz = 0$, applying the mean value theorem, for every $y \le \alpha_i - 2|I_i|$ we obtain the estimate

$$\begin{aligned}
|b_i * \varphi_t(y)| &\le \frac{1}{t}\int |b_i(z)| \left|\varphi\Big(\frac{y-z}{t}\Big) - \varphi\Big(\frac{y-\alpha_i}{t}\Big)\right| dz \\
&\le \frac{C}{t}\int_{I_i} |b_i(z)| \left|\frac{z-\alpha_i}{t}\right| \Big(\frac{t}{t+\alpha_i-y}\Big)^4 dz \\
&\le C|I_i| \frac{t^2}{(t+\alpha_i-y)^4} \int_{I_i} |f(z)|\, dz.
\end{aligned}$$

Then, by the definition of $F(x)$, (17) is majorized by

$$\begin{aligned}
\frac{C}{\mu} \sum_{i\ge 1} \int_{I_i} |f(z)|\, dz \int_{\widetilde{\Omega}^{\mathrm{c}}} |I_i| \int_0^\infty \int_{x<y<\beta_i,\, y\notin I_i^*} \Big(\frac{t}{t+y-x}\Big)^{\lambda'} \\
\times \frac{1}{(t+\alpha_i-y)^4}\, dy\, dt\, w(x)\, dx,
\end{aligned} \tag{18}$$

where $1 < \lambda' < \inf(\lambda, 2)$. Now, applying Lemma 11 with $k = 4$, we find that (18) is bounded by

$$\frac{C}{\mu} \sum_{i\ge 1} \int_{I_i} |f(z)|\, dz \int_{-\infty}^{\alpha_i - 4|I_i|} \frac{|I_i|^{\lambda'-1}}{(\alpha_i - x)^{\lambda'}}\, w(x) \chi_{\widetilde{\Omega}^{\mathrm{c}}}(x)\, dx.$$

The inner integral is bounded by $CM^-(w\chi_{\widetilde{\Omega}^{\mathrm{c}}})(\alpha_i)$. It is easy to verify that, by Hölder's inequality and (12),

$$\frac{1}{\mu}\int_{I_i}|f| \le \frac{1}{\mu^p}\int_{I_i}|f|^p.$$

Thus, we obtain

$$w(\{x \notin \widetilde{\Omega} : g^1(x) > \mu\}) \le \frac{C}{\mu^p}\sum_i \int_{I_i} |f(z)|^p\,dz\, M^-(w\chi_{\widetilde{\Omega}^c})(\alpha_i) \le \frac{C}{\mu^p}\int_{-\infty}^{\infty} |f(z)|^p M^- w(z)\,dz. \tag{19}$$

Now, let us consider $g^2(x)$. By (12), there exists an integer k_0 such that $|I_j| \le \|f\|_p^p \mu^{-p} \le 2^{k_0}$ for every $j \ge 1$. Let $A_k = \{j : 2^{k-1} < |I_j| \le 2^k\}$, $k \le k_0$. We can write

$$\bigcup_{j\ge 1} I_j^* = \bigcup_{k\le k_0}\bigcup_{j\in A_k} E_j^*,$$

where $E_j^* = I_j^* \setminus \bigcup_{l>k}\bigcup_{s\in A_l} I_s^*$ for each $j \in A_k$. We observe that if $I_i^* \cap E_j^*$ is not empty then $I_i^* \subset I_j'$, where I_j' is the interval with the same center of I_j and with measure $20|I_j|$. For each $x \notin \widetilde{\Omega}$, we have

$$g^2(x)^2 = \sum_{k\le k_0}\sum_{j\in A_k}\int_0^\infty \int_{x<y,\, y\in E_j^*} \left(\frac{t}{t+y-x}\right)^\lambda \Big|\sum_{i\,:\,y\in I_i^*} b_i * \varphi_t(y)\Big|^2 \frac{dy\,dt}{t^2}.$$

We observe that if $x \notin \widetilde{\Omega}^c$, $x < y$ and $y \in E_j^*$ then $x < \alpha_j - 4|I_j|$ and $t + y - x \ge (\alpha_j - x) - (\alpha_j - y) \ge (\alpha_j - x)/2$. Then

$$g^2(x)^2 \le C\sum_{k\le k_0}\sum_{j\in A_k,\, x<\alpha_j}\frac{1}{(\alpha_j - x)^\lambda} \times \int_0^\infty \int_{x<y,\, y\in E_j^*} t^{\lambda-2}\Big|\sum_{i\,:\,y\in I_i^*} b_i * \varphi_t(y)\Big|^2 dy\,dt. \tag{20}$$

