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Shift-invariant functionals on Banach sequence spaces

by

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To the memory of Aleksander Pełczyński

Abstract. The present paper is a continuation of [23], from which we know that the theory of traces on the Marcinkiewicz operator ideal

$$\mathfrak{M}(H) := \left\{ T \in \mathfrak{L}(H) : \sup_{1 \le m < \infty} \frac{1}{\log m + 1} \sum_{n=1}^{m} a_n(T) < \infty \right\}$$

can be reduced to the theory of shift-invariant functionals on the Banach sequence space

$$\mathfrak{w}(\mathbb{N}_0) := \left\{ c = (\gamma_l) : \sup_{0 \le k < \infty} \frac{1}{k+1} \sum_{l=0}^{k} |\gamma_l| < \infty \right\}.$$

The final purpose of my studies, which will be finished in [24], is the following. Using the density character as a measure, I want to determine the size of some subspaces of the dual $\mathfrak{M}^*(H)$. Of particular interest are the sets formed by the Dixmier traces and the Connes–Dixmier traces (see [2], [4], [6], and [13]).

As an intermediate step, the corresponding subspaces of $\mathfrak{w}^*(\mathbb{N}_0)$ are treated. This approach has a significant advantage, since non-commutative problems turn into commutative ones.

Notation and terminology. Standard notation and terminology of Banach space theory are adopted from [22]. In particular, X and Y denote real or complex Banach spaces, while H is a separable infinite-dimensional complex Hilbert space (identified with ℓ_2). Operators and functionals are always supposed to be linear and continuous (bounded). The symbol I stands for identity maps. The zero element of a Banach space is denoted by o.

An operator $J\colon X \to Y$ is called an *injection* if there exists some $\varrho > 0$ such that $\|Jx\| \ge \varrho \|x\|$ for all $x \in X$. A *metric* injection even satisfies the condition $\|Jx\| = \|x\|$.

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An operator $Q\colon X \to Y$ is called a *surjection* if there exists some $\varrho > 0$ such that $\|y\| \geq \varrho \inf\{\|x\| : Qx = y\}$ for all $y \in Y$. A *metric* surjection even satisfies the condition $\|y\| = \inf\{\|x\| : Qx = y\}$. Note that the preceding concepts are dual to each other; see [20, pp. 26–27].

Surjections $Q\colon X \to Y$ are just the operators whose range is all of Y. On the other hand, a one-to-one operator $J\colon X \to Y$ need not be an injection.

We distinguish between $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$. The letters m and n always stand for natural numbers different from 0, while h, i, j, k, l range over $\mathbb{N}_0$.

Throughout, $a = (\alpha_h)$, $b = (\beta_k)$, $c = (\gamma_l)$, and $z = (\zeta_i)$ denote real or complex sequences; $e = (1, 1, 1, \dots)$. Given any functional λ on a sequence space, we simply write $\lambda(\alpha_h)$ instead of $\lambda((\alpha_h))$.

1. The density character of a Banach space. The results presented in this section are well known, but spread over the literature. For the convenience of the reader, I have included some proofs.

We denote the *cardinality* of any set S by $|S|$. Concerning arithmetic of cardinal numbers we refer to [5, pp. 102–107]:

$$|A| \cdot |B| := |A \times B| \quad \text{and} \quad |A|^{|B|} := |\text{set of all functions from } B \text{ into } A|.$$

The *density character* of a Banach space X is the smallest cardinality of all dense subsets,

$$\operatorname{dense}(X) := \inf\{|D| : D \text{ is dense in } X\}.$$

The infimum is attained, since the class of all cardinalities is well-ordered.

Let $\varrho > 0$. A subset A of X is called *ϱ-separated* if

$$\|x_1 - x_2\| \geq \varrho \quad \text{whenever } x_1, x_2 \in A \text{ and } x_1 \neq x_2.$$

At first glance, it looks not so obvious that $\operatorname{dense}(X)$ is the largest cardinality of all ϱ-separated subsets. However, this is indeed true. The following result was, for the first time, proved by Gohberg–Kreĭn [9, Lemma 6.1] and rediscovered by Kottman [12, pp. 566–567].

LEMMA 1.1.

(1) *If A is ϱ-separated for some $\varrho > 0$, then* $|A| \leq \operatorname{dense}(X)$.

(2) *For every $\varrho > 0$ there exists a ϱ-separated subset A such that* $|A| = \operatorname{dense}(X)$.

Proof. We consider the non-trivial case that $X \neq \{\mathsf{o}\}$.

(1) For every dense subset D, the intersections $D \cap \{x + \frac{1}{2}\varrho U_X\}$ with $x \in A$ and $U_X := \{u \in X : \|u\| < 1\}$ are non-empty and mutually disjoint. Hence $|A| \leq |D|$, which yields $|A| \leq \operatorname{dense}(X)$.

(2) The collection of all ϱ-separated subsets A is inductively ordered by inclusion. So Zorn's lemma ensures the existence of maximal elements. Fix

such a maximal A. Then $|A| \geq \aleph_0$. Assume that

$$D := \Big\{ \sum_{i=1}^{n} \xi_i x_i : \xi_i \text{ rational}, \; x_i \in A, \; n = 1, 2, \ldots \Big\}$$

fails to be dense in X. Then $\overline{D}$ is a proper closed subspace. By the Riesz lemma [21, p. 139], we find $x_0 \in X$ such that $\|x - x_0\| \geq \varrho$ for all $x \in \overline{D}$. Hence A can be enlarged by adding x_0. This contradiction shows that D is indeed a dense subset. Thus $\text{dense}(X) \leq |D| \leq \aleph_0^3 \cdot |A| = |A|$. ∎

The density character has the following elementary properties: For all closed subspaces N of X, we know that

$$\text{dense}(N) \leq \text{dense}(X), \qquad \text{dense}(X/N) \leq \text{dense}(X),$$

and

$$\text{dense}(X) \leq \text{dense}(N) \cdot \text{dense}(X/N).$$

Moreover,

$$\text{dense}(X) \leq \text{dense}(X^*) \leq 2^{\text{dense}(X)}.$$

Thus the density character provides a (coarse) tool to measure the size of a Banach space.

Remark. The *dimension* of a Banach space X is defined as the smallest cardinality of all subsets D whose linear span is dense in X. Note, however, that apart from the finite-dimensional case, we get $\dim(X) = \text{dense}(X)$.

For later use, we mention that the estimate $\text{dense}(X/N) \leq \text{dense}(X)$ has the following consequence.

Lemma 1.2. *If there exists a surjection from X onto Y, then*

$$\text{dense}(Y) \leq \text{dense}(X).$$

To determine the density character of $\mathfrak{l}_\infty(\mathbb{N}_0)$ we need the *Stone–Čech compactification* $\beta\mathbb{N}_0$, whose points can be identified with the *ultrafilters* $\mathcal{U}$ on $\mathbb{N}_0$ or the *non-trivial multiplicative functionals* φ on $\mathfrak{l}_\infty(\mathbb{N}_0)$. The relationship between both objects is given as follows:

The ultrafilter $\mathcal{U}_\varphi$ corresponding to φ consists of all subsets $\mathbb{A}$ of $\mathbb{N}_0$ such that $\varphi(e_{\mathbb{A}}) = 1$, where $e_{\mathbb{A}}$ denotes the characteristic sequence of $\mathbb{A}$.

Conversely, with every ultrafilter $\mathcal{U}$ one associates the functional

$$\varphi_{\mathcal{U}}(a) := \mathcal{U}\text{-}\lim_h \alpha_h \qquad \text{for all } a = (\alpha_h) \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

In particular, $h \in \mathbb{N}_0$ generates the *principal* ultrafilter $\mathcal{U}_h := \{\mathbb{A} : h \in \mathbb{A}\}$ and the multiplicative functional $\varphi_h(a) := \alpha_h$, respectively.

Non-principal ultrafilters, also named *free*, are characterized by the property that all of their members are infinite sets.

Recall that $\beta\mathbb{N}_0$ becomes a compact Hausdorff space with respect to the weak* topology induced by $\mathfrak{l}_\infty^*(\mathbb{N}_0)$. The main result says that $\mathfrak{l}_\infty(\mathbb{N}_0)$ can be identified with $C(\beta\mathbb{N}_0)$, the Banach space of all continuous functions on $\beta\mathbb{N}_0$.

For the purpose of this paper, the following fact is most important:

$$|\beta\mathbb{N}_0 \setminus \mathbb{N}_0| = |\text{set of all free ultrafilters on } \mathbb{N}_0| = 2^{2^{\aleph_0}};$$

see [25], [17], and [8, pp. 130–131, 139].

A functional $\varphi \in \mathfrak{l}_\infty^*(\mathbb{N}_0)$ is said to be *singular* if it vanishes on all sequences with finite support. The set of all singular functionals, denoted by $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$, is a weakly* closed subspace of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$. A well-known result about annihilators shows that $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ can be identified with the dual of $\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)$. Sometimes we will use the fact that $\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)$ is just the quotient of $\mathfrak{l}_\infty(\mathbb{N}_0)$ modulo the null space of the seminorm

$$s(a \,|\, \mathfrak{l}_\infty) := \limsup_{h\to\infty} |\alpha_h|.$$

Note that $\varphi_{\mathcal{U}} \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ if and only if the ultrafilter $\mathcal{U}$ is free.

Next, we prove a classical result, which goes back to Fichtenholz–Kantorovitch [7, p. 81] and Nakamura–Kakutani [18, p. 227].

Proposition 1.3. $\mathrm{dense}(\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)) = \mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0)) = 2^{2^{\aleph_0}}$.

Proof. First of all, we check the upper estimate of $\mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0))$:

If $\mathbb{K}$ denotes the real or complex scalar field, then

$$|\mathfrak{l}_\infty(\mathbb{N}_0)| \le |\mathbb{K}|^{|\mathbb{N}_0|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\cdot\aleph_0} = 2^{\aleph_0}.$$

Thus

$$|\mathfrak{l}_\infty^*(\mathbb{N}_0)| \le |\mathbb{K}|^{|\mathfrak{l}_\infty(\mathbb{N}_0)|} \le (2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0\cdot 2^{\aleph_0}} = 2^{2^{\aleph_0}}.$$

Next, $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0) \subseteq \mathfrak{l}_\infty^*(\mathbb{N}_0)$ implies $\mathrm{dense}(\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)) \le \mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0))$.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be different free ultrafilters. Then there exists a subset $\mathbb{S}$ such that $\mathbb{S} \in \mathcal{U}_1$ and $\complement\mathbb{S} \in \mathcal{U}_2$. Define $z = (\zeta_i)$ by $\zeta_i := +1$ if $i \in \mathbb{S}$ and $\zeta_i := -1$ if $i \in \complement\mathbb{S}$. Now it follows from

$$\varphi_{\mathcal{U}_1}(z) - \varphi_{\mathcal{U}_2}(z) = \mathcal{U}_1\text{-}\lim_i \zeta_i - \mathcal{U}_2\text{-}\lim_i \zeta_i = 2$$

and

$$|\varphi_{\mathcal{U}_1}(z) - \varphi_{\mathcal{U}_2}(z)| \le \|\varphi_{\mathcal{U}_1} - \varphi_{\mathcal{U}_2} \,|\, \mathfrak{l}_\infty^*\|$$

that $\|\varphi_{\mathcal{U}_1} - \varphi_{\mathcal{U}_2} \,|\, \mathfrak{l}_\infty^*\| \ge 2$. This shows that $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ contains a 2-separated subset with cardinality $2^{2^{\aleph_0}}$. Hence $\mathrm{dense}(\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)) \ge 2^{2^{\aleph_0}}$. ∎

2. A quotient space. Given any fixed operator S on a Banach space X, the expression

$$u_S(a) := \inf\{\|a - x + Sx\| : x \in X\}$$

yields a semi-norm on X. Moreover, we know from [23, Props. 9.11 and 9.14] that

$$u_S(a) = \inf_{1\le n<\infty} \Big\| \frac{1}{n}\sum_{k=0}^{n-1} S^k a \Big\| = \lim_{n\to\infty} \Big\| \frac{1}{n}\sum_{k=0}^{n-1} S^k a \Big\|$$

whenever $\|S\| = 1$.

The quotient of X modulo the null space of u_S is denoted by $X/\!\!/S$. We stress that $X/\!\!/S$ is just the usual quotient $X/\overline{\mathcal{R}(I-S)}$, where $\mathcal{R}(I-S)$ denotes the range of $I-S$. The quotient map from X onto $X/\!\!/S$ is denoted by Q_S^X. Note that the dual $(X/\!\!/S)^*$ can be identified with the space of all S-invariant functionals on X; see [23, Prop. 9.9].

