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Dispersive and Strichartz estimates on H-type groups

by

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Abstract. Our purpose is to generalize the dispersive inequalities for the wave equation on the Heisenberg group, obtained in [1], to H-type groups. On those groups we get optimal time decay for solutions to the wave equation (decay as $t^{-p/2}$) and the Schrödinger equation (decay as $t^{(1-p)/2}$), p being the dimension of the center of the group. As a corollary, we obtain the corresponding Strichartz inequalities for the wave equation, and, assuming that $p > 1$, for the Schrödinger equation.

Introduction. Let T_t be a semigroup associated to a partial differential equation (PDE) such as the wave equation (4.1) or the Schrödinger equation (5.1) on $\mathbb{R}^Q$. By Fourier analysis we may write such operators explicitly as

$$T_t f(x) = \int_{\mathbb{R}^Q} e^{\mathbf{i}\phi(\xi,x,t)} f(\xi)\, d\xi. \tag{0.1}$$

These oscillatory integrals are well known; in particular, by stationary phase we get dispersive estimates

$$\|T_t f\|_\infty \le C(f) t^{-\alpha/2},$$

where $\alpha = Q - 1$ for the (half-)wave equation semigroup and $\alpha = Q$ for the Schrödinger semigroup, and the time decay is optimal. For $Q > 2$ these dispersive estimates imply all the Strichartz estimates (see Corollary 0.2) (see [27, 28, 13, 18, 16, 19]).

Strichartz estimates are a very useful tool in local and global existence problems for nonlinear PDEs [3, 24, 5, 30, 33, 12]. On the other hand, the dispersive estimates associated to wave equations have been used to prove pointwise Fourier convergence (see [21]). As expected, these estimates can be generalized to many other spaces, such as regular domains, Lie groups or compact Riemannian manifolds (see for instance [4, 1]).

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In 2000 H. Bahouri, P. Gérard, et C. Xu (see [1]) found sharp dispersive estimates and Strichartz inequalities for the wave equation on the Heisenberg group. Such estimates do not exist for the Schrödinger equation (cf. [1]).

In this paper we work on "bigger" spaces, like H-type groups, on which we will show that the Schrödinger equation is dispersive.

Theorem 0.1. *Let $\triangle$ be the Kohn sub-Laplacian associated to an H-type group G. Consider the semigroups $S_t = e^{\mathbf{i}t\triangle}$ and $W_t = e^{\mathbf{i}t\sqrt{\triangle}}$. We have the sharp estimates*

$$\|S_t f\|_\infty \leq C|t|^{-(p-1)/2}\|f\|_{\dot{B}^{N-(p-1)/2}_{1,1}}, \qquad \|W_t f\|_\infty \leq C|t|^{-p/2}\|f\|_{\dot{B}^{N-p/2}_{1,1}},$$

where p is the dimension of the center of G and $\dot{B}^{\varrho}_{p,r}$ is the homogeneous Besov space introduced in Section 2.

In comparison with Theorem 1.2 of [1], we have obtained a double improvement. On the one hand, the time decay is sharper for the wave equation solution. On the other hand, a new phenomenon appears for the Schrödinger equation solutions on H-type groups with $p > 1$: they become dispersive.

As a corollary, and following the work of Keel and Tao ([18]), which extends that of Ginibre and Velo ([13]), we get a very useful estimate on the solutions of the Schrödinger and wave equations

$$\text{(WE)} \quad \begin{cases} \triangle u + \partial_{tt} u = f, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1, \end{cases} \qquad \text{(SE)} \quad \begin{cases} \mathbf{i}\triangle u - \partial_t u = f, \\ u|_{t=0} = u_0. \end{cases}$$

Corollary 0.2. *Let N be the homogeneous dimension of the H-type group G, and p the dimension of its center. Let q, r be such that*

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - 1.$$

If w is the solution of the non-homogeneous wave equation (WE), *then for all $q \in [(2N-p)/p, \infty]$,*

$$\|w\|_{L^q_t L^r} \leq C(\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|f\|_{L^1_t L^2}).$$

Here $\dot{H}^1$ is the homogeneous Sobolev space of order 1, *associated to the sub-Laplacian $\triangle$.*

Moreover, if $p > 1$ and u is the solution of the non-homogeneous Schrödinger equation (SE), *then for all $q \in \left[\frac{2N+1-p}{p-1}, \infty\right]$,*

$$\|u\|_{L^q_t L^r} \leq C(\|u_0\|_{\dot{H}^1} + \|f\|_{L^1_t \dot{H}^1}).$$

Note that only the size of the center of the group increases the number of admissible exponents q in the Strichartz estimates. In particular, the Schrödinger equation is now dispersive and admits Strichartz estimates, since $p > 1$.

To avoid the particular behavior of the Schrödinger operator on the Heisenberg group (see [1, 25]), some authors have replaced the sub-Laplacian by the full Laplacian (see [11, 31, 32]).

The paper is organized as follows. In Section 1, we will rapidly introduce the structures of groups and operators that we are concerned with. We also recall the Fourier inversion theorem for spherical functions; we would like to thank an anonymous referee who informed us about this tool. In Section 2, we recall the construction of homogeneous Besov spaces that will appear in our estimates. Before starting the proofs of the main theorems, Section 3 will be dedicated to technical lemmas concerning Laguerre polynomials and Bessel functions appearing in the inverse Fourier transform formula. Finally, Section 4, for the wave equation, and Section 5, for the Schrödinger equation, will develop the proofs of the main and intermediate results for the dispersive and Strichartz estimates. In particular Lemma 4.1 is a mixed time and space estimate, which might be useful for weighted Strichartz estimates.

1. H-type groups and spherical Fourier transform

1.1. *H-type groups.* Let G be a connected, simply connected 2-step graded group. Suppose its Lie algebra $\mathcal{G}$ is endowed with a scalar product such that we have the orthogonal direct sum

$$\mathcal{G} = \mathcal{G}_1 \oplus^{\perp} \mathcal{G}_2,$$

where $\mathcal{G}_1$ spans the Lie algebra $\mathcal{G}$ and $\mathcal{G}_2 = [\mathcal{G}_1, \mathcal{G}_1]$.

Let $\mathcal{B}_0 = (X_1, \dots, X_n)$ be an orthonormal basis of $\mathcal{G}_1$ and $\mathcal{F}_0 = (T_1, \dots, T_p)$ an orthonormal basis of $\mathcal{G}_2$. By means of these bases we shall identify $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\mathbb{R}^n$ and $\mathbb{R}^p$.

S^{p-1} will indiscriminately denote the unit sphere in the euclidean space $\mathbb{R}^p$ or in $\mathcal{G}_2$ endowed with the induced scalar product of $\mathcal{G}$.