If we define $D_j = \bigcup_{i\,:\,E_j^*\cap I_i^*\ne\emptyset} I_i$ and $b^j(x) = |b(x)|\chi_{D_j}(x)$ then, for every $y \in E_j^*$, we obtain

$$\begin{aligned}\Big|\sum_{i\,:\,y\in I_i^*} b_i * \varphi_t(y)\Big| &\le \sum_{i\,:\,y\in I_i^*}\int_{I_i} |b(z)|\,|\varphi_t(y-z)|\,dz \\ &\le \int_{\bigcup_{i\,:\,E_j^*\cap I_i^*\ne\emptyset} I_i} |b(z)|\,|\varphi_t(y-z)|\,dz \\ &\le \int_{D_j} |b(z)|\,|\varphi_t(y-z)|\,dz = (b^j * |\varphi|_t)(y).\end{aligned}$$

Consequently, by (20), we have

$$g^2(x)^2 \le C \sum_{k\le k_0} \sum_{j\in A_k,\, x<\alpha_j} \frac{1}{(\alpha_j - x)^\lambda} \times \int_0^\infty \int_{x<y,\, y\in E_j^*} t^{\lambda-2} |(b^j * |\varphi|_t)(y)|^2 \, dy\, dt. \tag{21}$$

We claim that

$$\int_0^\infty \int_{E_j^*} t^{\lambda-2} |(b^j * |\varphi|_t)(y)|^2 \, dy\, dt \le C|E_j^*|^{\lambda-2/p} \|b^j\|_p^2. \tag{22}$$

In fact, by Fubini's theorem, we have

$$\int_0^\infty t^{\lambda-2}|(b^j * |\varphi|_t)(y)|^2\,dt = \int_y^\infty b^j(z) \int_y^\infty b^j(w) \int_0^\infty t^{\lambda-4} |\varphi|\left(\frac{y-z}{t}\right) |\varphi|\left(\frac{y-w}{t}\right) dt\, dw\, dz.$$

Since $\varphi \in \mathcal{S}$, and $\lambda < 3$,

$$\begin{aligned}\int_0^\infty t^{\lambda-4} |\varphi|\left(\frac{y-z}{t}\right) |\varphi|\left(\frac{y-w}{t}\right) dt &\le C \int_0^\infty t^{\lambda-4} \frac{1}{\left(1+\frac{z-y}{t}\right)^2} \frac{1}{\left(1+\frac{w-y}{t}\right)^2}\, dt \\ &\le C \int_0^\infty \frac{t^{\lambda-4}}{\left(1+\frac{z+w-2y}{t}\right)^2}\, dt = C_\lambda (z+w-2y)^{\lambda-3}.\end{aligned}$$

Then the left hand side of (22) is bounded by

$$\begin{aligned} C \int_{E_j^*} \int_y^\infty b^j(z) \int_y^\infty b^j(w) \frac{1}{(z+w-2y)^{3-\lambda}}\, dw\, dz\, dy &\le C' \int_{E_j^*} \int_y^\infty \frac{b^j(z)}{(z-y)^{(3-\lambda)/2}}\, dz \int_y^\infty \frac{b^j(w)}{(w-y)^{(3-\lambda)/2}}\, dw\, dy \\ &\le C' \int_{E_j^*} |I^+_{(\lambda-1)/2}(b^j)(y)|^2\, dy, \end{aligned}$$

where $I^+_{(\lambda-1)/2}$ denotes the one-sided fractional integral operator of order $(\lambda-1)/2$. In the case $1 < p < 2$ and $\lambda = 2/p$, since, as is well known, $I^+_{(\lambda-1)/2}$ is a bounded operator from L^p to L^2, it follows that (22) holds.

For $2 < \lambda < 3$, the operator $I^+_{(\lambda-1)/2}$ maps L^1 into weak-$L^{2/(3-\lambda)}$. Then, by Kolmogorov's condition (see [GRu, p. 485]), we obtain (22).