3. Shift-invariant functionals on $\mathfrak{l}_\infty(\mathbb{N}_0)$. This section can be regarded as a preparation for Section 4, in which the situation is more involved.

The *shift operators* acting on the sequences $b = (\beta_k)$ with $k \in \mathbb{N}_0$ are defined by

$$S_-\colon (\beta_k) \mapsto (\beta_1, \beta_2, \beta_3, \dots) \quad \text{and} \quad S_+\colon (\beta_k) \mapsto (0, \beta_0, \beta_1, \dots).$$

We call $\lambda \in \mathfrak{l}_\infty^*(\mathbb{N}_0)$ *shift-invariant* if

$$\lambda(S_-b) = \lambda(b) \quad \text{and} \quad \lambda(S_+b) = \lambda(b) \qquad \text{for all } b \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

By [23, Prop. 6.1], it suffices to verify the condition above either for S_- or S_+; the other one follows automatically.

Banach limits are a special kind of shift-invariant functionals that have two additional properties. They are positive and normalized:

$$\lambda(\beta_k) \ge 0 \quad \text{if } \beta_k \ge 0 \quad \text{and} \quad \lambda(e) = 1, \qquad \text{where } e = (1,1,1,\dots).$$

The latter concept was invented by Banach [1, p. 34] and Mazur [14, p. 103].

All shift-invariant functionals form a weakly* closed subspace of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$, denoted by $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$. We know from Section 2 that $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ can be identified with the dual of $\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-$, the quotient of $\mathfrak{l}_\infty(\mathbb{N}_0)$ modulo the null space of the seminorm

$$u_{S_-}(b\,|\,\mathfrak{l}_\infty) := \inf\big\{\|b - y + S_-y\,|\,\mathfrak{l}_\infty\| : y \in \mathfrak{l}_\infty(\mathbb{N}_0)\big\}.$$

Note that $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \subset \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$.

The *Cesàro operator* $C\colon \mathfrak{l}_\infty(\mathbb{N}_0) \to \mathfrak{l}_\infty(\mathbb{N}_0)$ is given by

$$C\colon (\beta_k) \mapsto \Big(\frac{1}{h+1}\sum_{k=0}^{h} \beta_k\Big).$$

For the convenience of the reader, we compile a list of some elementary facts.

Lemma 3.1.

(1) $$Cy \in \mathfrak{c}_0(\mathbb{N}_0) \quad \textit{for all } y \in \mathfrak{c}_0(\mathbb{N}_0),$$

(2) $$Cy - CS_-y \in \mathfrak{c}_0(\mathbb{N}_0) \quad \textit{for all } y \in \mathfrak{l}_\infty(\mathbb{N}_0),$$

(3) $$CS_-y - S_-Cy \in \mathfrak{c}_0(\mathbb{N}_0) \quad \textit{for all } y \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

As observed by Mazur [15, p. 173], every singular functional ψ defines a shift-invariant functional

$$C^*\psi : b \mapsto \psi(Cb) \quad \text{for all } b \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

This fact was already contained in lecture notes of von Neumann that circulated in a small group of insiders since 1940/41; see [19, p. 31].

The shift-invariant functionals obtained in this way are called *Mazur functionals*. They form a subspace of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$, denoted by $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$.

Next, we adapt the Cesàro operator C to the shift-invariant setting.

Lemma 3.2. *There exists a* (*unique*) *operator* C_0 *for which the diagram*

$$\begin{tikzcd}
\mathfrak{l}_\infty(\mathbb{N}_0) \arrow[r, "C"] \arrow[d, "Q^{\mathfrak{l}_\infty}_{S_-}"'] & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[d, "Q^{\mathfrak{l}_\infty}_{\mathfrak{c}_0}"] \\
\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_- \arrow[r, "C_0"] & \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)
\end{tikzcd}$$

commutes.

Proof. We know from Lemma 3.1(2) that $Cy - CS_-y \in \mathfrak{c}_0(\mathbb{N}_0)$ for all $y \in \mathfrak{l}_\infty(\mathbb{N}_0)$. Thus

$$s(Cb\,|\,\mathfrak{l}_\infty) \le s(Cb - Cy + CS_-y\,|\,\mathfrak{l}_\infty) + s(Cy - CS_-y\,|\,\mathfrak{l}_\infty) \le \|b - y + S_-y\,|\,\mathfrak{l}_\infty\|,$$

which proves that

$$s(Cb\,|\,\mathfrak{l}_\infty) \le u_{S_-}(b\,|\,\mathfrak{l}_\infty) \quad \text{for all } b \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

Hence the required C_0 is well-defined. ∎

Remark. Using the identifications

$$[\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-]^* \equiv \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \quad \text{and} \quad [\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)]^* \equiv \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0),$$

we may regard the dual operator C_0^* as a map from $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ into $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$, which is obtained by restricting C^*. Then the range $\mathcal{R}(C_0^*)$ coincides with $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$.

We now refine the diagram given in Lemma 3.2. To this end, let

$$\mathcal{N} := \{y_0 \in \mathfrak{l}_\infty(\mathbb{N}_0) : Cy_0 \in \mathfrak{c}_0(\mathbb{N}_0)\}$$

and note that the norm of $\mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}$ is induced by the seminorm

$$p(b\,|\,\mathfrak{l}_\infty) := \inf\{\|b - y_0\,|\,\mathfrak{l}_\infty\| : y_0 \in \mathcal{N}\}.$$

Lemma 3.3. *The operator C_0 admits a (unique) decomposition, where Q_0 is a quotient map, while C_{00} is one-to-one:*

$$\begin{tikzcd}
\mathfrak{l}_\infty(\mathbb{N}_0) \arrow[rr, "C"] \arrow[d, "Q^{\mathfrak{l}_\infty}_{S_-}"'] & & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[d, "Q^{\mathfrak{l}_\infty}_{\mathfrak{c}_0}"] \\
\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_- \arrow[rr, "C_0"] \arrow[dr, "Q_0"'] & & \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0) \\
& \mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}. \arrow[ur, "C_{00}"'] &
\end{tikzcd}$$

Proof. We know from Lemma 3.1(2) that $Cy - CS_-y \in \mathfrak{c}_0(\mathbb{N}_0)$ for all $y \in \mathfrak{l}_\infty(\mathbb{N}_0)$. Hence $y - S_-y \in \mathcal{N}$, which implies that

$$p(b \,|\, \mathfrak{l}_\infty) \le u_{S_-}(b \,|\, \mathfrak{l}_\infty) := \inf\{\|b-y+S_-y\| : y \in \mathfrak{l}_\infty(\mathbb{N}_0)\} \quad \text{for all } b \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

Thus the quotient map $Q_0 : \mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_- \to \mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}$ is well-defined.

Since

$$s(Cb \,|\, \mathfrak{l}_\infty) = s(Cb - Cy_0 \,|\, \mathfrak{l}_\infty) \le \|b - y_0 \,|\, \mathfrak{l}_\infty\| \quad \text{whenever } y_0 \in \mathcal{N},$$

we have

$$s(Cb \,|\, \mathfrak{l}_\infty) \le p(b \,|\, \mathfrak{l}_\infty) \quad \text{for all } b \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

This estimate ensures the existence of $C_{00} : \mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N} \to \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)$. ∎

Define the sequences $b^{\langle m\rangle} = (\beta_k^{\langle m\rangle})$ by

$$\beta_k^{\langle m\rangle} := \begin{cases} +1 & \text{if } \quad\;\; m2^i \le k < (m+1)2^i, \\ -1 & \text{if } (m+1)2^i \le k < (m+2)2^i, \\ 0 & \text{otherwise.} \end{cases} \qquad i = 0,1,2,\ldots,$$

To ensure that $(m+2)2^i \le m2^{i+1}$, we let $m \ge 2$.

Lemma 3.4. $p(b^{\langle m\rangle} \,|\, \mathfrak{l}_\infty) = 1$ *and* $s(Cb^{\langle m\rangle} \,|\, \mathfrak{l}_\infty) = \frac{1}{m+1}$.

Proof. Assume that $p(b^{\langle m\rangle} \,|\, \mathfrak{l}_\infty) < 1$ for some m. Then we may choose $\varrho \in \mathbb{R}$ and $y_0 = (\eta_{0,k}) \in \mathcal{N}$ such that

$$p(b^{\langle m\rangle} \,|\, \mathfrak{l}_\infty) < \varrho < 1 \quad \text{and} \quad \|b^{\langle m\rangle} - y_0 \,|\, \mathfrak{l}_\infty\| \le \varrho.$$

Hence

$$1 - \eta_{0,k} \le \varrho \quad \text{if } m2^i \le k < (m+1)2^i,$$

which implies

$$\frac{1}{2^i} \sum_{k=m2^i}^{(m+1)2^i-1} \eta_{0,k} \ge 1 - \varrho.$$

We now obtain

$$
\begin{aligned}
\frac{1}{m2^i}\sum_{k=0}^{m2^i-1}\eta_{0,k} - \frac{1}{(m+1)2^i}\sum_{k=0}^{(m+1)2^i-1}\eta_{0,k} &= \\
&= \left(\frac{1}{m2^i}-\frac{1}{(m+1)2^i}\right)\sum_{k=0}^{m2^i-1}\eta_{0,k} - \frac{1}{(m+1)2^i}\sum_{k=m2^i}^{(m+1)2^i-1}\eta_{0,k} \\
&\le \frac{1}{m+1}\,\frac{1}{m2^i}\sum_{k=0}^{m2^i-1}\eta_{0,k} - \frac{1-\varrho}{m+1}.
\end{aligned}
$$

Since

$$\lim_{h\to\infty}\frac{1}{h+1}\sum_{k=0}^{h}\eta_{0,k}=0,$$

letting $i\to\infty$ yields a contradiction, $0\le-\frac{1-\varrho}{m+1}$.

The non-negative sequence $a^{\langle m\rangle}=(\alpha_h^{\langle m\rangle}):=Cb^{\langle m\rangle}$ attains its local maxima at the indices $(m+1)2^i-1$. Thus it follows from

$$\alpha^{\langle m\rangle}_{(m+1)2^i-1}=\frac{1}{m+1}$$

that $s(Cb^{\langle m\rangle}\,|\,\mathfrak{l}_\infty)=\frac{1}{m+1}$. ∎

Proposition 3.5. *$\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$ fails to be a closed subspace of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$.*

Proof. Lemma 3.4 shows that the one-to-one operator

$$C_{00}:\mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}\to\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)$$

defined in Lemma 3.3 is not an injection. Hence Banach's inverse mapping theorem tells us that $\mathcal{R}(C_0)=\mathcal{R}(C_{00})$ cannot be closed. Therefore, by the closed range theorem, the same is true for $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)=\mathcal{R}(C_0^*)$; see the remark after Lemma 3.2. ∎

As just shown, $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$ fails to be complete. Thus we pass to the closed hull $\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}$. Unfortunately, there remains an open question concerning the weakly* closed hull.

Problem 3.6. *Which of the relations*

$$\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}=\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}\quad\textit{or}\quad\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}\subset\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$$

is true?

So, as a precaution, we have to distinguish between $\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}$ and $\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$.

In what follows, we determine the size of the Banach spaces

$$\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}\subseteq\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}\subset\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)\subset\mathfrak{l}_\infty^*(\mathbb{N}_0)$$

and the size of their 'differences'.

LEMMA 3.7. *If*

$$J_e : z = (\zeta_i) \mapsto b = (\beta_k) := \sum_{i=0}^{\infty} \zeta_i e_{2i+1},$$

then $(I - S_-)J_e$ *is a metric injection from* $\mathfrak{l}_\infty(\mathbb{N}_0)$ *into* $\mathfrak{l}_\infty(\mathbb{N}_0)$.

Proof. Since

$$J_e z = (0, \zeta_0, 0, \zeta_1, \ldots),$$

we have

$$(I - S_-)J_e z = (-\zeta_0, +\zeta_0, -\zeta_1, +\zeta_1, \ldots). \ \blacksquare$$

PROPOSITION 3.8. $\mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0)/\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)) \ge 2^{2^{\aleph_0}}$.

Proof. By Lemma 1.1 and Proposition 1.3, there exists a 2-separated subset A of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$ with $|A| = \mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0)) = 2^{2^{\aleph_0}}$. Lemma 3.7 tells us that $J_e^*(I - S_-)^*$ is a metric surjection from $\mathfrak{l}_\infty^*(\mathbb{N}_0)$ onto $\mathfrak{l}_\infty^*(\mathbb{N}_0)$. So, for every $\varphi \in A$, we may choose a $\psi \in \mathfrak{l}_\infty^*(\mathbb{N}_0)$ such that $\varphi = J_e^*(I - S_-)^*\psi$. The ψ's obtained in this way form a subset of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$, denoted by B.