Let $\triangle = -\sum_{i=1}^{n} X_i^2$ be the Kohn sub-Laplacian of G. This essentially self-adjoint positive operator does not depend on the choice of $\mathcal{B}_0$, and by a well known result, due to Hörmander, it is subelliptic.

Every point $g \in G$ will be described by its canonical coordinates (x, s) relative to the basis $\mathcal{B}_0 \cup \mathcal{F}_0$:

$$(x, s) = \exp\Big(\sum_{i=1}^{n} x_i X_i + \sum_{j=1}^{p} s_j T_j\Big)$$

with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $s = (s_1, \dots, s_p) \in \mathbb{R}^p$.

The group law is then an easy calculation thanks to the Hausdorff–Campbell formula

$$(x, s) \circ (x', s') = \left(x + x', s + s' + \frac{1}{2}\,[xX, x'X]\right) \quad \text{with } xX = \sum_{i=1}^{n} x_i X_i.$$

G is also endowed with a left-invariant Haar measure $d\mu = dx \otimes dt$ (cf. [6]). This expression is clearly independent of the choice of the bases $\mathcal{B}_0$ and $\mathcal{F}_0$.

Following Folland and Stein [10], we will exploit the canonical homogeneous structure, given by the family of dilations $\{\delta_r\}_{r>0}$,

$$\delta_r(x, s) := (rx, r^2 s).$$

We then define the homogeneous dimension $N := n + 2p$.

A direct calculation shows that

$$\triangle(f \circ \delta_r) = r^2 (\triangle f) \circ \delta_r,$$
$$\int_G r^N \varphi(\delta_r \cdot g)\, d\mu(g) = \int_G \varphi(g)\, d\mu(g).$$

Let us now characterize the Heisenberg type (H-type) groups. Following Kaplan's construction (cf. [17, 7, 8]) this generalization of the Heisenberg groups is characterized by the J *functional*

$$J : \mathcal{G}_2 \to \mathcal{A}_n(\mathcal{G}_1),$$

where $\mathcal{A}_n(V)$ is the set of all skew-symmetric linear maps on a linear space V. The functional is defined by

$$\forall X, X' \in \mathcal{G}_1,\ \forall T \in \mathcal{G}_2, \qquad T \cdot [X, X'] = J(T)X \cdot X',$$

and we say that G is of *H-type* if

$$J(T)^2 = -|T|^2 \,\mathrm{Id}_{\mathcal{G}_1}. \tag{H}$$

It then follows that $\dim \mathcal{G}_1 = n = 2d$ and

$$|J(T)X| = |T|\,|X|.$$

Equivalently (H) means that, for every unit vector $X \in \mathcal{G}_1$, $\mathrm{ad}(X)$ induces a surjective isometry from $\ker(\mathrm{ad}(X))^\perp$ onto $\mathcal{G}_2$. Then $\mathcal{G}_2 + \mathbb{R}X \subset \ker(\mathrm{ad}(X))$, and in particular

$$p \leq 2d - 1.$$

More generally for any choice of a p-dimensional space $\mathcal{G}_1$, we can always find infinitely many $\mathcal{G}_2$ spaces such that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ has an H-type structure.

Example 1.1 (H-type group of topological dimension 7). Consider $\mathcal{G}_1 = \mathbb{R}^4$ and $\mathcal{G}_2 = \mathbb{R}^3$ endowed with the canonical scalar product, and let $J(T)$ (with $T = (T_1, T_2, T_3) \in \mathcal{G}_2$) be the morphism whose associated matrix in the canonical basis of $\mathcal{G}_1$ is

$$[J(T)] = \begin{pmatrix} 0 & T_1 & T_2 & T_3 \\ -T_1 & 0 & -T_3 & T_2 \\ -T_2 & T_3 & 0 & -T_1 \\ -T_3 & -T_2 & T_1 & 0 \end{pmatrix}.$$

Then J equips $\mathcal{G} = \mathcal{G}_1 \oplus^\perp \mathcal{G}_2$ with an H-type structure.

1.2. *Spherical Fourier transform.* Korányi, Damek and Ricci (see [9] and [20]) have computed the spherical fonctions associated to the Gelfand pair $(G, O(n))$ (we identify $O(n)$ with $O(n) \otimes \mathrm{Id}_p$). They involve, as on the Heisenberg group, the Laguerre polynomials

$$L_m^{(q)}(t) = \sum_{k=0}^{m} (-1)^k \binom{m+q}{m-k} \frac{t^k}{k!}, \qquad t \geq 0,\ m, q \in \mathbb{N}. \tag{1.1}$$

For convenience let $\mathcal{L}_m^{(q)}(\xi) = L_m^{(q)}(\xi) \exp[-\xi/2]$

We say that a function φ is *radial* if it is invariant under the action of the orthogonal group $O(n)$, more precisely,

$$\forall S \in O(n),\ \forall (x, s) \in G, \qquad \varphi((S \cdot x, s)) = \varphi((x, s)).$$

In particular the set $L^1_{\mathrm{rad}}(G)$ of L^1 radial functions endowed with the convolution product

$$\varphi * \psi(h) = \int_G \varphi(g)\psi(g^{-1}h)\, d\mu(g)$$

is a commutative algebra.

Definition 1.2. Let $f \in L^1_{\mathrm{rad}}(G)$. We define

$$\widehat{f}_m(\lambda) = \binom{m+d-1}{m}^{-1} \int_{\mathbb{R}^{2d+p}} e^{\mathbf{i}\lambda\cdot s} f((\omega, s)) \mathcal{L}_m^{(d-1)}\left(\frac{|\lambda|}{2}\, \|\omega\|^2\right) d\omega\, ds$$

Put $\mathcal{F}[f](\lambda, m) = \widehat{f}_m(\lambda)$. We call $\mathcal{F}$ the *spherical Fourier transform.*

By a direct computation we have

$$\mathcal{F}(\varphi * \psi) = \mathcal{F}\varphi \cdot \mathcal{F}\psi.$$

Thanks to a partial integration on the sphere S^{p-1} we deduce from the Plancherel theorem on Heisenberg groups its analogue for H-type groups. In particular

Proposition 1.3. *For all radial functions f in the Schwartz class ($f \in \mathcal{S}_{\mathrm{rad}}$) such that*

$$\sum_m \binom{m+d-1}{m} \int_{\mathbb{R}^p} |\widehat{f}_m(\lambda)|\, |\lambda|^d\, d\lambda < \infty,$$

for instance if $\widehat{f}_m(\lambda) = Q(b_m(\lambda))$ with $Q \in \mathcal{S}(\mathbb{R}_+^*)$, we have

$$f((\omega, s)) = \left(\frac{1}{2\pi}\right)^{d+p} \sum_{m\in\mathbb{N}} \int_{\mathbb{R}^p} \exp[-\mathbf{i}\lambda \cdot s]\, \widehat{f}_m(\lambda) \mathcal{L}_m^{(d-1)}\left(\frac{|\lambda|}{2}\, \|\omega\|^2\right) |\lambda|^d\, d\lambda,$$

the sum being convergent in L^∞ norm.

