STUDIA MATHEMATICA 221 (2) (2014)

Diagonals of projective tensor products and orthogonally additive polynomials

by

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Abstract. Let E be a Banach space with 1-unconditional basis. Denote by $\Delta(\hat{\otimes}_{n,\pi} E)$ (resp. $\Delta(\hat{\otimes}_{n,s,\pi} E)$) the main diagonal space of the n-fold full (resp. symmetric) projective Banach space tensor product, and denote by $\Delta(\hat{\otimes}_{n,|\pi|} E)$ (resp. $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$) the main diagonal space of the n-fold full (resp. symmetric) projective Banach lattice tensor product. We show that these four main diagonal spaces are pairwise isometrically isomorphic, and in addition, that they are isometrically lattice isomorphic to $E_{[n]}$, the completion of the n-concavification of E. Using these isometries, we also show that the norm of any (vector valued) continuous orthogonally additive homogeneous polynomial on E equals the norm of its associated symmetric linear operator.

1. Introduction. For every continuous n-homogeneous polynomial P and its associated symmetric n-linear operator T_P we have the Polarization Inequalities: $\|P\| \le \|T_P\| \le (n^n/n!)\|P\|$. It is known that $\|T_P\| = \|P\|$ for every polynomial P with any Hilbert space as its domain and any Banach space as its range (see [6, Proposition 1.44] and [7]), while there is a polynomial P with ℓ_1 as its domain such that $\|T_P\| = (n^n/n!)\|P\|$ (see [6, Example 1.39] and [7]). It is of interest to find which n-homogeneous polynomials P satisfy $\|T_P\| = \|P\|$.

A homogeneous polynomial P on a vector lattice is called *orthogonally additive* if $P(x+y) = P(x) + P(y)$ whenever x and y are disjoint. In this paper, E will be a Banach space with a 1-unconditional basis (such E is a Banach lattice with coordinatewise order). We show that for any (vector valued) continuous orthogonally additive homogeneous polynomial P on E its associated symmetric linear operator T_P satisfies $\|T_P\| = \|P\|$. To obtain this result from the linearization of orthogonally additive n-homogeneous polynomials given by Benyamini, Lassalle, and Llavona in [2], it suffices to show that $\Delta(\hat{\otimes}_{n,\pi} E)$, the main diagonal space of the n-fold full projective Banach space tensor product, is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,s,\pi} E)$, the

2010 *Mathematics Subject Classification*: 46G25, 47H60, 46B28.
Key words and phrases: homogeneous polynomials, multilinear operators, tensor products.

DOI: 10.4064/sm221-2-1 © Instytut Matematyczny PAN, 2014

main diagonal space of the n-fold symmetric projective Banach space tensor product. It is this new result that we prove in this paper, and we emphasize that our proof does not depend on the degree of homogeneity. To get the announced isometry, we first show that $\Delta(\hat{\otimes}_{n,\pi} E)$ is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,|\pi|} E)$, the main diagonal space of the n-fold full projective Banach lattice tensor product. Secondly, we show that $\Delta(\hat{\otimes}_{n,s,\pi} E)$ is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$, the main diagonal space of the n-fold symmetric projective Banach lattice tensor product. Finally, by using Banach lattice structure, we show that $\Delta(\hat{\otimes}_{n,|\pi|} E)$ is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$, which, therefore, implies that $\Delta(\hat{\otimes}_{n,\pi} E)$ is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,s,\pi} E)$. As a consequence, we also show that each of these four main diagonal spaces is isometrically isomorphic to $E_{[n]}$, the completion of the n-concavification of E.

2. Preliminaries. For a Banach space X, let $\otimes_n X$ denote the *n-fold algebraic tensor product* of X. The *projective tensor norm* on $\otimes_n X$ is defined by

$$\|u\|_\pi = \inf\Big\{\sum_{k=1}^m \|x_{1,k}\|\cdots\|x_{n,k}\| : u = \sum_{k=1}^m x_{1,k}\otimes\cdots\otimes x_{n,k} \in \otimes_n X\Big\},\qquad u\in\otimes_n X.$$

Let $\hat{\otimes}_{n,\pi} X$ denote the completion of $(\otimes_n X, \|\cdot\|_\pi)$, called the *$n$-fold projective tensor product* of X. For $x_1\otimes\cdots\otimes x_n \in \otimes_n X$, let $x_1\otimes_s\cdots\otimes_s x_n$ denote its symmetrization, that is,

$$x_1\otimes_s\cdots\otimes_s x_n = \frac{1}{n!}\sum_{\sigma\in\pi(n)} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)},$$

where $\pi(n)$ is the group of permutations of $\{1,\dots,n\}$. We write $\otimes_{n,s} X$ for the *n-fold symmetric algebraic tensor product* of X, that is, the linear span of $\{x_1\otimes_s\cdots\otimes_s x_n : x_1,\dots,x_n\in X\}$ in $\otimes_n X$. Each $u\in\otimes_{n,s} X$ has a representation $u=\sum_{k=1}^m \lambda_k x_k\otimes_s\cdots\otimes_s x_k$ where $\lambda_1,\dots,\lambda_m$ are scalars and $x_1,\dots,x_m$ are vectors in X. The *symmetric projective tensor norm* on $\otimes_{n,s} X$ is defined by

$$\|u\|_{s,\pi} = \inf\Big\{\sum_{k=1}^m |\lambda_k|\cdot\|x_k\|^n : u=\sum_{k=1}^m \lambda_k x_k\otimes_s\cdots\otimes_s x_k \in \otimes_{n,s} X\Big\},\qquad u\in\otimes_{n,s} X.$$

Let $\hat{\otimes}_{n,s,\pi} X$ denote the completion of $(\otimes_{n,s} X, \|\cdot\|_{s,\pi})$, called the *$n$-fold symmetric projective tensor product* of X.