Given different members $\varphi_1 = J_e^*(I - S_-)^*\psi_1$ and $\varphi_2 = J_e^*(I - S_-)^*\psi_2$ of A, it follows from Lemma 3.7 that

$$\begin{aligned}\|\varphi_1 - \varphi_2 - J_e^*(I - S_-)^*\lambda \,|\, \mathfrak{l}_\infty^*\| &= \|J_e^*(I - S_-)^*(\psi_1 - \psi_2 - \lambda) \,|\, \mathfrak{l}_\infty^*\| \\ &\le \|\psi_1 - \psi_2 - \lambda \,|\, \mathfrak{l}_\infty^*\|\end{aligned}$$

for every $\lambda \in \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$. Next, we take $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$ such that

$$|\varphi_1(z) - \varphi_2(z)| \ge \tfrac{1}{2}\|\varphi_1 - \varphi_2 \,|\, \mathfrak{l}_\infty^*\| \ge 1 \quad \text{and} \quad \|z \,|\, \mathfrak{l}_\infty\| = 1.$$

Then

$$\begin{aligned}\|\psi_1 - \psi_2 - \lambda \,|\, \mathfrak{l}_\infty^*\| &\ge \|\varphi_1 - \varphi_2 - J_e^*(I - S_-)^*\lambda \,|\, \mathfrak{l}_\infty^*\| \\ &\ge |\varphi_1(z) - \varphi_2(z) - J_e^*(I - S_-)^*\lambda(z)| \\ &= |\varphi_1(z) - \varphi_2(z) - \lambda((I - S_-)J_e z)| \\ &= |\varphi_1(z) - \varphi_2(z)| \ge 1.\end{aligned}$$

This shows that the canonical image of B is 1-separated in $\mathfrak{l}_\infty^*(\mathbb{N}_0)/\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$. Moreover, $|B| = |A| = 2^{2^{\aleph_0}}$. $\blacksquare$

REMARK. When preparing this paper, I was in doubt whether the formalism of annihilators [26, pp. 95–99], which requires some additional knowledge, should be employed. Finally, I had chosen a more direct and longer approach. Only the proof of Proposition 4.20 was given via annihilators. The referee, who deserves a big 'Thank You' for his careful work, disliked my decision. As a compromise, I add a modified proof. The proofs of Propositions 3.13 and 4.10 are changed in the same way, while the proof of Proposition 4.16 is kept old-fashioned.

Proof (second version). As $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ is the annihilator of $M := \overline{\mathcal{R}(I - S_-)}$, we have the identifications

$$M^* \equiv \mathfrak{l}_\infty^*(\mathbb{N}_0)/M^\perp \equiv \mathfrak{l}_\infty^*(\mathbb{N}_0)/\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0).$$

Lemma 3.7 tells us that $(I - S_-)J_e$ is an injection from $\mathfrak{l}_\infty(\mathbb{N}_0)$ into M. Hence $J_e^*(I - S_-)^*$ is a surjection from M^* onto $\mathfrak{l}_\infty^*(\mathbb{N}_0)$. The required conclusion now follows from Lemma 1.2 and Proposition 1.3. ■

Now we present a construction which provides the basic tool of this paper. Let $h_i := 2^{i+2}$ and $d_i \in \mathbb{N}$ (to be specified later) such that $i + 1 \le d_i \le 2^i$. Consider the sequences $s^{[i]} = (\sigma_k^{[i]})$ with

$$\sigma_k^{[i]} := \begin{cases} 0 & \text{if} \quad k < h_i, \\ +1 & \text{if} \quad h_i \le k < h_i + d_i, \\ -1 & \text{if} \quad h_i + d_i \le k < h_i + 2d_i, \\ 0 & \text{if} \quad h_i + 2d_i \le k. \end{cases}$$

Since $h_i + 2d_i < h_{i+1}$, the supports of the $s^{[i]}$'s are mutually disjoint. Because of this fact, the next result is obvious.

Lemma 3.9. *The rule*

$$J_s \colon z = (\zeta_i) \mapsto b = (\beta_k) := \sum_{i=0}^{\infty} \zeta_i s^{[i]}$$

defines a metric injection from $\mathfrak{l}_\infty(\mathbb{N}_0)$ *into* $\mathfrak{l}_\infty(\mathbb{N}_0)$.

Next, we establish a counterpart of Lemma 3.2.

Lemma 3.10. *There exists a (unique) metric injection $J_{s,0}$ such that the diagram*

$$\begin{tikzcd} \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[r, "J_s"] \arrow[d, "Q_{\mathfrak{c}_0}^{\mathfrak{l}_\infty}"'] & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[d, "Q_{S_-}^{\mathfrak{l}_\infty}"] \\ \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0) \arrow[r, "J_{s,0}"] & \mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_- \end{tikzcd}$$

commutes.

Proof. Since, by [23, Lemma 9.17],

$$\begin{aligned} u_{S_-}(J_s z \,|\, \mathfrak{l}_\infty) &\le u_{S_-}(J_s(z - x) \,|\, \mathfrak{l}_\infty) + u_{S_-}(J_s x \,|\, \mathfrak{l}_\infty) \\ &\le \|J_s(z - x) \,|\, \mathfrak{l}_\infty\| \le \|z - x \,|\, \mathfrak{l}_\infty\| \end{aligned}$$

for all sequences x with finite support, we get $u_{S_-}(J_s z \,|\, \mathfrak{l}_\infty) \le s(z \,|\, \mathfrak{l}_\infty)$. Thus $J_{s,0}$ is well-defined.

According to Section 2,

$$u_{S_-}(b \,|\, \mathfrak{l}_\infty) = \inf_{1 \le n < \infty} \sup_{0 \le h < \infty} \frac{1}{n} \Big| \sum_{k=0}^{n-1} \beta_{h+k} \Big|.$$

Fix n and let $b = (\beta_k) := J_s(\zeta_i)$. If $j \ge n-1$, then it follows from

$$\frac{1}{n}\sum_{k=0}^{n-1} \beta^{[j]}_{h_j+k} = \frac{1}{n}\sum_{k=0}^{n-1} \zeta_j \sigma^{[j]}_{h_j+k} = \zeta_j$$

that

$$\sup_{0\le h<\infty} \frac{1}{n}\Big|\sum_{k=0}^{n-1} \beta_{h+k}\Big| \ge \frac{1}{n}\Big|\sum_{k=0}^{n-1} \beta_{h_j+k}\Big| = |\zeta_j|.$$

Hence

$$\sup_{0\le h<\infty} \frac{1}{n}\Big|\sum_{k=0}^{n-1} \beta_{h+k}\Big| \ge \sup_{j\ge n-1} |\zeta_j| \ge \limsup_{j\to\infty} |\zeta_j|,$$

which proves that $u_{S_-}(J_s z \,|\, \mathfrak{l}_\infty) \ge s(z \,|\, \mathfrak{l}_\infty)$. So $J_{s,0}$ is a metric injection. ■

The operator $J_s \colon \mathfrak{l}_\infty(\mathbb{N}_0) \to \mathfrak{l}_\infty(\mathbb{N}_0)$ depends on the choice of (d_i). In what follows, we need only the limiting cases $d_i = i+1$ and $d_i = 2^i$.

Lemma 3.11. *Let $a := Cb$ and $b := J_s z$ for $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$.*

(1) *If $d_i = i+1$, then $a \in \mathfrak{c}_0(\mathbb{N}_0)$.*
(2) *If $d_i = 2^i$, then $\alpha_{h_i+d_i-1} = \frac{1}{5}\zeta_i$.*

Proof. Recall that $h_i = 2^{i+2}$ and $\beta_k = \sum_{i=0}^{\infty} \zeta_i \sigma^{[i]}_k$.
(1) If $h_i \le h < h_{i+1}$, then

$$|\alpha_h| = \frac{1}{h+1}\Big|\sum_{k=0}^{h} \beta_k\Big| = \frac{1}{h+1}\Big|\sum_{k=h_i}^{h} \zeta_i \sigma^{[i]}_k\Big| \le \frac{d_i}{h_i+1}|\zeta_i| = \frac{i+1}{2^{i+2}+1}|\zeta_i|.$$

Therefore $CJ_s(\zeta_i) \in \mathfrak{c}_0(\mathbb{N}_0)$.
(2) Indeed,

$$\alpha_{h_i+d_i-1} = \frac{1}{h_i+d_i}\sum_{k=0}^{h_i+d_i-1} \beta_k = \frac{1}{h_i+d_i}\sum_{k=h_i}^{h_i+d_i-1} \zeta^{[i]}_i \sigma^{[i]}_k = \frac{d_i}{h_i+d_i}\zeta_i = \frac{1}{5}\zeta_i. \ ■$$

Proposition 3.12. $\operatorname{dense}\big(\overline{\mathfrak{l}^{\mathrm{mf}}_\infty(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}}$.

Proof. Specify the operator J_s by letting $d_i := 2^i$.
With every free ultrafilter $\mathcal{U}$ we associate the singular functional

$$\psi_{\mathcal{U}}(a) := \mathcal{U}\text{-}\lim_i \alpha_{h_i+d_i-1} \quad \text{for all } a \in \mathfrak{l}_\infty(\mathbb{N}_0),$$

which in turn generates the Mazur functional $\kappa_{\mathcal{U}} := C^*\psi_{\mathcal{U}}$. If $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$ and $a := CJ_s z$, then it follows from $\alpha_{h_i+d_i-1} = \frac{1}{5}\zeta_i$ (Lemma 3.11) that

$$\kappa_{\mathcal{U}}(J_s z) = \psi_{\mathcal{U}}(CJ_s z) = \mathcal{U}\text{-}\lim_i \alpha_{h_i+d_i-1} = \tfrac{1}{5}\mathcal{U}\text{-}\lim_i \zeta_i.$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be different free ultrafilters. Then there exists a subset $\mathbb{S}$ such that $\mathbb{S} \in \mathcal{U}_1$ and $\complement\mathbb{S} \in \mathcal{U}_2$. Define $z = (\zeta_i)$ by $\zeta_i := +1$ if $i \in \mathbb{S}$ and $\zeta_i := -1$ if $i \in \complement\mathbb{S}$. We infer from

$$\kappa_{\mathcal{U}_1}(J_s z) - \kappa_{\mathcal{U}_2}(J_s z) = \tfrac{1}{5}\mathcal{U}_1\text{-}\lim_i \zeta_i - \tfrac{1}{5}\mathcal{U}_2\text{-}\lim \zeta_i = \tfrac{2}{5}$$

and Lemma 3.9 that

$$\tfrac{2}{5} = |\kappa_{\mathcal{U}_1}(J_s z) - \kappa_{\mathcal{U}_2}(J_s z)| \le \|\kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2} \,|\, \mathfrak{l}_\infty^*\|.$$

So the $\kappa_{\mathcal{U}}$'s form a 2/5-separated subset of $\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}$. Since the set of all free ultrafilters on $\mathbb{N}_0$ has cardinality $2^{2^{\aleph_0}}$, the estimate $\mathrm{dense}\big(\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}}$ follows from Lemma 1.1. ∎

The observation that $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$ is a proper subset of $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ was already made by Jerison [10, p. 80]. Now we show that the difference between both spaces is very big.

Proposition 3.13. $\mathrm{dense}\big(\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)/\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}\big) \ge 2^{2^{\aleph_0}}$.

Proof. Specify the operator J_s by letting $d_i := i + 1$.

Using the identifications

$$[\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-]^* \equiv \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \quad \text{and} \quad [\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)]^* \equiv \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0),$$

we may regard $J_{s,0}^*$ as a restriction of J_s^*. Hence, by Lemma 3.10, the surjection J_s^* induces a surjection from $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ onto $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$.

If $\kappa \in \overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$, then there exists a net $(\psi_\iota)_{\iota\in\mathbb{I}}$ in $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ such that $(C^*\psi_\iota)_{\iota\in\mathbb{I}}$ converges to κ in the weak* topology of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$. Since Lemma 3.11 implies that $CJ_s z \in \mathfrak{c}_0(\mathbb{N}_0)$ for $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$, we get

$$\kappa(J_s z) = \lim_{\iota\in\mathbb{I}} C^*\psi_\iota(J_s z) = \lim_{\iota\in\mathbb{I}} \psi_\iota(CJ_s z) = 0.$$

Therefore $J_{s,0}^*\kappa = o$, which means that $\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$ is included in the null space of $J_{s,0}^*$. Consequently, $J_{s,0}^*$ induces a surjection from $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$ onto $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$. The required conclusion now follows from Lemma 1.2 and Proposition 1.3. ∎

Theorem 3.14. *All of the Banach spaces*

$$\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)} \subseteq \overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*} \subset \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \subset \mathfrak{l}_\infty^*(\mathbb{N}_0),$$

$$\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)} \subset \mathfrak{l}_\infty^*(\mathbb{N}_0)/\,\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)},$$

$$\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*} \subset \mathfrak{l}_\infty^*(\mathbb{N}_0)/\,\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\mathrm{w}^*},$$

and

$$\mathfrak{l}_\infty^*(\mathbb{N}_0)/\,\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$$

have the same density character, namely $2^{2^{\aleph_0}}$.