Following the computation in [20] and [9], for all $f \in \mathcal{S}_{\mathrm{rad}}$ we have

$$\triangle f \in \mathcal{S}_{\mathrm{rad}}, \qquad \mathcal{F}[\triangle f](\lambda, m) = b_m(\lambda)\widehat{f}_m(\lambda)$$

with $b_m(\lambda) = (2m+d)|\lambda|$. Therefore for a bounded continuous function of the sub-Laplacian $h(\triangle)$ we have

$$\forall f \in \mathcal{S}_{\mathrm{rad}}(G), \qquad \mathcal{F}[h(\triangle)f](\lambda, m) = h(b_m(\lambda))\widehat{f}_m(\lambda).$$

2. Homogeneous Besov spaces. In this section we recall the construction, given in [1], of homogeneous Besov spaces. These spaces are well known, we will only list the principal results we will need (see [1]).

Let $R^* \in C_0^\infty(C_0)$ with $C_0 = \{\tau \in \mathbb{R} : 1/2 \le |\tau| \le 4\}$ be even and such that

$$0 \le R^* \le 1, \qquad \forall \tau \in \mathbb{R} - \{0\}, \quad \sum_{j\in\mathbb{Z}} R^*(2^{-2j}\tau) = 1.$$

As $R^* \in C_{\mathrm{c}}^\infty(\mathbb{R}_+^*)$, Hulanicki proved that the kernel φ of $R^*(\triangle)$ belongs to the Schwartz class (see [15]). In particular

$$\widehat{\varphi}_m(\lambda) = R_m^*(|\lambda|) := R^*((2m+n)\tau).$$

For $j \in \mathbb{Z}$, we define

$$\varphi_j((x,y,s)) = 2^{Nj}\varphi(\delta_{2^j}(x,y,s)), \qquad \forall f \in \mathcal{S}'(G), \quad \Delta_j f = f * \varphi_j$$

Obviously $\mathcal{F}\varphi_j(\lambda, m) = R_m^*(2^{-2j}|\lambda|)$.

Then, following [1], we can introduce (see [2] for clarifications):

Definition 2.1. Let $1 \le q, r \le \infty$ and $\varrho < N/q$. We define the *homogeneous Besov space* $\dot{B}^\varrho_{q,r}$ as the set of distributions $u \in \mathcal{S}'(G)$ such that

$$\|u\|_{\dot{B}^\varrho_{q,r}} = \Big(\sum_{j\in\mathbb{Z}} 2^{jr\varrho}\,\|\Delta_j u\|^r_{L^q(G)}\Big)^{1/r} < \infty$$

and $u = \sum_{j\in\mathbb{Z}} \Delta_j u$ in $\mathcal{S}'$.

Definition 2.2. Let $\varrho < N/2$. The *homogeneous Sobolev space* $\dot{H}^\varrho$ is

$$\dot{H}^\varrho = \dot{B}^\varrho_{2,2}.$$

This definition is equivalent to

$$u \in \dot{H}^\varrho \Leftrightarrow \triangle^{\varrho/2}u \in L^2,$$

and the associated norms are of course equivalent.

3. Technical lemmas

3.1. *Oscillating integrals.* Our goal is to apply oscillating integral estimates to the the wave and Schrödinger operators. These integrals naturally appear thanks to the Fourier transform, and the estimates will, in general, be a consequence of this classical result:

Proposition 3.1 (Stationary phase estimate). *Let $\phi \in C^\infty(]a,b[)$ be such that*

$$|\phi''(u)| \geq 1 \qquad on\ ]a,b[.$$

Then for any collection of functions $\psi_t \in C^\infty(]a,b[)$, where t is a real parameter,

$$\Big|\int_a^b e^{\mathbf{i}t\phi(u)}\psi_t(u)\,du\Big| \leq Ct^{-1/2}\Big[\|\psi_t\|_\infty + \int_a^b |\psi_t'(u)|\,du\Big].$$

C being a constant not depending on t, a or b and independent of ψ_t and ϕ.

A proof can be found in [26, Chapter VIII]. We will also need some other classical results, to be found in [26] or [14].

3.2. *Laguerre polynomials.* Laguerre polynomials have been introduced for the spherical Fourier transform.

A classical estimate is given by (see [29])

$$\forall q \in \mathbb{N}, \quad |\mathcal{L}_m^{(q)}| \leq \binom{m+q}{m} \leq C_q(m+1)^q. \tag{3.1}$$

Recall that $\mathcal{L}_m^{(q)}(\xi) = L_m^{(q)}(\xi)\exp[-\xi/2]$. Just as in [1] we will need to estimate their derivatives:

Lemma 3.2.

$$\Big|\Big(\xi\,\frac{d}{d\xi}\Big)^N(\mathcal{L}_m^{(d-1)}(\xi))\Big| \leq C_{N,d}(2m+d)^{d-1/4}$$

for all $0 \leq N \leq d$.

Proof. It suffices to continue the calculations done in [1] for $N = 0, 1$. For $q \in \mathbb{N}$ and $m \geq q$ let

$$L_m^{(-q)} := \sum_{k=0}^{n-q}\binom{n-q}{k}\frac{(-x)^{k+q}}{(k+q)!} = (-x)^q\,\frac{(n-q)!}{n!}\,L_{n-q}^{(q)}.$$

For $q \in \mathbb{Z}$, set $\mathcal{L}_m^{(q)}(\xi) := L_m^{(q)}(\xi)\exp[-\xi/2]$.

Fix $q \in \mathbb{Z}$ and $m \geq (-q)_+ + 1$; since

$$\begin{aligned}
\frac{d}{d\xi}L_m^{(q)}(\xi) &= mL_m^{(q)}(\xi) - (m+q)L_{m-1}^{(q)}(\xi),\\
\xi L_m^{(q)}(\xi) &= -(m+1)L_{m+1}^{(q)}(\xi) + (2m+q+1)L_m^{(q)}(\xi) - (m+q)L_{m-1}^{(q)}(\xi),\\
L_m^{(q-1)}(\xi) &= L_m^{(q)}(\xi) - L_{m-1}^{(q)}(\xi),
\end{aligned}$$

we get

$$\Big(\xi\,\frac{d}{d\xi}\Big)\mathcal{L}_m^{(q)} = \frac{m+1}{2}\,(\mathcal{L}_{m+1}^{(q-1)} + \mathcal{L}_m^{(q-1)}) - \frac{q+1}{2}\,\mathcal{L}_m^{(q)} - \frac{q-1}{2}\,\mathcal{L}_{m-1}^{(q)}.$$

Then, by induction on $N \in \mathbb{N}$,

$$\forall q \in \mathbb{Z},\ \forall m \geq 2N - q, \qquad \left(\xi \frac{d}{d\xi}\right)^N \mathcal{L}_m^{(q)}(\xi) = \sum_{k=-N}^{N} \sum_{p=0}^{N} B_{k,p}^{q,N}(m) \mathcal{L}_{m-k}^{(q-p)}$$

with $|B_{k,p}^{q,N}(m)| \leq C_{k,p}^{q,N}(m+1)^p$. So we get (3.1) for $0 \leq N \leq d-1$.