On the other hand, since $\int |b_i(y)|^p\,dy \le (2\mu)^p|I_i|$, we have

$$\|b^j\|_p \le \Big(\sum_{i\,:\,E_j^*\cap I_i^*\ne\emptyset} (2\mu)^p|I_i|\Big)^{1/p} \le 2\mu|I_j'|^{1/p} = C\mu|I_j|^{1/p}.$$

Therefore, by (21) and (22) we get

$$g^2(x)^2 \le C'\mu^2 \sum_{k\le k_0}\ \sum_{j\in A_k,\, x<\alpha_j} \frac{|I_j|^\lambda}{(\alpha_j - x)^\lambda}.$$

Consequently,

$$\begin{aligned}
(23)\qquad w(\{x\notin \widetilde{\Omega} : g^2(x) > \mu\}) &\le C\sum_j |I_j|^\lambda \int\limits_{-\infty}^{\alpha_j-4|I_j|} \frac{w(x)\chi_{\widetilde{\Omega}^c}(x)}{(\alpha_j - x)^\lambda}\,dx \\
&\le \frac{C}{\mu^p}\sum_j \int\limits_{I_j} |f(z)|^p\,dz\, M^-(w\chi_{\widetilde{\Omega}^c})(\alpha_j) \\
&\le \frac{C}{\mu^p}\int\limits_{-\infty}^{\infty} |f(z)|^p M^- w(z)\,dz.
\end{aligned}$$

From (15), (16), (19) and (23) we deduce that (13) holds for $\lambda = 2/p$ if $1 < p < 2$ and for $2 < \lambda < 3$ if $p = 1$. Taking into account that if $\lambda_1 \le \lambda_2$ then $g^+_{\lambda_2,\varphi}(f)(x) \le g^+_{\lambda_1,\varphi}(f)(x)$, the proof of the theorem is complete. ∎

We now deduce Theorem C from Theorems A and B.

Proof of Theorem C. The case $p = 2$ and $\lambda > 1$ was considered in Theorem A. Let $1 < p < 2$ and $2/p < \lambda < 2$. We have $\lambda = 2/q$ with $1 < q < p$. Then, by Theorem B, $g^+_{\lambda,\varphi}$ maps $L^q(M^-w)$ into weak-$L^q(w)$. Since $g^+_{\lambda,\varphi}$ is bounded from $L^2(M^-w)$ to $L^2(w)$, by interpolation, we get the assertion for $\lambda < 2$. The case $\lambda \ge 2$ follows by simple arguments. ∎

The following remark shows that for $\lambda = 2$ and $p = 1$, a weak type inequality as in Theorem B cannot be valid.

Remark. Let $\varphi \ne 0$ belong to $\mathcal{S}$ with $\operatorname{supp}(\varphi) \subset [-1, 0]$ and $\int \varphi(x)\,dx = 0$. There exists $f \in L^1$ such that $g^+_{2,\varphi}(f)(x) = \infty$ for every x belonging to an unbounded set.

In fact, we consider

$$f(t) = \Big(\frac{1}{|t|\ln^{3/2}(1/|t|)} - c\Big)\chi_{[-1/2,0]}(t),$$

where c is the unique constant such that $\int f(t)\,dt = 0$. For every $x < -4$, we have

$$g^{+}_{2,\varphi}(f)(x)^2 \geq \frac{1}{(1-x)^2}\int_0^1\int_{-2}^0 |f * \varphi_t(y)|^2\,dy\,dt. \tag{24}$$

The support of $f * \varphi_t$ is contained in $(-\infty, 0]$ and the fractional integral $I_{1/2}(f) \notin L^2$ (see [Z, p. 232]). Then Plancherel's theorem yields

$$\begin{aligned} A := \int_0^\infty\int_{-\infty}^0 |f * \varphi_t(y)|^2\,dy\,dt &= \int_0^\infty\int_{-\infty}^\infty |\widehat{\varphi}(ty)|^2|\widehat{f}(y)|^2\,dy\,dt \\ &\geq C_\varphi \int_{-\infty}^\infty \frac{|\widehat{f}(y)|^2}{|y|}\,dy \\ &= C_\varphi \int_{-\infty}^\infty |I_{1/2}(f)(y)|^2\,dy = \infty. \end{aligned}$$