Proof. The upper estimates follow from

$$\mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0)) \le 2^{2^{\aleph_0}} \qquad \text{(Proposition 1.3)},$$

while the lower estimates are implied by

$$\mathrm{dense}\big(\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}} \qquad \text{(Proposition 3.12)},$$

$$\mathrm{dense}\big(\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*}\big) \ge 2^{2^{\aleph_0}} \qquad \text{(Proposition 3.13)},$$

and

$$\mathrm{dense}(\mathfrak{l}_\infty^*(\mathbb{N}_0)/\,\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)) \ge 2^{2^{\aleph_0}} \qquad \text{(Proposition 3.8)}. \ \blacksquare$$

Remark. A long time ago, the formula $\mathrm{dense}(\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)) = 2^{2^{\aleph_0}}$ was proved in [3, p. 199].

4. Shift-invariant functionals on $\mathfrak{w}(\mathbb{N}_0)$. The Banach space $\mathfrak{w}(\mathbb{N}_0)$ consists of all sequences $c = (\gamma_l)$ for which

$$\|c \,|\, \mathfrak{w}\| := \sup_{0\le k<\infty} \frac{1}{k+1} \sum_{l=0}^{k} |\gamma_l|$$

is finite.

A functional $\varphi \in \mathfrak{w}^*(\mathbb{N}_0)$ is said to be *singular* if it vanishes on all sequences with finite support. The set of all singular functionals, denoted by $\mathfrak{w}^{\mathrm{sgf}}(\mathbb{N}_0)$, is a weakly* closed subspace of $\mathfrak{w}^*(\mathbb{N}_0)$.

A functional $\mu \in \mathfrak{w}^*(\mathbb{N}_0)$ is called *shift-invariant* if

$$\mu(S_- c) = \mu(c) \quad \text{and} \quad \mu(S_+ c) = \mu(c) \qquad \text{for all } c \in \mathfrak{w}(\mathbb{N}_0).$$

By [23, Prop. 2.3], it suffices to verify the condition above either for S_- or S_+; the other one follows automatically.

All shift-invariant functionals form a weakly* closed subspace of $\mathfrak{w}^*(\mathbb{N}_0)$, denoted by $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$. We know from Section 2 that $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$ can be identified with the dual of $\mathfrak{w}(\mathbb{N}_0)/\!\!/S_-$, the quotient of $\mathfrak{w}(\mathbb{N}_0)$ modulo the null space of the seminorm

$$u_{S_-}(c \,|\, \mathfrak{w}) := \inf\{\|c - z + S_- z \,|\, \mathfrak{w}\| : z \in \mathfrak{w}(\mathbb{N}_0)\}.$$

Note that $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) \subset \mathfrak{w}^{\mathrm{sgf}}(\mathbb{N}_0)$.

The *Cesàro operator* $C_{\mathfrak{w}} : \mathfrak{w}(\mathbb{N}_0) \to \mathfrak{l}_\infty(\mathbb{N}_0)$ is given by

$$C_{\mathfrak{w}} : (\gamma_l) \mapsto \bigg(\frac{1}{k+1} \sum_{l=0}^{k} \gamma_l\bigg).$$

Now we are able to introduce two special kinds of shift-invariant functionals on $\mathfrak{w}(\mathbb{N}_0)$:

A *Dixmier functional* has the form $\mu = C_{\mathfrak{w}}^* \lambda$ with $\lambda \in \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$.

A *Connes–Dixmier functional* has the form $\mu = C_{\mathfrak{w}}^* C^* \psi$ with $\psi \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$.

The space consisting of all Dixmier functionals is denoted by $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$, and $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$ stands for the space of all Connes–Dixmier functionals.

Next, we adapt the Cesàro operator $C_{\mathfrak{w}}$ to the shift-invariant setting; see [23, Lemma 9.18].

LEMMA 4.1. *There exists a* (*unique*) *operator* $C_{\mathfrak{w},0}$ *for which the diagram*

$$\begin{tikzcd} \mathfrak{w}(\mathbb{N}_0) \arrow[r, "C_{\mathfrak{w}}"] \arrow[d, "Q_{S_-}^{\mathfrak{w}}"'] & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[d, "Q_{S_-}^{\mathfrak{l}_\infty}"] \\ \mathfrak{w}(\mathbb{N}_0)/\!\!/S_- \arrow[r, "C_{\mathfrak{w},0}"] & \mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_- \end{tikzcd}$$

commutes.

REMARK. Using the identifications

$$[\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-]^* \equiv \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \quad \text{and} \quad [\mathfrak{w}(\mathbb{N}_0)/\!\!/S_-]^* \equiv \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0),$$

the dual operator $C_{\mathfrak{w},0}^*$ may be regarded as a map from $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ into $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$, which is obtained by restricting $C_{\mathfrak{w}}^*$. The range $\mathcal{R}(C_{\mathfrak{w},0}^*)$ coincides with $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$. Analogously, because

$$[\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)]^* \equiv \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0),$$

Lemmas 3.2 and 4.1 show that $C_{\mathfrak{w},0}^* C_0^*$ may be regarded as a map from $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ into $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$, which is obtained by restricting $C_{\mathfrak{w}}^* C^*$. The range $\mathcal{R}(C_{\mathfrak{w},0}^* C_0^*)$ coincides with $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$.

To prove Proposition 4.4 below, we need an analogue of Lemma 3.1(3), which is taken from [23, Lemma 6.3].

LEMMA 4.2. $C_{\mathfrak{w}} S_- z - S_- C_{\mathfrak{w}} z \in \mathfrak{c}_0(\mathbb{N}_0)$ *for all* $z \in \mathfrak{w}(\mathbb{N}_0)$.

Next, we extend Lemma 3.3. To this end, let

$$\mathcal{N}_\diamond := \{z_0 \in \mathfrak{w}(\mathbb{N}_0) : C C_{\mathfrak{w}} z_0 \in \mathfrak{c}_0(\mathbb{N}_0)\},$$
$$\mathcal{N} := \{y_0 \in \mathfrak{l}_\infty(\mathbb{N}_0) : C y_0 \in \mathfrak{c}_0(\mathbb{N}_0)\}.$$

Note that the norms of $\mathfrak{w}(\mathbb{N}_0)/\mathcal{N}_\diamond$ and $\mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}$ are induced by the seminorms

$$q(c \,|\, \mathfrak{w}) := \inf\{\|c - z_0 \,|\, \mathfrak{w}\| : z_0 \in \mathcal{N}_\diamond\},$$
$$p(b \,|\, \mathfrak{l}_\infty) := \inf\{\|b - y_0 \,|\, \mathfrak{l}_\infty\| : y_0 \in \mathcal{N}\},$$

respectively.

LEMMA 4.3. *The operator $C_0 C_{\mathfrak{w},0}$ admits a (unique) decomposition, where $Q_\diamond$ and Q_0 are quotient maps, while $C_{\mathfrak{w},\diamond}$ and C_{00} are one-to-one:*

$$\begin{tikzcd}
\mathfrak{w}(\mathbb{N}_0) \arrow[rr, "C_{\mathfrak{w}}"] \arrow[d, "Q^{\mathfrak{w}}_{S_-}"'] & & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[rr, "C"] \arrow[d, "Q^{\mathfrak{l}_\infty}_{S_-}"'] & & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[d, "Q^{\mathfrak{l}_\infty}_{\mathfrak{c}_0}"] \\
\mathfrak{w}(\mathbb{N}_0)/\!\!/S_- \arrow[rr, "C_{\mathfrak{w},0}"'] \arrow[dr, "Q_\diamond"'] & & \mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_- \arrow[rr, "C_0"'] \arrow[dr, "Q_0"'] & & \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0) \\
& \mathfrak{w}(\mathbb{N}_0)/\mathcal{N}_\diamond \arrow[rr, "C_{\mathfrak{w},\diamond}"'] & & \mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}. \arrow[ur, "C_{00}"'] &
\end{tikzcd}$$

Proof. The following reasoning is based on the proof of Lemma 3.3.

If $z \in \mathfrak{w}(\mathbb{N}_0)$, then we infer from Lemmas 3.1 and 4.2 that

$$CC_{\mathfrak{w}}z - CS_-C_{\mathfrak{w}}z \in \mathfrak{c}_0(\mathbb{N}_0) \quad \text{and} \quad CS_-C_{\mathfrak{w}}z - CC_{\mathfrak{w}}S_-z \in \mathfrak{c}_0(\mathbb{N}_0).$$

Hence $CC_{\mathfrak{w}}z - CC_{\mathfrak{w}}S_-z \in \mathfrak{c}_0(\mathbb{N}_0)$, which means that $z - S_-z \in \mathcal{N}_\diamond$. Therefore

$$q(c\,|\,\mathfrak{w}) \le u_{S_-}(c\,|\,\mathfrak{w}) = \inf\{\|c - z + S_-z\,|\,\mathfrak{w}\| : z \in \mathfrak{w}(\mathbb{N}_0)\} \quad \text{for all } c \in \mathfrak{w}(\mathbb{N}_0).$$

This estimate ensures the existence of $Q_\diamond : \mathfrak{w}(\mathbb{N}_0)/\!\!/S_- \to \mathfrak{w}(\mathbb{N}_0)/\mathcal{N}_\diamond$.

It follows from $C_{\mathfrak{w}}(\mathcal{N}_\diamond) \subseteq \mathcal{N}$ that

$$p(C_{\mathfrak{w}}c\,|\,\mathfrak{l}_\infty) \le q(c\,|\,\mathfrak{w}) \quad \text{for all } c \in \mathfrak{w}(\mathbb{N}_0).$$

Thus the operator $C_{\mathfrak{w},\diamond} : \mathfrak{w}(\mathbb{N}_0)/\mathcal{N}_\diamond \to \mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N}$ is well-defined. ∎

PROPOSITION 4.4. $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$ *fails to be a closed subspace of* $\mathfrak{w}^*(\mathbb{N}_0)$.

Proof. Using the sequences $b^{\langle m\rangle} = (\beta_k^{\langle m\rangle})$ defined before Lemma 3.4, we let $c^{\langle m\rangle} = (\gamma_l^{\langle m\rangle}) := C_{\mathfrak{w}}^{-1} b^{\langle m\rangle}$. That is, $\gamma_l^{\langle m\rangle} = \beta_l^{\langle m\rangle} + l(\beta_l^{\langle m\rangle} - \beta_{l-1}^{\langle m\rangle})$ or, more precisely,

$$\gamma_l^{\langle m\rangle} := \begin{cases} m2^i + 1 & \text{if} \quad l = m2^i, \\ +1 & \text{if} \quad m2^i < l < (m+1)2^i, \\ -2(m+1)2^i - 1 & \text{if} \quad l = (m+1)2^i, \qquad i = 0,1,2,\ldots, \\ -1 & \text{if} \quad (m+1)2^i < l < (m+2)2^i, \\ (m+2)2^i & \text{if} \quad l = (m+2)2^i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from

$$\sum_{l=m2^i}^{(m+2)2^i} |\gamma_l^{\langle m\rangle}| = (4m+6)2^i$$

that the sequences $c^{\langle m\rangle}$ belong to $\mathfrak{w}(\mathbb{N}_0)$.

We know from Lemma 3.4 and the preceding proof that

$$q(c^{\langle m\rangle}\,|\,\mathfrak{w}) \ge p(C_{\mathfrak{w}}c^{\langle m\rangle}\,|\,\mathfrak{l}_\infty) = p(b^{\langle m\rangle}\,|\,\mathfrak{l}_\infty) = 1$$

and

$$s(CC_{\mathfrak{w}}c^{\langle m\rangle} \mid \mathfrak{l}_\infty) = s(Cb^{\langle m\rangle} \mid \mathfrak{l}_\infty) = \frac{1}{m+1}.$$

Therefore the one-to-one operator

$$C_{00}C_{\mathfrak{w},\diamond} : \mathfrak{w}(\mathbb{N}_0)/\mathcal{N}_\diamond \to \mathfrak{l}_\infty(\mathbb{N}_0)/\mathcal{N} \to \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)$$

is not an injection. Hence Banach's inverse mapping theorem tells us that $\mathcal{R}(C_0C_{\mathfrak{w},0}) = \mathcal{R}(C_{00}C_{\mathfrak{w},\diamond})$ cannot be closed. Therefore, by the closed range theorem, the same follows for $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0) = \mathcal{R}(C^*_{\mathfrak{w},0}C^*_0)$; see the remark after Lemma 4.1. ∎

As just shown, $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$ fails to be complete. Thus we pass to the closed hull $\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}$. Unfortunately, there remains an open question concerning the weakly* closed hull.