For the case $N = d$, we use an estimate proved by Rooney in [23, 22]:

$$|\mathcal{L}_m^{(-1)}| \leq \frac{\sqrt{2(2m)!}}{2^m m!} \leq C(m+1)^{-1/4}. \ \blacksquare$$

3.3. *Bessel functions.* Let J_ν be the Bessel function of order $\nu > -1/2$,

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2)\pi^{1/2}} \int_{-1}^{1} e^{\mathbf{i}rt}(1-t^2)^{\nu-1/2}\, dt.$$

By m-fold integration by parts we obtain

Lemma 3.3. *For any $m \in \mathbb{N}$,*

$$J_{m+1/2}(r) = r^{-1/2} \sum_{j=0}^{m} (a_j e^{\mathbf{i}r} + b_j e^{-\mathbf{i}r}) r^{-j},$$

where a_j and b_j are complex coefficients.

Lemma 3.4. *For any $m \in \mathbb{N}$,*

$$J_m(r) = e^{\mathbf{i}r}\left[\frac{\alpha_+}{r^{1/2}} + \phi_+(r)\right] + e^{-\mathbf{i}r}\left[\frac{\alpha_-}{r^{1/2}} + \phi_-(r)\right],$$

where $\phi_\pm \in \mathcal{S}(\mathbb{R}_+^*)$ are such that

$$\forall r > 0, \quad |\phi_\pm(r)| \leq Cr^{-1/2}, \quad |\phi'_\pm(r)| \leq Cr^{-3/2}. \tag{3.2}$$

Proof. For the Bessel function of integer order we get (see [26, p. 338])

$$J_m(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{\mathbf{i}r\sin\theta} e^{-\mathbf{i}m\theta}\, d\theta.$$

Following Stein, let ψ_+ (resp. ψ_-) be a C^∞ function with support small enough around $\pi/2$ (resp. $3\pi/2$). Let ψ_0 be such that

$$\psi_+ + \psi_0 + \psi_- = 1.$$

With natural notations we get

$$J_m(r) = e^{\mathbf{i}r} J_m^+(r) + J_m^0(r) + e^{-\mathbf{i}r} J_m^-(r),$$

in particular the oscillating integral

$$J_m^+(r) = \int_a^b e^{\mathbf{i}r\varphi(\theta)} \chi(\theta)\, d\theta$$

with $\chi \in C_{\mathrm{c}}^{\infty}(]a,b[)$ ($]a,b[$ being a small interval around $\pi/2$). Since $\varphi(\theta) = \sin\theta - 1$,

$$\varphi(\pi/2) = \varphi'(\pi/2) = 0 \quad \text{and} \quad \varphi''(\pi/2) \neq 0.$$

So by [26, Proposition 3, p. 334]), we get

$$J_m^{+}(r) = \alpha_+ r^{-1/2} + \psi_+(r)$$

with $\psi_+(r) = O(r^{-3/2})$ and $\psi'_+(r) = O(r^{-5/2})$. J_m^+ being analytic, an easy computation shows that ψ_+ has the good behaviour. The same is true for

$$J_m^{-}(r) = \alpha_- r^{-1/2} + \psi_-(r).$$

Finally, since J_m^0 has no critical point in its phase (thus is in the Schwartz class), we conclude that $\phi_+(r) = \psi_+(r) + J_m^0(r)e^{-\mathbf{i}r}$ and $\phi_-(r) = \psi_-(r)$ satisfy (3.2). ∎

4. The wave equation. Looking at the wave equation on G,

$$\begin{cases} \triangle u + \partial_{tt} u = 0, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1, \end{cases} \tag{4.1}$$

we naturally introduce the wave operator

$$W_t = \exp[\mathbf{i}t\sqrt{\triangle}]$$

which is connected to the general solution u of (4.1) by

$$u(g,t) = \frac{dA_t}{dt}\, u_0 + A_t u_1$$

with $A_t = \frac{\sin(t\sqrt{\triangle})}{\sqrt{\triangle}}$, and $dA_t/dt = \cos(t\sqrt{\triangle})$.

LEMMA 4.1 (Sharp space-time dispersion). *The kernel φ of $R^*(\triangle)$ introduced in Section 2 satisfies the estimate*

$$\sup_{x,y} |W_t\varphi((x,y,s))| \le C|t|^{-1/2}|s|^{(1-p)/2}.$$

Proof. By Fourier inversion we see that $W_t\varphi((x,y,s))$ is proportional to

$$\Omega := \sum_{m\in\mathbb{N}} \int_{S^{p-1}} \int_0^{\infty} e^{-\mathbf{i}\lambda\varepsilon\cdot s + \mathbf{i}t\sqrt{\lambda(2m+d)}} R_m^*(\lambda)\mathcal{L}_m^{(d-1)}\left(\lambda\,\frac{\|\omega\|^2}{2}\right)\frac{d\lambda\, d\sigma(\varepsilon)}{\lambda^{1-d-p}} \tag{4.2}$$

with $\omega = (x,s)$. By a partial integration, the expression after the S^{p-1} integral sign is very similar to an integral computed in [1]. Thus, integrating this result (see proof of Lemma 4.1 in [1], or see below) over S^{p-1}, we get

$$\sup_{g\in G} |W_t\varphi(g)| \le C\min\{1, |t|^{-1/2}\}.$$

Hence it suffices to show the lemma for $p > 1$ and $|s| > 1$. This time we integrate first over S^{p-1}:

$$\Omega = \int_0^\infty \widehat{d\sigma}(\lambda s) \sum_{m\in\mathbb{N}} e^{\mathbf{i}t\sqrt{\lambda(2m+d)}} R_m^*(\lambda) \mathcal{L}_m^{(d-1)}\left(\lambda \frac{\|\omega\|^2}{2}\right) \lambda^{d+p-1} d\lambda.$$

Now,

$$\widehat{d\sigma}(\xi) = \int_{S^{p-1}} e^{-\mathbf{i}x\cdot\xi}\, d\sigma(x) = 2\pi\left(\frac{|\xi|}{2\pi}\right)^{(2-p)/2} J_{(p-2)/2}(|\xi|)$$

(see [26, p. 347]).