Applying the mean value theorem, for every $y \leq -2$ we obtain

$$\begin{aligned} |f * \varphi_t(y)| &\leq \frac{1}{t}\int_{-1/2}^0 |f(z)|\left|\varphi\Big(\frac{y-z}{t}\Big) - \varphi\Big(\frac{y}{t}\Big)\right| dz \\ &\leq \frac{1}{t}\int_{-1/2}^0 |f(z)|\,\frac{|z|}{t}\,C_\varphi\Big(\frac{t}{t+|y|}\Big)^2 dz \leq C\frac{1}{(t+|y|)^2}. \end{aligned}$$

Using these inequalities we get

$$A_1 := \int_0^\infty\int_{-\infty}^{-2} |f * \varphi_t(y)|^2\,dy\,dt \leq C\int_0^\infty\int_{-\infty}^{-2} \frac{1}{(t+|y|)^4}\,dy\,dt < \infty.$$

Since $|f * \varphi_t(y)| \leq \frac{1}{t}\|\varphi\|_\infty\|f\|_1$, we have

$$A_2 := \int_1^\infty\int_{-2}^0 |f * \varphi_t(y)|^2\,dy\,dt \leq C\int_1^\infty\int_{-2}^0 \frac{1}{t^2}\,dy\,dt < \infty.$$

By (24) and the estimates obtained for A, A_1, and A_2 it follows that $g^{+}_{2,\varphi}(f)(x) = \infty$ for every $x < -4$.

To prove Theorem D, we proceed as in Theorem 1.10 of [P, p. 150].

Proof of Theorem D. More generally, we shall prove that

$$\int_{-\infty}^\infty g^{+}_{\lambda,\varphi}(f)(x)^p w(x)\,dx \leq C\int_{-\infty}^\infty |f(x)|^p M^{-}_B(w^{2/p})(x)^{p/2}\,dx,$$

where B is a Young function that satisfies

$$\int_c^\infty \left(\frac{t^{p/2}}{B(t)}\right)^{(p/2)'-1} \frac{dt}{t} < \infty. \tag{25}$$

In the case $B(t) \approx t^{p/2}(1+\ln^+ t)^{[p/2]}$, we get Theorem D.

Let $r = p/2$. We have

$$I = \|g_{\lambda,\varphi}^+(f)\|_{L^p(w)}^2 = \|g_{\lambda,\varphi}^+(f)^2 w^{1/r}\|_{L^r} = \int_{-\infty}^{\infty} g_{\lambda,\varphi}^+(f)(x)^2 w(x)^{1/r} g(x)\,dx,$$

for some $g \in L^{r'}$ with unit norm. We recall that

$$M^-(g_1g_2)(x) \le M_B^-(g_1)(x) M_{\overline{B}}^-(g_2)(x),$$

where $\overline{B}$ is the complementary function to B. Then Theorem A and Hölder's inequality yield

$$\begin{aligned}
I &\le C \int_{-\infty}^{\infty} |f(x)|^2 M^-(w^{1/r}g)(x)\,dx \\
&\le C \int_{-\infty}^{\infty} |f(x)|^2 M_B^-(w^{1/r})(x) M_{\overline{B}}^-(g)(x)\,dx \\
&\le C\Big(\int_{-\infty}^{\infty} |f(x)|^p M_B^-(w^{1/r})(x)^{p/2}\,dx\Big)^{2/p} \Big(\int_{-\infty}^{\infty} M_{\overline{B}}^-(g)(x)^{r'}\,dx\Big)^{1/r'} \\
&= C\|f\|_{L^p(v)}^2 \|M_{\overline{B}}^-(g)\|_{L^{r'}},
\end{aligned}$$

where $v = M_B^-(w^{1/r})(x)^r$. By Theorem 2.6 in [RiRoT], if B satisfies (25), then

$$I \le C\|f\|_{L^p(v)}^2 \|g\|_{L^{r'}} \le C\|f\|_{L^p(v)}^2.$$

It is easy to check that $M_B^-(w^{1/r})(x)^r = M_{\widetilde{B}}^-(w)(x)$, where $\widetilde{B}(t) = B(t^{1/r})$. If $\widetilde{B}(t) = t(1+\ln^+ t)^{[r]}$ then B satisfies (25), and by Proposition 2.15 in [RiRoT] there exist two constants C_1 and C_2 such that

$$C_1 M_{\widetilde{B}}^-(w)(x) \le (M^-)^{[r]+1} w(x) \le C_2 M_{\widetilde{B}}^-(w)(x),$$

which completes the proof. ∎

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