Problem 4.5. *Which of the relations*

$$\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} = \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*} \quad \textit{or} \quad \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} \subset \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$$

is true?

In the case of $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$, my knowledge is even more unsatisfactory.

Problem 4.6. *Does $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$ fail to be a closed subspace of $\mathfrak{w}^*(\mathbb{N}_0)$?*

Remark. By the closed graph theorem, it suffices to show that the range $\mathcal{R}(C_{\mathfrak{w},0})$ is not closed in $\mathfrak{l}_\infty(\mathbb{N}_0)/\!/S_-$ (see Lemma 4.1).

Problem 4.7. *Which of the relations*

$$\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)} = \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*} \quad \textit{or} \quad \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*}$$

is true?

Unfortunately, the open questions raised above force us to distinguish between $\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}$ and $\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*}$ as well as between $\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}$ and $\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$.

In what follows, we determine the size of the spaces

$$\begin{matrix} \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)} \\ \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*} \end{matrix} \subset \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) \subset \mathfrak{w}^*(\mathbb{N}_0),$$

and the size of their 'differences'; see also Section 5.

Proposition 4.8. $\mathrm{dense}(\mathfrak{w}^*(\mathbb{N}_0)) \le 2^{2^{\aleph_0}}$.

Proof. If $\mathbb{K}$ denotes the real or complex scalar field, then

$$|\mathfrak{w}(\mathbb{N}_0)| \le |\mathbb{K}|^{|\mathbb{N}_0|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\cdot\aleph_0} = 2^{\aleph_0}.$$

Thus

$$|\mathfrak{w}^*(\mathbb{N}_0)| \le |\mathbb{K}|^{|\mathfrak{w}(\mathbb{N}_0)|} \le (2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0\cdot 2^{\aleph_0}} = 2^{2^{\aleph_0}}. \ \blacksquare$$

We proceed to a counterpart of Lemma 3.7.

LEMMA 4.9. *If*

$$J_d\colon z = (\zeta_i) \mapsto c = (\gamma_l) := \sum_{i=0}^{\infty} 2^i \zeta_i e_{2^{i+1}},$$

then $(I - S_-)J_d$ *is an injection from* $\mathfrak{l}_\infty(\mathbb{N}_0)$ *into* $\mathfrak{w}(\mathbb{N}_0)$ *such that*

$$\tfrac{2}{3}\|z \,|\, \mathfrak{l}_\infty\| \le \|(I - S_-)J_d z \,|\, \mathfrak{w}\| \le 2\|z \,|\, \mathfrak{l}_\infty\|.$$

Proof. If $2^j \le k+1 < 2^{j+1}$, then it follows from

$$(\delta_l) := (I - S_-)J_d z = (0, -\zeta_0, +\zeta_0, -2\zeta_1, +2\zeta_1, 0, \dots, 0, -2^i\zeta_i, +2^i\zeta_i, 0, \dots)$$

that

$$\frac{1}{k+1}\sum_{l=0}^{k} |\delta_l| \le \frac{1}{2^j}\sum_{i=0}^{j-1} 2\cdot 2^i|\zeta_i| \le \frac{2(2^j-1)}{2^j}\|z \,|\, \mathfrak{l}_\infty\|.$$

Therefore

$$\|(I - S_-)J_d z \,|\, \mathfrak{w}\| = \sup_{0\le k<\infty} \frac{1}{k+1}\sum_{l=0}^{k} |\delta_l| \le 2\|z \,|\, \mathfrak{l}_\infty\|.$$

On the other hand, for $j \ge 1$,

$$\|(I - S_-)J_d z \,|\, \mathfrak{w}\| \ge \frac{1}{2^j+1}\sum_{l=0}^{2^j} |\delta_l| \ge \frac{1}{2^j+1}(|\delta_{2^j-1}| + |\delta_{2^j}|) \ge \frac{2^j}{2^j+1}|\zeta_{j-1}|.$$

Hence $\|(I - S_-)J_d z \,|\, \mathfrak{w}\| \ge \frac{2}{3}\|z \,|\, \mathfrak{l}_\infty\|$. ∎

Next, we establish an analogue of Proposition 3.8.

PROPOSITION 4.10. $\mathrm{dense}(\mathfrak{w}^*(\mathbb{N}_0)/\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)) \ge 2^{2^{\aleph_0}}$.

Proof. Since $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$ is the annihilator of $M := \overline{\mathcal{R}(I - S_-)}$, we have the identification

$$M^* \equiv \mathfrak{w}^*(\mathbb{N}_0)/M^\perp \equiv \mathfrak{w}^*(\mathbb{N}_0)/\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0).$$

By Lemma 4.9, we can regard $(I - S_-)J_d$ as an injection from $\mathfrak{l}_\infty(\mathbb{N}_0)$ into M. Then $J_d^*(I - S_-)^*$ becomes a surjection from M^* onto $\mathfrak{l}_\infty^*(\mathbb{N}_0)$. The required conclusion now follows from Lemma 1.2 and Proposition 1.3. ∎

Now we extend the basic construction described before Lemma 3.9. To this end, let $t^{[i]} = (\tau_l^{[i]}) := C_{\mathfrak{w}}^{-1}s^{[i]}$. That is, $\tau_l^{[i]} = \sigma_l^{[i]} + l(\sigma_l^{[i]} - \sigma_{l-1}^{[i]})$ or, more precisely,

$$\tau_l^{[i]} := \begin{cases} 0 & \text{if} \quad l < h_i, \\ h_i + 1 & \text{if} \quad l = h_i, \\ +1 & \text{if} \quad h_i < l < h_i + d_i, \\ -2h_i - 2d_i - 1 & \text{if} \quad l = h_i + d_i, \\ -1 & \text{if} \quad h_i + d_i < l < h_i + 2d_i, \\ h_i + 2d_i & \text{if} \quad l = h_i + 2d_i, \\ 0 & \text{if} \quad h_i + 2d_i < l. \end{cases}$$

Since $h_i + 2d_i < h_{i+1}$, the supports of the $t^{[i]}$'s are mutually disjoint.

LEMMA 4.11. *The rule*

$$J_t \colon z = (\zeta_i) \mapsto c = (\gamma_l) := \sum_{i=0}^{\infty} \zeta_i t^{[i]}$$

defines an injection from $\mathfrak{l}_\infty(\mathbb{N}_0)$ *into* $\mathfrak{w}(\mathbb{N}_0)$ *such that*

$$3\|z \,|\, \mathfrak{l}_\infty\| \le \|J_t z \,|\, \mathfrak{w}\| \le 11\|z \,|\, \mathfrak{l}_\infty\|.$$

Proof. Recall that $h_i = 2^{i+2}$ and $i + 1 \le d_i \le 2^i$. It follows from

$$\sum_{l=h_i}^{h_i+2d_i} |\tau_l^{[i]}| = 4h_i + 6d_i \le 22 \cdot 2^i$$

that

$$\sum_{l=0}^{h_j+2d_j} |\gamma_l| \le \sum_{i=0}^{j} |\zeta_i|(4h_i + 6d_i) \le 22\|z \,|\, \mathfrak{l}_\infty\| \sum_{i=0}^{j} 2^i \le 22 \cdot 2^{j+1}\|z \,|\, \mathfrak{l}_\infty\|.$$

If $k \ge h_0 = 4$, then there exists j such that $h_j \le k < h_{j+1}$. Hence

$$\frac{1}{k+1} \sum_{l=0}^{k} |\gamma_l| \le \frac{1}{h_j + 1} \sum_{l=0}^{h_j+2d_j} |\gamma_l| \le 11\|z \,|\, \mathfrak{l}_\infty\|.$$

Since the estimate above is trivial for $k \le 3$, we obtain

$$\|J_t z \,|\, \mathfrak{w}\| \le 11\|z \,|\, \mathfrak{l}_\infty\| \quad \text{for all } z \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

On the other hand, we infer from

$$\sum_{l=0}^{h_j+2d_j} |\gamma_l| \ge \sum_{l=h_j}^{h_j+2d_j} |\zeta_j|\,|\tau_l^{[j]}| = (4h_j + 6d_j)|\zeta_j|$$

that

$$\|J_t z \,|\, \mathfrak{w}\| \ge \frac{1}{h_j + 2d_j + 1} \sum_{l=0}^{h_j+2d_j} |\gamma_l| \ge \frac{4h_j + 6d_j}{h_j + 2d_j + 1}|\zeta_j| \ge 3|\zeta_j|.$$

Thus $\|J_t z \,|\, \mathfrak{w}\| \ge 3\|z \,|\, \mathfrak{l}_\infty\|$. ∎

LEMMA 4.12. $C_{\mathfrak{w}} J_t = J_s$.

Proof. The equation above follows from $C_{\mathfrak{w}} t^{[i]} = s^{[i]}$. ∎

Next, we transfer Lemma 3.10 from J_s to J_t.

LEMMA 4.13. *There exists a* (*unique*) *injection* $J_{t,0}$ *such that the diagram*

$$\begin{tikzcd}
\mathfrak{l}_\infty(\mathbb{N}_0) \arrow[r, "J_t"] \arrow[d, "Q_{\mathfrak{c}_0}^{\mathfrak{l}_\infty}"'] & \mathfrak{w}(\mathbb{N}_0) \arrow[d, "Q_{S_-}^{\mathfrak{w}}"] \\
\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0) \arrow[r, "J_{t,0}"] & \mathfrak{w}(\mathbb{N}_0)/\!\!/S_-
\end{tikzcd}$$

commutes.

Proof. Since, by [23, Lemma 9.17],

$$\begin{aligned}
u_{S_-}(J_t z \,|\, \mathfrak{w}) &= u_{S_-}(J_t(z-x) \,|\, \mathfrak{w}) + u_{S_-}(J_t x \,|\, \mathfrak{w}) \\
&\le \|J_t(z-x) \,|\, \mathfrak{w}\| \le 11\|z-x \,|\, \mathfrak{l}_\infty\|
\end{aligned}$$

for all sequences x with finite support, we get $u_{S_-}(J_t z \,|\, \mathfrak{w}) \le 11 s(z \,|\, \mathfrak{l}_\infty)$. Thus $J_{t,0}$ is well-defined.

Combining the diagram just obtained with that in Lemma 4.1 yields

$$C_{\mathfrak{w}} J_t = J_s$$

$$\begin{tikzcd}
\mathfrak{l}_\infty(\mathbb{N}_0) \arrow[r, "J_t"] \arrow[d, "Q_{\mathfrak{c}_0}^{\mathfrak{l}_\infty}"'] & \mathfrak{w}(\mathbb{N}_0) \arrow[r, "C_{\mathfrak{w}}"] \arrow[d, "Q_{S_-}^{\mathfrak{w}}"] & \mathfrak{l}_\infty(\mathbb{N}_0) \arrow[d, "Q_{S_-}^{\mathfrak{l}_\infty}"] \\
\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0) \arrow[r, "J_{t,0}"] & \mathfrak{w}(\mathbb{N}_0)/\!\!/S_- \arrow[r, "C_{\mathfrak{w},0}"] & \mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-.
\end{tikzcd}$$

$$C_{\mathfrak{w},0} J_{t,0} = J_{s,0}$$

We know from Lemma 3.10 that $J_{s,0}$ is an injection. So $J_{t,0}$ must be an injection as well. ∎

For later reference, we formulate a byproduct of the preceding proof.

LEMMA 4.14. $C_{\mathfrak{w},0} J_{t,0} = J_{s,0}$.

The following result is analogous to Proposition 3.12.

PROPOSITION 4.15. $\mathrm{dense}\big(\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}}$.

Proof. Specify the operators J_s and J_t by letting $d_i := 2^i$.