Case 1: *p odd.* By Lemma 3.3 we have

$$J_{(p-2)/2}(r) = r^{-1/2} e^{\mathbf{i}r} \sum_{j<p/2-1/2} a_j r^{-j} + r^{-1/2} e^{-\mathbf{i}r} \sum_{j<p/2-1/2} b_j r^{-j}.$$

Then it suffices to study

$$I_{m,j}^{\pm} = \int_0^\infty (\lambda|s|)^{(1-p)/2-j} e^{\mathbf{i}(\pm\lambda|s|+t\sqrt{\lambda(2m+d)})} R_m^*(\lambda) \mathcal{L}_m^{(d-1)}\left(\lambda\frac{\|\omega\|^2}{2}\right) \lambda^{d+p-1}\, d\lambda.$$

Let $s = ts'$, $\mu = (2m+d)\lambda$. By this change of variables

$$I_{m,j}^{\pm} = |ts'|^{(1-p)/2-j} \times \int_{C_0} e^{\mathbf{i}tg_m^{\pm}(\mu)} R^*(\mu) \mathcal{L}_m^{(d-1)}\left(\mu\, \frac{\|\omega\|^2}{2(2m+d)}\right) \frac{\mu^{d+(p-1)/2-j}}{(2m+d)^{d+(p+1)/2-j}}\, d\mu$$

with

$$g_m^{\pm}(\mu) = \pm\frac{\mu}{(2m+d)}\, |s'| + \sqrt{\mu}.$$

Following [1] the stationary phase (Lemma 3.1) gives us

$$|I_{m,j}^{\pm}| \le C|s|^{(1-p)/2}(2m+d)^{-(p+3)/2+j} t^{-1/2}.$$

To conclude it suffices to sum these estimates since

$$\sum_{m=0}^{\infty} (2m+d)^{-(p+3)/2+j} < \infty.$$

Case 2: *p even.* This time, by Lemma 3.4,

$$J_{(p-2)/2}(r) = e^{\mathbf{i}r}\left[\frac{\alpha_+}{r^{1/2}} + \phi_+(r)\right] + e^{-\mathbf{i}r}\left[\frac{\alpha_-}{r^{1/2}} + \phi_-(r)\right],$$

so it suffices to study

$$I_m^\pm = \int_0^\infty (\lambda|s|)^{(1-p)/2} e^{\mathbf{i}(\pm\lambda|s|+t\sqrt{\lambda(2m+d)})} R_m^*(\lambda)\mathcal{L}_m^{(d-1)}\left(\lambda\,\frac{\|\omega\|^2}{2}\right)\lambda^{d+p-1}\,d\lambda,$$

$$\Upsilon_m^\pm = \int_0^\infty (\lambda|s|)^{(2-p)/2}\phi_\pm(\lambda|s|) e^{\mathbf{i}(\pm\lambda|s|+t\sqrt{\lambda(2m+d)})} \times R_m^*(\lambda)\mathcal{L}_m^{(d-1)}\left(\lambda\,\frac{\|\omega\|^2}{2}\right)\lambda^{d+p-1}\,d\lambda.$$

We recognize here the expression of $I_{m,0}^\pm$, so

$$|I_m^\pm| \le C|ts'|^{(1-p)/2}(2m+d)^{-(p+3)/2}t^{-1/2}. \tag{4.3}$$

Let us detail the computation for $\Upsilon_m^\pm$. Set $s = ts'$ and $\mu = (2m+d)\lambda$. Then

$$\Upsilon_m^\pm = |ts'|^{(2-p)/2}\int_{C_0} e^{\mathbf{i}tg_m^\pm(\mu)} f(\mu)\,d\mu$$

with

$$f(\mu) = \phi_\pm\left(\frac{\mu|ts'|}{(2m+d)}\right)R^*(\mu)\mathcal{L}_m^{(d-1)}\left(\mu\,\frac{\|\omega\|^2}{2(2m+d)}\right)\frac{\mu^{d+p/2}}{(2m+d)^{d+p/2+1}},$$

where $g_m^\pm(\mu)$ is as in Case 1.

Estimate of $\|f\|_\infty$. By Lemma 3.4,

$$|f(\mu)| \le \frac{(2m+d)^{-(d+p/2+1)}}{\sqrt{\mu|ts'|/(2m+d)}}\,|\mu^{d+p/2}R^*(\mu)|\left|\mathcal{L}_m^{(d-1)}\left(\frac{\mu}{2(2m+d)}\|\omega\|^2\right)\right|.$$

So, recalling that $R^* \in C_0^\infty(C_0)$, Lemma 3.2 gives us

$$\forall\mu\in C_0,\qquad |f(\mu)| \le C_d|ts'|^{-1/2}(2m+d)^{-(p+3)/2}.$$

Estimate of $\int_{C_0}|f'(\mu)|\,d\mu$. Let $f'(\mu) = \alpha(\mu)+\beta(\mu)+\gamma(\mu)$ with

$$\alpha(\mu) = \frac{d}{d\mu}\left(\phi_\pm\left(\frac{\mu|ts'|}{(2m+d)}\right)\right)\frac{\mu^{d+p/2}}{(2m+d)^{d+p/2+1}}\,R^*(\mu)\mathcal{L}_m^{(d-1)}\left(\mu\frac{\|\omega\|^2}{2(2m+d)}\right)$$

$$\beta(\mu) = \phi_\pm\left(\frac{\mu|ts'|}{(2m+d)}\right)\frac{d}{d\mu}\left(\frac{\mu^{d+p/2}}{(2m+d)^{d+p/2+1}}R^*(\mu)\right)\mathcal{L}_m^{(d-1)}\left(\mu\frac{\|\omega\|^2}{2(2m+d)}\right)$$

$$\gamma(\mu) = \phi_\pm\left(\frac{\mu|ts'|}{(2m+d)}\right)\frac{\mu^{d+p/2}}{(2m+d)^{d+p/2+1}}\,R^*(\mu)\frac{d}{d\mu}\left(\mathcal{L}_m^{(d-1)}\left(\mu\frac{\|\omega\|^2}{2(2m+d)}\right)\right).$$

From Lemma 3.2, Lemma 3.4 and the equalities

$$\gamma(\mu) = \phi_\pm\left(\frac{\mu|ts'|}{2m+d}\right)\frac{\mu^{d+p/2-1}}{(2m+d)^{d+p/2+1}}R^*(\mu)\left[\xi\,\frac{d}{d\xi}\mathcal{L}_m^{(d-1)}(\xi)\right]_{\xi=\frac{\mu\|\omega\|^2}{2(2m+d)}},$$

$$\frac{d}{d\mu}\left(\phi_\pm\left(\frac{\mu|ts'|}{(2m+d)}\right)\right) = \frac{|ts'|}{(2m+d)}\phi_\pm'\left(\frac{\mu|ts'|}{(2m+d)}\right),$$

we get

$$\int_{C_0} |f'(\mu)|\, d\mu \le C_d |ts'|^{-1/2}(2m+d)^{-(p+3)/2}.$$

Thus, the stationary phase estimate on $\Upsilon_m^{\pm}$ gives

$$|\Upsilon_m^{\pm}| \le C|ts'|^{(1-p)/2}(2m+d)^{-(p+3)/2}t^{-1/2}. \tag{4.4}$$

To conclude we just have to sum the two kinds of estimates (4.3) and (4.4). ∎

This estimate is sharp in the joint space-time cone

$$\{(s,t)\in \mathbb{R}^p \times \mathbb{R} : s = Ct\}.$$

It should be of interest in the study of weighted Strichartz estimates.