With every free ultrafilter $\mathcal{U}$ we associate the singular functional

$$\psi_{\mathcal{U}}(a) := \mathcal{U}\text{-}\lim_i \alpha_{h_i+d_i-1} \quad \text{for all } a \in \mathfrak{l}_\infty(\mathbb{N}_0),$$

which in turn generates the Connes–Dixmier functional $\kappa_{\mathcal{U}} := C_{\mathfrak{w}}^* C^* \psi_{\mathcal{U}}$. If $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$ and $a := CJ_s z := CC_{\mathfrak{w}} J_t z$, then it follows from $\alpha_{h_i+d_i-1} = \frac{1}{5}\zeta_i$ (Lemma 3.11) that

$$\kappa_{\mathcal{U}}(J_t z) = \psi_{\mathcal{U}}(CC_{\mathfrak{w}} J_t z) = \mathcal{U}\text{-}\lim_i \alpha_{h_i+d_i-1} = \tfrac{1}{5}\mathcal{U}\text{-}\lim_i \zeta_i.$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be different free ultrafilters. Then there exists a subset $\mathbb{S}$ such that $\mathbb{S} \in \mathcal{U}_1$ and $\complement\mathbb{S} \in \mathcal{U}_2$. Define $z = (\zeta_i)$ by $\zeta_i := +1$ if $i \in \mathbb{S}$ and $\zeta_i := -1$ if $i \in \complement\mathbb{S}$. We infer from

$$\kappa_{\mathcal{U}_1}(J_t z) - \kappa_{\mathcal{U}_2}(J_t z) = \tfrac{1}{5}\mathcal{U}_1\text{-}\lim_i \zeta_i - \tfrac{1}{5}\mathcal{U}_2\text{-}\lim \zeta_i = \tfrac{2}{5}$$

and from Lemma 4.11 that

$$\tfrac{2}{5} = |\kappa_{\mathcal{U}_1}(J_t z) - \kappa_{\mathcal{U}_2}(J_t z)| \le 11\|\kappa_{\mathcal{U}_1} - \kappa_{\mathcal{U}_2} \,|\, \mathfrak{w}^*\|.$$

So the $\kappa_{\mathcal{U}}$'s form a 2/55-separated subset of $\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}$. Since the set of all free ultrafilters on $\mathbb{N}_0$ has cardinality $2^{2^{\aleph_0}}$, the estimate $\mathrm{dense}\big(\overline{\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}}$ follows from Lemma 1.1. ■

Next, we establish an analogue of Proposition 3.13.

Proposition 4.16.

$$\mathrm{dense}\big(\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}}, \quad \mathrm{dense}\big(\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*}/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}\big) \ge 2^{2^{\aleph_0}}.$$

Proof. Specify the operators J_s and J_t by letting $d_i := i+1$.

Using the identifications

$$[\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-]^* \equiv \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0), \quad [\mathfrak{w}(\mathbb{N}_0)/\!\!/S_-]^* \equiv \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0),$$

and

$$[\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)]^* \equiv \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0),$$

we may regard $J_{s,0}^*$, $J_{t,0}^*$, and $C_{\mathfrak{w},0}^*$ as restrictions of J_s^*, J_t^*, and $C_{\mathfrak{w}}^*$, respectively. Hence, by Lemmas 3.10 and 4.14, the surjection J_s^* induces a surjection from $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ onto $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ via $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$:

$$J_{s,0}^*\colon \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \xrightarrow{C_{\mathfrak{w},0}^*} \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) \xrightarrow{J_{t,0}^*} \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0).$$

By Lemma 1.1 and Proposition 1.3, there exists a 2-separated subset A of $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ with $|A| = \mathrm{dense}(\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)) = 2^{2^{\aleph_0}}$. So, for every $\varphi \in A$, we may choose a $\lambda \in \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ such that $\varphi = J_t^* C_{\mathfrak{w}}^* \lambda$. The functionals $\mu := C_{\mathfrak{w}}^*\lambda$ form a subset of $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$, denoted by B.

If $\nu \in \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$, then there exists a net $(\psi_\iota)_{\iota\in\mathbb{I}}$ in $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ such that $(C_{\mathfrak{w}}^* C^* \psi_\iota)_{\iota\in\mathbb{I}}$ converges to ν in the weak* topology of $\mathfrak{w}^*(\mathbb{N}_0)$. Since Lemmas 3.11 and 4.12 imply $CC_{\mathfrak{w}}J_t z = CJ_s z \in \mathfrak{c}_0(\mathbb{N}_0)$ for $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$, we get

$$\nu(J_t z) = \lim_{\iota\in\mathbb{I}} C_{\mathfrak{w}}^* C^* \psi_\iota(J_t z) = \lim_{\iota\in\mathbb{I}} \psi_\iota(CC_{\mathfrak{w}}J_t z) = 0.$$

Given different members $\varphi_1 = J_t^* C_{\mathfrak{w}}^*\lambda_1$ and $\varphi_2 = J_t^* C_{\mathfrak{w}}^*\lambda_2$ of A, we let $\mu_1 := C_{\mathfrak{w}}^*\lambda_1$ and $\mu_2 := C_{\mathfrak{w}}^*\lambda_2$. It follows from Lemma 4.11 that

$$\|\varphi_1 - \varphi_2 - J_t^*\nu \,|\, \mathfrak{l}_\infty^*\| = \|J_t^*(\mu_1 - \mu_2 - \nu) \,|\, \mathfrak{l}_\infty^*\| \le 11\|\mu_1 - \mu_2 - \nu \,|\, \mathfrak{w}^*\|.$$

Next, we take $z \in \mathfrak{l}_\infty(\mathbb{N}_0)$ such that

$$|\varphi_1(z) - \varphi_2(z)| \ge \tfrac{1}{2}\|\varphi_1 - \varphi_2 \,|\, \mathfrak{l}_\infty^*\| \ge 1 \quad \text{and} \quad \|z \,|\, \mathfrak{l}_\infty\| = 1.$$

Hence

$$\begin{aligned}11\|\mu_1-\mu_2-\nu\,|\,\mathfrak{w}^*\| &\ge \|\varphi_1-\varphi_2-J_t^*\nu\,|\,\mathfrak{l}_\infty^*\| \ge |\varphi_1(z)-\varphi_2(z)-J_t^*\nu(z)| \\ &= |\varphi_1(z)-\varphi_2(z)-\nu(J_tz)| = |\varphi_1(z)-\varphi_2(z)| \ge 1.\end{aligned}$$

Since B is contained in $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$, the canonical image of B is $1/11$-separated in $\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}/\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}$ and, all the more, in $\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*}/\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$. Moreover, $|B|=|A|=2^{2^{\aleph_0}}$. ∎

REMARK. In my opinion, the preceding proof (though a little bit longer) is more transparent than the following argument:

Keep in mind that $J_{s,0}^*$, $J_{t,0}^*$, and $C_{\mathfrak{w},0}^*$ are restrictions of J_s^*, J_t^*, and $C_{\mathfrak{w}}^*$, respectively. As shown above,

$$J_{s,0}^*\colon \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0) \xrightarrow{C_{\mathfrak{w},0}^*} \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) \xrightarrow{J_{t,0}^*} \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$$

is a surjection. Since, by definition, $C_{\mathfrak{w}}^*$ maps $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ onto $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$, the restriction of $J_{t,0}^*$ to $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$ remains surjective. We also know that $J_{t,0}^*$ vanishes on $\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$. So $J_{t,0}^*$ induces surjections from $\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}/\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}$ and $\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*}/\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\mathrm{w}^*}$ onto $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$. The required conclusions now follow from Lemma 1.2 and Proposition 1.3.

LEMMA 4.17. *The rule*

$$J_a\colon z=(\zeta_i)\mapsto x=(\xi_p)=\sum_{i=1}^{\infty}\zeta_i\sum_{p\in\Delta_i}e_p,$$

where $\Delta_i:=\{p\in\mathbb{N}: 2^i\le p<2^{i+1}\}$, *defines a metric injection from* $\mathfrak{l}_\infty(\mathbb{N}_0)$ *into* $\mathfrak{l}_\infty(\mathbb{N})$.

Letting $l_p:=2p^2$ and $d_p:=p$ for $p=1,2,\ldots$, we consider the sequences $r^{[p]}=(\varrho_l^{[p]})$ given by

$$\varrho_l^{[p]}:=\begin{cases}0 & \text{if} \quad l<l_p,\\ +1 & \text{if} \quad l_p\le l<l_p+d_p,\\ 0 & \text{if} \quad l_p+d_p\le l<l_p+2d_p,\\ -1 & \text{if} \quad l_p+2d_p\le l<l_p+3d_p,\\ 0 & \text{if} \quad l_p+3d_p\le l.\end{cases}$$

LEMMA 4.18. *The rule*

$$J_r\colon x=(\xi_p)\mapsto c=(\gamma_l):=\sum_{p=1}^{\infty}\xi_p r^{[p]}$$

defines an operator from $\mathfrak{l}_\infty(\mathbb{N})$ *into* $\mathfrak{w}(\mathbb{N}_0)$ *such that* $\|J_r\colon \mathfrak{l}_\infty\to\mathfrak{w}\|=1$. *Moreover,* $C_{\mathfrak{w}}J_rx\in\mathfrak{c}_0(\mathbb{N}_0)$.

Proof. Since $l_p + 3d_p < l_{p+1}$, the supports of the $r^{[p]}$'s are mutually disjoint. Thus

$$\sum_{p=1}^{\infty} |\varrho_l^{[p]}| \le 1 \quad \text{for } l = 0, 1, \ldots,$$

which implies

$$\frac{1}{k+1} \sum_{l=0}^{k} |\gamma_l| \le \frac{1}{k+1} \sum_{l=0}^{k} \sum_{p=1}^{\infty} |\xi_p|\,|\varrho_l^{[p]}| \le \|x \,|\, \mathfrak{l}_\infty\|.$$

Therefore $\|J_r x \,|\, \mathfrak{w}\| \le \|x \,|\, \mathfrak{l}_\infty\|$.

The non-negative sequence $C_{\mathfrak{w}} r^{[p]}$ has support $[l_p, l_p+3d_p-2]$ and attains its maximum at the index $l_p + d_p - 1$. Hence it follows from

$$\frac{1}{l_p+d_p} \sum_{l=0}^{l_p+d_p-1} \varrho_l^{[p]} = \frac{d_p}{l_p+d_p} = \frac{1}{2p+1}$$

that $C_{\mathfrak{w}} J_r x \in \mathfrak{c}_0(\mathbb{N}_0)$. ■

LEMMA 4.19. *There exist* (*unique*) *operators* $J_{a,0}$ *and* $J_{r,0}$ *for which the diagram*

$$\begin{array}{ccccc}
\mathfrak{l}_\infty(\mathbb{N}_0) & \xrightarrow{\quad J_a \quad} & \mathfrak{l}_\infty(\mathbb{N}) & \xrightarrow{\quad J_r \quad} & \mathfrak{w}(\mathbb{N}_0) \\
{\scriptstyle Q_{\mathfrak{c}_0}^{\mathfrak{l}_\infty}}\big\downarrow & & \big\downarrow{\scriptstyle Q_{\mathfrak{c}_0}^{\mathfrak{l}_\infty}} & & \big\downarrow{\scriptstyle Q_{S_-}^{\mathfrak{w}}} \\
\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0) & \xrightarrow{J_{a,0}} & \mathfrak{l}_\infty(\mathbb{N})/\mathfrak{c}_0(\mathbb{N}) & \xrightarrow{J_{r,0}} & \mathfrak{w}(\mathbb{N}_0)/\!\!/S_-
\end{array}$$

commutes. Moreover, $J_{r,0}J_{a,0}$ *is an injection such that*

$$\tfrac{1}{4} s(z \,|\, \mathfrak{l}_\infty) \le u_{S_-}(J_r J_a z \,|\, \mathfrak{w}) \le s(z \,|\, \mathfrak{l}_\infty).$$

Proof. The existence of $J_{a,0}$ is obvious.