Following [1], Theorem 0.1 is a direct consequence of

LEMMA 4.2. *The kernel φ of $R^*(\triangle)$ introduced in Section 2 satisfies the estimate*

$$\sup_g |W_t\varphi(g)| \le C|t|^{-p/2}.$$

Proof. To improve the time decay, we will try to apply p times a non-critical phase estimate.

Fixing $s \in \mathbb{R}^p \setminus \{0\}$ and $t > 1$, recall that

$$W_t\varphi(g) = C\sum_{m=0}^{\infty} \int_{S^{p-1}} I_{\varepsilon,m}\, d\sigma(\varepsilon)$$

and the integral $I_{\varepsilon,m}$ has been computed in the proof above:

$$I_{\varepsilon,m} = \int_{C_0} \exp[\mathbf{i}tg_{m,\varepsilon}(\mu)]\Psi(\mu)\, d\mu$$

with

$$g_{m,\varepsilon}(\mu) = \sqrt{\mu} - \frac{\mu}{(2m+d)t}\,\varepsilon s,$$

$$\Psi(\mu) = \frac{\mu^{d+p-1}}{(2m+d)^{d+p}}\, R^*(\mu)\mathcal{L}_m^{(d-1)}\biggl(\mu\,\frac{\|\omega\|^2}{2(2m+d)}\biggr).$$

CASE 1: $|s| \ge \frac{1}{4\sqrt{2}}(2m+d)t$. Inspecting the proof of Lemma 4.1 we get

$$\Bigl|\int_{S^{p-1}} I_{\varepsilon,m}\, d\sigma(\varepsilon)\Bigr| \le C|s|^{(1-p)/2-j}(2m+d)^{-(p+3)/2+j}t^{-1/2}$$

(with $0 \le j < p/2 - 1/2$ when p is even and $j = 0$ when p is odd). So

$$\Bigl|\int_{S^{p-1}} I_{\varepsilon,m}\, d\sigma(\varepsilon)\Bigr| \le C(2m+d)^{-(p+1)}t^{-p/2}.$$

CASE 2: $|s| \le \frac{1}{8}(2m+d)t$. The phase $g_{m,\varepsilon}$ has no critical point on C_0:

$$g'_{m,\varepsilon}(\mu) = \frac{1}{2\sqrt{\mu}} - \frac{\varepsilon \cdot s}{(2m+d)t} \ge \frac{1}{4} - \frac{|s|}{(2m+d)t} \ge \frac{1}{8}.$$

By Q-fold integration by parts, we get

$$I_{\varepsilon,m} = (\mathbf{i}t)^{-Q} \int\limits_{C_0} \exp[\mathbf{i}t g_{m,\varepsilon}(\mu)] D^Q(\Psi(\mu))\, d\mu,$$

where the differential operator D is defined by

$$D\psi = \frac{d}{d\mu} \frac{\psi(\mu)}{g'_{m,\varepsilon}(\mu)}.$$

But by a direct induction,

$$D^Q \psi = \sum_{k=Q}^{2Q} \sum_{\wr\alpha\wr = k} C(\alpha,k,Q) \frac{\psi^{(\alpha_1)}(g''_{m,\varepsilon})^{\alpha_2} \cdots (g^{(Q+1)}_{m,\varepsilon})^{\alpha_{Q+1}}}{(g'_{m,\varepsilon})^k}$$

with $\alpha = (\alpha_1, \dots, \alpha_{Q+1}) \in \{0,\dots,Q\} \times \mathbb{N}^Q$ et $\wr\alpha\wr = \sum_{j=1}^{Q+1} j\alpha_j$.

Now, because $g^{(l)}_{m,\varepsilon}(\mu) = -\frac{(2l)!}{2^l l!}\mu^{-l+1/2}$ for $l \ge 2$, we get $|g^{(l)}_{m,\varepsilon}(\mu)| \le \frac{(2l)!}{\sqrt{2}l!}$ on C_0 and so

$$|I_{\varepsilon,m}| \le Ct^{-Q} \sup_{0 \le k \le Q} \|\Psi^{(k)}\|_\infty,$$

uniformly with respect to $\varepsilon \in S^{p-1}$. On the other hand, by induction we get

$$\Psi^{(k)}(\mu) = (2m+d)^{-d-p} \sum_{k=0}^{Q} \sum_{\alpha,\beta \in \mathcal{F}} A(Q,\alpha,\beta,k) \times \mu^\alpha \frac{d}{d\mu}(\mu^\beta R^*(\mu)) \left(\mu \frac{d}{d\mu}\right)^k \Lambda(\mu),$$

where

$$\Lambda(\mu) = \mathcal{L}_m^{(d-1)}\left(\mu \frac{\|\omega\|^2}{2(2m+d)}\right)$$

and $\mathcal{F} \subset [p+d-Q-1, p+d+Q-1] \cap \mathbb{Z}$. Since, by Lemma 3.2,

$$\left|\left(\mu \frac{d}{d\mu}\right)^k \Lambda(\mu)\right| = \left|\left[\left(\mu \frac{d}{d\mu}\right)^k \mathcal{L}_m^{(d-1)}\right]\left(\mu \frac{\|\omega\|^2}{2(2m+d)}\right)\right| \le C_{d,k}(2m+d)^{d-1/4}$$

it follows that

$$\left| \int\limits_{S^{p-1}} I_{\varepsilon,m}\, d\sigma(\varepsilon)\right| \le C(2m+d)^{-(p+1/4)} t^{-Q}.$$

But since $p \le 2d-1$, this estimate remains true on replacing the power $Q = d$ by $p/2 \le Q$.

Conclusion. Putting together the two estimates, we get by a straightforward summation

$$|W_t\varphi(g)| \le C\sum_{m=0}^{\infty}(2m+d)^{-(p+1/4)}t^{-p/2} \le Ct^{-p/2}. \blacksquare$$

As in [1] we have

THEOREM 4.3 (Sharp time dispersion). *If u is a solution of the wave equation* (4.1), *then*

$$\|u(t)\|_\infty \le C|t|^{-p/2}(\|u_0\|_{B_{1,1}^{N-p/2}} + \|u_1\|_{B_{1,1}^{N-1-p/2}}),$$

and this time decay is sharp in time.