Since, by [23, Lemma 9.17] and Lemma 4.18,

$$\begin{aligned} u_{S_-}(J_r x \,|\, \mathfrak{w}) &\le u_{S_-}(J_r(x-x_0) \,|\, \mathfrak{w}) + u_{S_-}(J_r x_0 \,|\, \mathfrak{w}) \\ &\le \|J_r(x-x_0) \,|\, \mathfrak{w}\| \le \|x - x_0 \,|\, \mathfrak{l}_\infty\| \end{aligned}$$

for all sequences x_0 with finite support, we get $u_{S_-}(J_r x \,|\, \mathfrak{w}) \le s(x \,|\, \mathfrak{l}_\infty)$. Thus $J_{r,0}$ is well-defined and

$$u_{S_-}(J_r J_a z \,|\, \mathfrak{w}) \le s(J_a z \,|\, \mathfrak{l}_\infty) = s(z \,|\, \mathfrak{l}_\infty).$$

Let

$$c = (\gamma_l) := J_r J_a z = \sum_{i=0}^{\infty} \zeta_i \sum_{p \in \Delta_i} r^{[p]}.$$

To obtain a lower estimate of

$$u_{S_+}(J_r J_a z \,|\, \mathfrak{w}) = u_{S_+}(c \,|\, \mathfrak{w}) = \inf_{1 \le n < \infty} \bigg\| \frac{1}{n} \sum_{k=0}^{n-1} S_+^k c \,\bigg|\, \mathfrak{w} \bigg\|$$

we define

$$A_n c = (\gamma_{l,n}) := \frac{1}{n}\sum_{k=0}^{n-1} S_+^k c = \Big(\frac{1}{n}\sum_{k=0}^{n-1}\gamma_{l-k}\Big),$$

with the understanding that $\gamma_{l-k} := 0$ whenever $l-k<0$. Assuming that $2^i \ge n$, we consider the finite sets

$$\mathbb{L}_{i,n}^+ := \bigcup_{p\in\Delta_i}\{l\in\mathbb{N}_0 : l_p+n-1\le l<l_p+d_p\}$$

and

$$\mathbb{L}_{i,n}^- := \bigcup_{p\in\Delta_i}\{l\in\mathbb{N}_0 : l_p+2d_p+n-1\le l<l_p+3d_p\}.$$

Then

$$\gamma_{l,n}=\pm\zeta_i \quad \text{for all } l\in\mathbb{L}_{i,n}^\pm \quad\text{and}\quad |\mathbb{L}_{i,n}^\pm| = \sum_{p\in\Delta_i}(d_p-n+1)\ge 2^i(2^i-n+1).$$

Moreover, we have

$$l_p+3d_p<l_{p+1}\le l_{2^{i+1}} \qquad \text{whenever } p\in\Delta_i.$$

Hence

$$\begin{aligned}\|A_n c\,|\,\mathfrak{w}\| &\ge \frac{1}{l_{2^{i+1}}}\sum_{l=0}^{l_{2^{i+1}-1}}|\gamma_{l,n}| \ge \frac{1}{2\cdot(2^{i+1})^2}(|\mathbb{L}_{i,n}^+|+|\mathbb{L}_{i,n}^-|)|\zeta_i|\\ &\ge \frac{2^i-n+1}{2^{i+2}}|\zeta_i|.\end{aligned}$$

Passing to the limit as $i\to\infty$ yields

$$\|A_n c\,|\,\mathfrak{w}\| \ge \limsup_{i\to\infty}\tfrac{1}{4}|\zeta_i|.$$

Therefore, by [23, Prop. 9.16],

$$u_{S_-}(J_rJ_az\,|\,\mathfrak{w}) = u_{S_+}(J_rJ_az\,|\,\mathfrak{w}) = \inf_{1\le n<\infty}\|A_n c\,|\,\mathfrak{w}\| \ge \tfrac{1}{4}s(z\,|\,\mathfrak{l}_\infty). \ \blacksquare$$

Proposition 4.20. $\operatorname{dense}\big(\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*}\big)\ge 2^{2^{\aleph_0}}$.

Proof. Lemmas 4.18 and 4.19 tell us that $J_{r,0}J_{a,0}$ is an injection whose range is contained in the null space $\mathcal{N}(C_{\mathfrak{w},0})$ of $C_{\mathfrak{w},0}$. Hence it induces an injection

$$J\colon \mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)\to\mathcal{N}(C_{\mathfrak{w},0}).$$

In view of $[\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)]^*\equiv\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ and

$$\begin{aligned}\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\mathrm{w}^*} &\equiv (\mathfrak{w}(\mathbb{N}_0)/\!\!/S_-)^*/\overline{\mathcal{R}(C_{\mathfrak{w},0}^*)}^{\mathrm{w}^*}\\ &\equiv (\mathfrak{w}(\mathbb{N}_0)/\!\!/S_-)^*/\mathcal{N}(C_{\mathfrak{w},0})^\perp\equiv\mathcal{N}(C_{\mathfrak{w},0})^*,\end{aligned}$$

the dual operator

$$J^*\colon \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \to \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$$

is a surjection. Therefore, by Lemma 1.2 and Proposition 1.3,

$$\mathrm{dense}\big(\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*}\big) \ge \mathrm{dense}(\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)) \ge 2^{2^{\aleph_0}}. \;\blacksquare$$

THEOREM 4.21. *All of the Banach spaces*

$$\left.\begin{array}{c} \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)} \\ \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \end{array}\right. \subset \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) \subset \mathfrak{w}^*(\mathbb{N}_0),$$

$$\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} \subset \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} \subset \mathfrak{w}^*(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)},$$

$$\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*}/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \subset \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \subset \mathfrak{w}^*(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*},$$

$$\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)} \subset \mathfrak{w}^*(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)},$$

$$\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \subset \mathfrak{w}^*(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*},$$

and

$$\mathfrak{w}^*(\mathbb{N}_0)/\,\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$$

have the same density character, namely $2^{2^{\aleph_0}}$.

Proof. The upper estimates follow from

$$\mathrm{dense}(\mathfrak{w}^*(\mathbb{N}_0)) \le 2^{2^{\aleph_0}} \qquad \text{(Proposition 4.8)},$$

while the lower estimates are implied by

$$\begin{array}{ll} \mathrm{dense}\big(\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}} & \text{(Proposition 4.15)}, \\ \mathrm{dense}\big(\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}\big) \ge 2^{2^{\aleph_0}} & \text{(Proposition 4.16)}, \\ \mathrm{dense}\big(\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*}/\,\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*}\big) \ge 2^{2^{\aleph_0}} & \text{(Proposition 4.16)}, \\ \mathrm{dense}\big(\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)/\,\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*}\big) \ge 2^{2^{\aleph_0}} & \text{(Proposition 4.20)}, \\ \mathrm{dense}(\mathfrak{w}^*(\mathbb{N}_0)/\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)) \ge 2^{2^{\aleph_0}} & \text{(Proposition 4.10)}. \;\blacksquare \end{array}$$

5. Medium-sized subspaces of a Banach space. A closed subspace N of a Banach space X has precisely one of the following properties:

- N is *large*:

$$\mathrm{dense}(N) = \mathrm{dense}(X) \quad \text{and} \quad \mathrm{dense}(X/N) < \mathrm{dense}(X).$$

- N is *small*:

$$\mathrm{dense}(N) < \mathrm{dense}(X) \quad \text{and} \quad \mathrm{dense}(X/N) = \mathrm{dense}(X).$$

- N is *medium-sized*:

$$\mathrm{dense}(N) = \mathrm{dense}(X) \quad \text{and} \quad \mathrm{dense}(X/N) = \mathrm{dense}(X).$$

The fourth property, namely

$$\mathrm{dense}(N) < \mathrm{dense}(X) \quad \text{and} \quad \mathrm{dense}(X/N) < \mathrm{dense}(X),$$

cannot occur because $\mathrm{dense}(X) \le \mathrm{dense}(N) \cdot \mathrm{dense}(X/N)$.

For example, it follows from $\mathfrak{l}_\infty^*(\mathbb{N}_0) = \mathfrak{l}_1^{**}(\mathbb{N}_0) = \mathfrak{l}_1(\mathbb{N}_0) \oplus \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$, Proposition 1.3, Theorem 3.14, and $\mathrm{dense}(\mathfrak{l}_1(\mathbb{N}_0)) = \aleph_0$ that $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ is a large, $\mathfrak{l}_1(\mathbb{N}_0)$ is a small, and $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$ is a medium-sized subspace of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$.

Using the terminology above, we state an immediate consequence of Theorem 4.21, which summarizes the main results of this paper.

THEOREM 5.1. *In the pairs*

$$\overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}, \qquad \overline{\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \subset \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*},$$
$$\overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)} \subset \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0), \qquad \overline{\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)}^{\,\mathrm{w}^*} \subset \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0),$$

and

$$\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) \subset \mathfrak{w}^*(\mathbb{N}_0)$$

the left-hand members are medium-sized subspaces of the right-hand members.

6. Positive shift-invariant functionals on $\mathfrak{l}_\infty(\mathbb{N}_0)$. In the rest of this paper, we restrict our considerations to the real case.

Since $\mathfrak{c}_0(\mathbb{N}_0)$ is a closed ideal of the Banach lattice $\mathfrak{l}_\infty(\mathbb{N}_0)$, the quotient $\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0)$ becomes a Banach lattice as well [28, p. 85]. Its norm is induced by the seminorm

$$s(a) := \limsup_{h\to\infty} |\alpha_h|.$$

We know that $\mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$, the space of singular functionals, can be identified with the topological dual $(\mathfrak{l}_\infty(\mathbb{N}_0)/\mathfrak{c}_0(\mathbb{N}_0))^*$. Therefore it is a weakly* closed linear sublattice of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$.

Similarly, $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$, the space of shift-invariant functionals, coincides with $(\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-)^*$. Unfortunately, I do not know whether $\mathfrak{l}_\infty(\mathbb{N}_0)/\!\!/S_-$ becomes a lattice under its canonical ordering. Therefore we use another (and even more direct) argument to show that $\mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0)$ is a linear sublattice of $\mathfrak{l}_\infty^*(\mathbb{N}_0)$.

Recall from the theory of linear lattices that the positive part λ_+ of a functional $\lambda \in \mathfrak{l}_\infty^*(\mathbb{N}_0)$ is the linear extension of

$$\lambda_+(a) := \sup\{\lambda(a_0) : a \ge a_0 \ge \mathsf{o}\} \quad \text{for all } a \in \mathfrak{l}_\infty(\mathbb{N}_0) \text{ with } a \ge \mathsf{o}.$$

Since

$$\mu \vee \nu = (\mu - \nu)_+ + \nu \quad \text{whenever } \mu, \nu \in \mathfrak{l}_\infty^*(\mathbb{N}_0),$$

it suffices to prove the following result.

PROPOSITION 6.1. *If $\lambda \in \mathfrak{l}_\infty^*(\mathbb{N}_0)$ is shift-invariant, then so is λ_+.*

Proof. Since $a \geq a_0 \geq \mathsf{o}$ implies $S_\pm a \geq S_\pm a_0 \geq \mathsf{o}$, we have

$$\begin{aligned}\lambda_+(a) &= \sup\{\lambda(a_0) : a \geq a_0 \geq \mathsf{o}\}\\ &\leq \sup\{\lambda(b) : S_\pm a \geq S_\pm b \geq \mathsf{o}\}\\ &= \sup\{\lambda(S_\pm b) : S_\pm a \geq S_\pm b \geq \mathsf{o}\}\\ &\leq \sup\{\lambda(c_\pm) : S_\pm a \geq c_\pm \geq \mathsf{o}\} = \lambda_+(S_\pm a).\end{aligned}$$

It follows from

$$\lambda_+(a) \leq \lambda_+(S_+ a) \leq \lambda_+(S_- S_+ a) = \lambda_+(a)$$

that λ_+ is S_+-invariant, and therefore shift-invariant. ■

Note that the cone $\mathfrak{l}_{\infty,+}^{\mathrm{sif}}(\mathbb{N}_0) := \{\lambda \in \mathfrak{l}_{\infty}^{\mathrm{sif}}(\mathbb{N}_0) : \lambda \geq \mathsf{o}\}$ is weakly* closed.

The situation is unclear for $\mathfrak{l}_{\infty}^{\mathrm{mf}}(\mathbb{N}_0)$, the space of Mazur functionals. Of course, $\mathfrak{l}_{\infty}^{*}(\mathbb{N}_0)$ induces a partial ordering on $\mathfrak{l}_{\infty}^{\mathrm{mf}}(\mathbb{N}_0)$ and we may consider the cone

$$\mathfrak{l}_{\infty,+}^{\mathrm{mf}}(\mathbb{N}_0) := \{C^*\psi : \psi \in \mathfrak{l}_{\infty}^{\mathrm{sgf}}(\mathbb{N}_0),\, C^*\psi \geq \mathsf{o}\}$$

formed by the *positive* Mazur functionals. However, I doubt that $\mathfrak{l}_{\infty}^{\mathrm{mf}}(\mathbb{N}_0)$ is a linear sublattice of $\mathfrak{l}_{\infty}^{*}(\mathbb{N}_0)$.

Moreover, letting

$$\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0) := \{C^*\psi : \psi \in \mathfrak{l}_{\infty}^{\mathrm{sgf}}(\mathbb{N}_0),\, \psi \geq \mathsf{o}\}$$

yields another natural cone, whose members are referred to as *strictly positive* Mazur functionals. Obviously, every strictly positive Mazur functional is positive, which means that

$$\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0) \subseteq \mathfrak{l}_{\infty,+}^{\mathrm{mf}}(\mathbb{N}_0).$$

To show that the preceding inclusion is proper, we need some preparation.