Let us show as in [1] the sharpness of the estimate. Let $Q \in C_0^\infty(D_0)$ with $Q(1) = 1$, where D_0 is a small neighborhood of 1 such that $0 \notin D_0$. Then

$$\mathcal{F}u_0(\lambda)(\lambda, m) = Q(|\lambda|)\delta_{m0}$$

and $u_1 := 0$ determine a solution of (4.1),

$$u((\omega,s),t) = \cos(t\sqrt{\triangle})u_0 = C\int_{\mathbb{R}^p} e^{-\mathbf{i}\lambda\cdot s-|\lambda|\,\|\omega\|^2/4}\cos(t\sqrt{d|\lambda|})Q(|\lambda|)\,|\lambda|^d d\lambda.$$

Consider $u((0,ts),t)$ for a fixed s. This oscillating integral has a phase $\phi_\pm(\lambda) := -\lambda\cdot s \pm \sqrt{d|\lambda|}$ with a unique critical point $\lambda_0^\pm = \mp(d/\sqrt{2})s$, which is not degenerate. Indeed, the Hessian is equal to

$$\mp\frac{3\sqrt{d}}{4}\,|\lambda|^{-7/2}\Big(\lambda_i\lambda_j - \frac{2}{3}\,|\lambda|^2\delta_{ij}\Big)_{1\le i,j\le p} \sim \pm\frac{\sqrt{d}}{4}\,|\lambda|^{-3/2}\begin{pmatrix} 2 & & & \\ & \ddots & & \\ & & 2 & \\ & & & -1\end{pmatrix}.$$

So, applying [26, Proposition 6, p. 344] (asymptotic expansion of oscillating integrals) for $|s| = \sqrt{2}/d$ we get

$$u((0,ts),t) \sim C|t|^{-p/2}. \tag{4.5}$$

The Strichartz estimates follow as in [1].

4.1. *Strichartz inequalities.* These inequalities were established by R. Strichartz in the 70's [27]; they give local and global existence for nonlinear partial differential equations. In order to get these estimates Ginibre and Velo [13], or more generally Keel and Tao [18] make use of dispersive inequalities (and Hardy–Littlewood–Sobolev estimates combined with some interpolation). A dispersive estimate is not a necessary condition to reach Strichartz inequalities (see [4]), but it has become a very classical way to obtain them. Following [1, 13, 18] we will just list the intermediate results:

Let $\beta(r) = (N - p/2)(1/2 - 1/r)$, and denote by $\bar{q}$ the conjugate exponent of q. By a direct application of the Keel and Tao theorem ([18]) we get

PROPOSITION 4.4. *The wave operator W_t satisfies the estimates*

$$\|W_t g\|_{L^q_t \dot{B}^{-\beta(r)}_{r,2}} \le C\|g\|_{L^2}, \tag{4.6}$$

$$\Big\| \int_0^t W_{t-s} F(s)\, ds \Big\|_{L^{q_1}_t \dot{B}^{-\beta(r_1)}_{r_1,2}} \le C\|F\|_{L^{\bar{q}_2}_t \dot{B}^{\beta(r_2)}_{\bar{r}_2,2}} \tag{4.7}$$

with $2/q + p/r = p/2$ and $q, r \ge 2$ except for $(q, r, p) = (2, \infty, 2)$ (and the same holds for q_i and r_i).

By direct Besov spaces injections the $\beta(r)$ and $\beta(r_i)$ can be replaced by any ϱ_i such that

$$\frac{N}{r_i} + \frac{1}{q_i} - \varrho_i = \frac{N}{2}, \qquad \frac{2}{q_i} + \frac{p}{r_i} \le \frac{p}{2}.$$

Now we get a first result on homogeneous Besov spaces for the non-homogeneous wave equation

$$\begin{cases} \triangle u + \partial_{tt} u = f, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1. \end{cases} \tag{4.8}$$

We can compute the general solution w by setting

$$w(x,t) = u(x,t) + v(x,t),$$

where u and v are classically given by

$$u(t) = \frac{dA_t}{dt} u_0 + A_t u_1, \qquad v(t) = \int_0^t A_{t-s} f(s)\, ds.$$

$$\partial_t u(t) = -A_t \triangle u_0 + \frac{dA_t}{dt}\, u_1 \qquad \partial_t v(t) = \int_0^t \frac{dA_{t-s}}{dt} f(s)\, ds.$$

THEOREM 4.5 (Strichartz estimates on homogeneous Besov spaces). *Let $q_i, r_i \ge 2$ and $(q_i, r_i, p) \ne (2, \infty, 2)$ be such that*

$$\frac{N}{r_1} + \frac{1}{q_1} - \varrho_1 = \frac{N}{2} - 1, \qquad \frac{N}{r_2} + \frac{1}{q_2} - \varrho_2 = \frac{N}{2}, \qquad \frac{2}{q_i} + \frac{p}{r_i} \le \frac{p}{2}.$$

Then

$$\|u\|_{L^{q_1}_t \dot{B}^{\varrho_1}_{r_1,2}} + \|\partial_t u\|_{L^{q_1}_t \dot{B}^{\varrho_1 - 1}_{r_1,2}} \le C(\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}),$$

$$\|v\|_{L^{q_1}_t \dot{B}^{\varrho_1}_{r_1,2}} + \|\partial_t v\|_{L^{q_1}_t \dot{B}^{\varrho_1 - 1}_{r_1,2}} \le C\|f\|_{L^{\bar{q}_2}_t \dot{B}^{-\varrho_2}_{\bar{r}_2,2}}.$$

In particular by Sobolev injections we get the result announced in the introduction:

THEOREM 4.6 (Strichartz estimates on Lebesgue spaces). *If w is the solution of the non-homogeneous equation* (4.8), *then for any $q \in [(2N-p)/p, \infty]$ and r such that*

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - 1$$

we have the estimate

$$\|w\|_{L_t^q L^r} \leq C(\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|f\|_{L_t^1 L^2}).$$

This improves the result found in [1] thanks to the stronger dispersion of the wave operator.

5. The Schrödinger equation. As noticed by Bahouri, Gérard and Xu, the Schrödinger equation

$$\begin{cases} \mathbf{i}\triangle u - \partial_t u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{5.1}$$

has no dispersive estimate on the Heisenberg group $\mathbb{H}^n$, since, for a suitable u_0, we can have (see [1])

$$\forall q \in [1, \infty], \qquad \|u(t)\|_q = \|u_0\|_q.$$

We will show this behavior disappears when the center of the stratified group G becomes bigger. Typically this is true when we pass from a Heisenberg group to an H-type one.

By analogy with the wave operator, we will focus on the Schrödinger operator

$$S_t = e^{\mathbf{i}t\triangle}.$$

LEMMA 5.1. *If φ is as in Section* 2, *then*

$$\|S_t\varphi\|_\infty \leq C\min\{1, t^{(1-p)/2}\}.$$

Proof. The case of the Heisenberg group ($p = 1$) is clear. From now on, let $p > 1$. We will just adapt our computation (with the same notation) for the wave operator.