LEMMA 6.2. *There exists a sequence $b_\heartsuit \in \mathfrak{l}_\infty(\mathbb{N}_0)$ such that*

$$Cb_\heartsuit \geq \mathsf{o} \quad \textit{and} \quad s(Cb_\heartsuit - Cy \,|\, \mathfrak{l}_\infty) \geq 1 \quad \textit{for all positive } y \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

Proof. Let $h_i := 2 \cdot 2^i - 2$ and $k_i := 3 \cdot 2^i - 2$. Then $k_i - h_i = 2^i$ and $h_{i+1} - k_i = 2^i$. Define $b_\heartsuit = (\beta_k)$ by

$$\beta_k := \begin{cases} +8 & \text{if } h_i \leq k < k_i,\\ -8 & \text{if } k_i \leq k < h_{i+1}.\end{cases} \qquad i = 0, 1, 2, \ldots,$$

Then all terms α_h of $a_\heartsuit := Cb_\heartsuit$ are non-negative. In particular,

$$\alpha_{k_i - 1} = 8 \cdot 2^i / k_i \quad \text{and} \quad \alpha_{h_{i+1}-1} = 0.$$

Assuming that $s(a_\heartsuit - Cy \,|\, \mathfrak{l}_\infty) < 1$, we have

$$\frac{8 \cdot 2^i}{k_i} - \frac{1}{k_i} \sum_{k=0}^{k_i - 1} \eta_k \leq 1 \quad \text{and} \quad \frac{1}{h_{i+1}} \sum_{k=0}^{h_{i+1}-1} \eta_k \leq 1$$

for all i sufficiently large. In view of $\eta_k \geq 0$, it follows that

$$8 \cdot 2^i - k_i \leq \sum_{k=0}^{k_i-1} \eta_k \leq \sum_{k=0}^{h_{i+1}-1} \eta_k \leq h_{i+1}.$$

Hence

$$8 \cdot 2^i \leq h_{i+1} + k_i = 7 \cdot 2^i - 4.$$

Dividing by 2^i and letting $i \to \infty$ yields a contradiction. ∎

Now we are prepared to verify the proper inclusion

$$\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0) \subset \mathfrak{l}_{\infty,+}^{\mathrm{mf}}(\mathbb{N}_0).$$

THEOREM 6.3. *There exists a positive Mazur functional $\lambda_\heartsuit$ on $\mathfrak{l}_\infty(\mathbb{N}_0)$ that fails to be strictly positive.*

Proof. Define a sublinear functional on $\mathfrak{l}_\infty(\mathbb{N}_0)$ by letting

$$r(a \,|\, \mathfrak{l}_\infty) := \inf\{s(a - Cy \,|\, \mathfrak{l}_\infty) : y \in \mathfrak{l}_\infty(\mathbb{N}_0),\ y \geq \mathsf{o}\}.$$

Now we use the positive sequence $a_\heartsuit := Cb_\heartsuit$ constructed in the proof of the preceding lemma. Since $r(a_\heartsuit \,|\, \mathfrak{l}_\infty) \geq 1$, we have

$$\varphi(\xi a_\heartsuit) := \xi \leq r(\xi a_\heartsuit \,|\, \mathfrak{l}_\infty) \quad \text{for all } \xi \in \mathbb{R}.$$

The Hahn–Banach theorem yields an extension $\varphi_\heartsuit$ such that

$$\varphi_\heartsuit(a) \leq r(a \,|\, \mathfrak{l}_\infty) \leq s(a \,|\, \mathfrak{l}_\infty) \quad \text{for all } a \in \mathfrak{l}_\infty(\mathbb{N}_0).$$

Then it follows from

$$C^*\varphi_\heartsuit(b) = \varphi_\heartsuit(Cb) \leq r(Cb \,|\, \mathfrak{l}_\infty) = 0 \quad \text{if } b \geq \mathsf{o}$$

that the Mazur functional $\lambda_\heartsuit := -C^*\varphi_\heartsuit$ is positive. On the other hand, the existence of a representation $\lambda_\heartsuit = C^*\psi$ with some positive functional $\psi \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ would lead to a contradiction:

$$-1 = -\varphi_\heartsuit(a_\heartsuit) = -\varphi_\heartsuit(Cb_\heartsuit) = \lambda_\heartsuit(b_\heartsuit) = \psi(a_\heartsuit) \geq 0. \ ∎$$

Compared with Proposition 3.5, the following result looks surprising.

PROPOSITION 6.4. *The cone $\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0)$ is weakly* closed in $\mathfrak{l}_\infty^*(\mathbb{N}_0)$.*

Proof. Let $B(\mathfrak{l}_{\infty,++}^{\mathrm{mf}})$ and $B(\mathfrak{l}_{\infty,+}^{\mathrm{sgf}})$ consist of all functionals in $\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0)$ and $\mathfrak{l}_{\infty,+}^{\mathrm{sgf}}(\mathbb{N}_0) := \{\psi \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0) : \psi \geq \mathsf{o}\}$, respectively, whose norms are less than or equal to 1.

Given $\lambda \in B(\mathfrak{l}_{\infty,++}^{\mathrm{mf}})$, we choose $\psi \in \mathfrak{l}_{\infty,+}^{\mathrm{sgf}}(\mathbb{N}_0)$ such that $\lambda = C^*\psi$. Since

$$\|\psi \,|\, \mathfrak{l}_\infty^*\| = \psi(e) = \psi(Ce) = \lambda(e) \leq 1$$

implies $\psi \in B(\mathfrak{l}_{\infty,+}^{\mathrm{sgf}})$, we see that $B(\mathfrak{l}_{\infty,++}^{\mathrm{mf}})$ is the weakly* continuous image of the weakly* compact set $B(\mathfrak{l}_{\infty,+}^{\mathrm{sgf}})$ (Bourbaki–Alaoglu theorem). The required conclusion now follows by applying the Kreĭn–Šmulian theorem (see [16, p. 242] or [27, p. 152]). ∎

Unfortunately, I have no idea whether the preceding proposition remains true if $\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0)$ is replaced by $\mathfrak{l}_{\infty,+}^{\mathrm{mf}}(\mathbb{N}_0)$.

PROBLEM 6.5. *Does the cone $\mathfrak{l}_{\infty,+}^{\mathrm{mf}}(\mathbb{N}_0)$ fail to be closed in $\mathfrak{l}_\infty^*(\mathbb{N}_0)$?*

Finally, I stress that both cones $\mathfrak{l}_{\infty,+}^{\mathrm{mf}}(\mathbb{N}_0)$ and $\mathfrak{l}_{\infty,++}^{\mathrm{mf}}(\mathbb{N}_0)$ generate $\mathfrak{l}_\infty^{\mathrm{mf}}(\mathbb{N}_0)$.

7. Positive shift-invariant functionals on $\mathfrak{w}(\mathbb{N}_0)$. We begin with an analogue of Proposition 6.1, whose proof can be adopted word for word.

PROPOSITION 7.1. *If $\mu \in \mathfrak{w}^*(\mathbb{N}_0)$ is shift-invariant, then so is μ_+.*

As a consequence, we observe that $\mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0)$ is a linear sublattice of $\mathfrak{w}^*(\mathbb{N}_0)$. Note that the cone $\mathfrak{w}_+^{\mathrm{sif}}(\mathbb{N}_0) := \{\mu \in \mathfrak{w}^{\mathrm{sif}}(\mathbb{N}_0) : \mu \geq \mathsf{o}\}$ is weakly* closed.

The situation remains unclear for $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$ and $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$. Of course, $\mathfrak{w}^*(\mathbb{N}_0)$ induces partial orderings on both spaces. So we may consider the cones

$$\mathfrak{w}_+^{\mathrm{df}}(\mathbb{N}_0) := \{C_{\mathfrak{w}}^*\lambda : \lambda \in \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0),\ C_{\mathfrak{w}}^*\lambda \geq \mathsf{o}\}$$

and

$$\mathfrak{w}_+^{\mathrm{cdf}}(\mathbb{N}_0) := \{C^*C_{\mathfrak{w}}\psi : \psi \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0),\ C_{\mathfrak{w}}^*C^*\psi \geq \mathsf{o}\},$$

formed by the *positive* Dixmier and *positive* Connes–Dixmier functionals, respectively. However, I doubt that $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$ and $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$ are linear sublattices of $\mathfrak{w}^*(\mathbb{N}_0)$,

Moreover, letting

$$\mathfrak{w}_{++}^{\mathrm{df}}(\mathbb{N}_0) := \{C_{\mathfrak{w}}^*\lambda : \lambda \in \mathfrak{l}_\infty^{\mathrm{sif}}(\mathbb{N}_0),\ \lambda \geq \mathsf{o}\}$$

and

$$\mathfrak{w}_{++}^{\mathrm{cdf}}(\mathbb{N}_0) := \{C^*C_{\mathfrak{w}}\psi : \psi \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0),\ \psi \geq \mathsf{o}\}.$$

yields other natural cones. The members of $\mathfrak{w}_{++}^{\mathrm{df}}(\mathbb{N}_0)$ and $\mathfrak{w}_{++}^{\mathrm{cdf}}(\mathbb{N}_0)$ are called *strictly positive* Dixmier and *strictly positive* Connes–Dixmier functionals, respectively. Obviously, we have

$$\mathfrak{w}_{++}^{\mathrm{df}}(\mathbb{N}_0) \subseteq \mathfrak{w}_+^{\mathrm{df}}(\mathbb{N}_0) \quad \text{and} \quad \mathfrak{w}_{++}^{\mathrm{cdf}}(\mathbb{N}_0) \subseteq \mathfrak{w}_+^{\mathrm{cdf}}(\mathbb{N}_0).$$

A result of Kalton–Sukochev [11, p. 75] shows that the left-hand inclusion is proper; see also [23, Prop. 9.31].

THEOREM 7.2. *There exists a positive Dixmier functional on $\mathfrak{w}(\mathbb{N}_0)$ that fails to be strictly positive.*

The right-hand inclusion $\mathfrak{w}_{++}^{\mathrm{cdf}}(\mathbb{N}_0) \subseteq \mathfrak{w}_+^{\mathrm{cdf}}(\mathbb{N}_0)$ is proper as well. This can be checked by continuing the proof of Theorem 6.3.

THEOREM 7.3. *There exists a positive Connes–Dixmier functional on $\mathfrak{w}(\mathbb{N}_0)$ that fails to be strictly positive.*

Proof. Obviously, $\mu_\heartsuit := C_\mathfrak{w}^* \lambda_\heartsuit = -C_\mathfrak{w}^* C^* \varphi_\heartsuit$ is a positive Connes–Dixmier functional.

Use the positive sequence $a_\heartsuit := Cb_\heartsuit$ constructed in the proof of Lemma 6.2 and let $c_\heartsuit = (\gamma_l) := C_\mathfrak{w}^{-1} b_\heartsuit$. Since

$$\gamma_l := \begin{cases} +16h_i + 8 & \text{if} \quad l = h_i, \\ +8 & \text{if} \; h_i < l < k_i, \\ -16k_i - 8 & \text{if} \quad l = k_i, \\ -8 & \text{if} \; k_i < l < h_{i+1}, \end{cases} \qquad i = 0, 1, 2, \dots,$$

we get $c_\heartsuit \in \mathfrak{w}(\mathbb{N}_0)$. Then the existence of a representation $\mu_\heartsuit = C_\mathfrak{w}^* C^* \psi$ with some positive $\psi \in \mathfrak{l}_\infty^{\mathrm{sgf}}(\mathbb{N}_0)$ leads to a contradiction:

$$-1 = -\varphi_\heartsuit(a_\heartsuit) = -\varphi_\heartsuit(CC_\mathfrak{w} c_\heartsuit) = \mu_\heartsuit(c_\heartsuit) = \psi(a_\heartsuit) \geq 0. \; \blacksquare$$

The next result can be obtained by a slight modification of the proof of Proposition 6.4.

PROPOSITION 7.4. *The cones* $\mathfrak{w}_{++}^{\mathrm{df}}(\mathbb{N}_0)$ *and* $\mathfrak{w}_{++}^{\mathrm{cdf}}(\mathbb{N}_0)$ *are weakly* closed in* $\mathfrak{w}^*(\mathbb{N}_0)$.

Unfortunately, I have no idea whether the preceding proposition remains true for $\mathfrak{w}_{+}^{\mathrm{df}}(\mathbb{N}_0)$ and $\mathfrak{w}_{+}^{\mathrm{cdf}}(\mathbb{N}_0)$.

PROBLEM 7.5. *Do the cones* $\mathfrak{w}_{+}^{\mathrm{df}}(\mathbb{N}_0)$ *and* $\mathfrak{w}_{+}^{\mathrm{cdf}}(\mathbb{N}_0)$ *fail to be closed in* $\mathfrak{w}^*(\mathbb{N}_0)$*?*

Finally, I stress that both cones $\mathfrak{w}_{+}^{\mathrm{df}}(\mathbb{N}_0)$ and $\mathfrak{w}_{++}^{\mathrm{df}}(\mathbb{N}_0)$ generate $\mathfrak{w}^{\mathrm{df}}(\mathbb{N}_0)$. Similarly, $\mathfrak{w}_{+}^{\mathrm{cdf}}(\mathbb{N}_0)$ and $\mathfrak{w}_{++}^{\mathrm{cdf}}(\mathbb{N}_0)$ generate $\mathfrak{w}^{\mathrm{cdf}}(\mathbb{N}_0)$.

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