We will only consider the case where p is an odd integer and $t > 1$. Fix $s \neq 0$. We will try to estimate the L^∞ norm of the terms in the sum

$$S_t\varphi(\omega, s) = \sum_m K_m(\omega, s).$$

CASE 1: $|s| \leq \frac{1}{2}(2m + d)t$. Then

$$K_m\varphi(\omega, s) = \int_{S^{p-1}} \mathcal{I}_{\varepsilon,m}(\omega, s) \quad \text{with} \quad \mathcal{I}_{\varepsilon,m}(\omega, s) = \int_{C_0} e^{\mathbf{i}\eta_\varepsilon\mu}\Psi(\mu)\, d\mu,$$

where Ψ is computed in the proof of Lemma 4.2 and $\eta_\varepsilon = t - s\cdot\varepsilon/(2m+d)$. As $|\eta_\varepsilon| \geq t/2 > 0$, by Q-fold integration by parts, we get

$$\mathcal{I}_{\varepsilon,m}(\omega,s) = \frac{C}{\eta_\varepsilon^Q}\int\limits_{C_0} e^{\mathbf{i}\eta_\varepsilon\mu}\,\frac{d^Q}{d\mu^Q}\,\Psi(\mu)\,d\mu.$$

Since $\left\|\frac{d^Q}{d\mu^Q}\Psi\right\|_\infty \leq C(2m+d)^{-p-1/4}$ for all $Q \leq d$ (see proof of Lemma 4.2, Case 2), we get a uniform estimate (with respect to $\varepsilon \in S^{p-1}$)

$$|\mathcal{I}_{\varepsilon,m}(\omega,s)| \leq \frac{C}{t^Q}\,(2m+d)^{-p-1/4}.$$

Finally, we integrate on the sphere S^{p-1} with $Q = d \geq (p-1)/2$ to obtain

$$|K_m(\omega,s)| \leq \frac{C}{t^d}\,(2m+d)^{-p-1/4} \leq \frac{C}{t^{(p-1)/2}}\,(2m+d)^{-p-1/4}.$$

Case 2: $|s| \geq \frac{1}{2}(2m+d)t$. By the same calculation of the proof of Lemma 4.1 (Case 1), we get

$$K_m = \sum_{\pm,j<p/2-1/2} \mathcal{I}^{\pm}_{m,j}$$

with

$$\mathcal{I}^{\pm}_{m,j}(x,y,s) = \left(\frac{|s|}{2m+d}\right)^{(1-p)/2-j}\int\limits_{C_0} e^{\mathbf{i}\mu\eta_\pm}\Psi_j(\mu)\,d\mu,$$

where

$$\Psi_j(\mu) = \lambda^{(1-p)/2-j}\Psi(\mu), \quad \eta_\pm = \pm\frac{|s|}{2m+d} + t.$$

But since $\|\Psi_j\|_\infty \leq C(2m+d)^{-p-1}$, we have

$$|\mathcal{I}^{\pm}_{m,j}(\omega,s)| \leq Ct^{(1-p)/2-j}(2m+d)^{-p-1}.$$

Finally, summing over all j's and $\pm$ we get

$$|K_m(\omega,s)| \leq \frac{C}{t^{(p-1)/2}}\,(2m+d)^{-p-1} \leq \frac{C}{t^{(p-1)/2}}\,(2m+d)^{-p-1/4}.$$

Conclusion. By summing over $m \in \mathbb{N}$ the lemma is proved. ∎

By analogy, we get

Theorem 5.2 (Sharp dispersive estimate). *If u is the solution of the Schrödinger equation* (5.1), *then*

$$\|u(t)\|_\infty \leq C|t|^{(1-p)/2}\|u_0\|_{B^{N-(p-1)/2}_{1,1}},$$

and the result is sharp in time.

To convince ourselves of sharpness, we proceed as in the wave equation case. For the same Q, D_0 and u_0, and u the solution of (5.1), set

$$u((0,dte_1),t) = C\int_{\mathbb{R}^p} e^{\mathbf{i}t\phi(\lambda)}Q(|\lambda|)|\lambda|^d\,d\lambda$$

with $e_1 = (1,0,\dots,0) \in \mathbb{R}^p$. This time the phase $\phi(\lambda) = d(\lambda\cdot e_1 - |\lambda|)$ is degenerate:

$$\phi'(\lambda) := d(e_1 - \lambda/|\lambda|) = 0 \;\Leftrightarrow\; \lambda \in \mathbb{R}e_1.$$

But the Hessian of rank $p-1$ is

$$\operatorname{Hess}\phi(\lambda) = -\frac{d}{|\lambda|}\left(\delta_{ij} - \frac{\lambda_i\lambda_j}{|\lambda|}\right)_{ij} \sim -\frac{d}{|\lambda|}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0\end{pmatrix}.$$

We conclude thanks to a classical result on oscillating integrals (see [14]) or just by noticing that this integral can be seen as a solution of the wave equation in $\mathbb{R}^{p-1}$.

Combining this with the Strichartz machine, we get

Theorem 5.3 (Strichartz estimates on homogeneous Besov spaces). *Let $q_i, r_i \ge 2$ with $(q_i,r_i,p) \ne (2,\infty,3)$ be such that*

$$\frac{N}{r_i} + \frac{1}{q_i} - \varrho_i = \frac{N}{2}, \qquad \frac{2}{q_i} + \frac{p-1}{r_i} \le \frac{p-1}{2}.$$

If $p > 1$, then

$$\|S_t g\|_{L_t^{q_1}\dot{B}^{\varrho_1}_{r_1,2}} \le C\|g\|_{L^2},$$

$$\left\|\int_0^t S_{t-s}F(s)\,ds\right\|_{L_t^{q_1}\dot{B}^{\varrho_1}_{r_1,2}} \le C\|F\|_{L_t^{\bar{q}_2}\dot{B}^{-\varrho_2}_{\bar{r}_2,2}}.$$

Consider the non-homogeneous equation

$$\begin{cases} \mathbf{i}\triangle u - \partial_t u = f,\\ u|_{t=0} = u_0.\end{cases} \tag{5.2}$$

The solution is

$$u(x,t) = S_t[u_0](x) - \mathbf{i}\int_0^t S_{t-s}f(s)\,ds.$$

Again Sobolev injections give us the result announced in the introduction:

Corollary 5.4. *If $p > 1$ and u is the solution of (5.2), then for all $q \in [(2N+1-p)/(p-1),\infty]$ and for r such that*

$$\frac{1}{q} + \frac{N}{r} = \frac{N}{2} - 1$$

we get

$$\|u\|_{L^q_t L^r} \le C(\|u_0\|_{\dot H^1} + \|f\|_{L^1_t \dot H^1}).$$

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