For the basic knowledge about (symmetric) projective tensor products, we refer to [6], [7], and [16].

For a Banach lattice E, let $\bar{\otimes}_n E$ denote the *n-fold vector lattice tensor* *product* of E. The *positive projective tensor norm* on $\bar{\otimes}_n E$ is defined by

$$\|u\|_{|\pi|} = \inf\Big\{\sum_{k=1}^m \|x_{1,k}\|\cdots\|x_{n,k}\| : x_{i,k}\in E^+,\ |u| \le \sum_{k=1}^m x_{1,k}\otimes\cdots\otimes x_{n,k}\Big\},\quad u\in\bar{\otimes}_n E,$$

where E^+ denotes the positive cone of E. Let $\hat{\otimes}_{n,|\pi|} E$ denote the completion of $(\bar{\otimes}_n E, \|\cdot\|_{|\pi|})$, which is a Banach lattice, called the *positive n-fold projective tensor product* of E. Let $\bar{\otimes}_{n,s} E$ denote the *n-fold symmetric vector lattice tensor product* of E. The *positive symmetric projective tensor norm* on $\bar{\otimes}_{n,s} E$ is defined by

$$\|u\|_{s,|\pi|} = \inf\Big\{\sum_{k=1}^m |\lambda_k|\cdot\|x_k\|^n : x_k\in E^+,\ |u| \le \sum_{k=1}^m |\lambda_k| x_k\otimes_s\cdots\otimes_s x_k\Big\},\quad u\in\bar{\otimes}_{n,s} E.$$

Let $\hat{\otimes}_{n,s,|\pi|} E$ denote the completion of $(\bar{\otimes}_{n,s} E, \|\cdot\|_{s,|\pi|})$, which is a Banach lattice, called the *positive n-fold symmetric projective tensor product* of E.

For the basic knowledge about (symmetric) vector lattice tensor products and positive (symmetric) projective tensor products, we refer to [8], [9] and [18] (also see [3]).

3. Diagonals of projective tensor products. In this section we assume that X is a Banach space with a 1-unconditional basis $\{e_i : i\in\mathbb{N}\}$. Gelbaum and Lamadrid [10] showed that $\{e_i\otimes e_j : (i,j)\in\mathbb{N}^2\}$ with the square order is a basis of $\hat{\otimes}_{2,\pi} X$ (it is not necessarily an unconditional basis). For instance, Kwapień and Pełczyński [14] proved that $\{e_i\otimes e_j : (i,j)\in\mathbb{N}^2\}$ is not an unconditional basis of $\hat{\otimes}_{2,\pi}\ell_2$. In general, Grecu and Ryan [11] established that $\{e_{i_1}\otimes\cdots\otimes e_{i_n} : (i_1,\dots,i_n)\in\mathbb{N}^n\}$ with the order defined in [11] is a basis of $\hat{\otimes}_{n,\pi} X$. They also showed that $\{e_{i_1}\otimes_s\cdots\otimes_s e_{i_n} : (i_1,\dots,i_n)\in\mathbb{N}^n,\ i_1\ge\cdots\ge i_n\}$ with the order defined in [11] is a basis of $\hat{\otimes}_{n,s,\pi} X$.

Let $\Delta(\hat{\otimes}_{n,\pi} X)$ (resp. $\Delta(\hat{\otimes}_{n,s,\pi} X)$) denote the *main diagonal space* of $\hat{\otimes}_{n,\pi} X$ (resp. $\hat{\otimes}_{n,s,\pi} X$), that is, the closed subspace spanned in $\hat{\otimes}_{n,\pi} X$ (resp. in $\hat{\otimes}_{n,s,\pi} X$) by the *tensor diagonal* $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$. A combination of [12, Theorem 3.12] and [5, Lemma 2] yields the following lemma.

Lemma 3.1. *The tensor diagonal* $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ *is a* 1*-unconditional basis of* $\Delta(\hat{\otimes}_{n,\pi} X)$*, and the projection* $Q:\hat{\otimes}_{n,\pi} X\to\Delta(\hat{\otimes}_{n,\pi} X)$ *defined by*

$$Q(e_{i_1}\otimes\cdots\otimes e_{i_n}) = \begin{cases} e_{i_1}\otimes\cdots\otimes e_{i_n} & \text{if } i_1=\cdots=i_n,\\ 0 & \text{otherwise},\end{cases}$$

is bounded with $\|Q\|\le 1$.

We will use the following Rademacher averaging formula to show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$. For this formula see [17, Lemma 2.22] and [13, Lemma 2.3].

RADEMACHER AVERAGING. *Let $Z_1, \ldots, Z_n$ be vector spaces and $x_{i,k} \in Z_i$ for $i = 1, \ldots, n$ and $k = 1, \ldots, m$. Then*

$$\sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} = \int_0^1 \Big(\sum_{k=1}^{m} r_k(t) x_{1,k}\Big) \otimes \cdots \otimes \Big(\sum_{k=1}^{m} r_k(t) x_{n,k}\Big)\, dt,$$

where (r_k) is the sequence of Rademacher functions on $[0,1]$.

LEMMA 3.2. *The tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$ and the projection $Q_s : \hat{\otimes}_{n,s,\pi} X \to \Delta(\hat{\otimes}_{n,s,\pi} X)$ defined by*

$$Q_s(e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}) = \begin{cases} e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} & \text{if } i_1 = \cdots = i_n, \\ 0 & \text{otherwise}, \end{cases}$$

is bounded with $\|Q_s\| \le 1$.

Proof. Since $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n,\ i_1 \ge \cdots \ge i_n\}$ is a basis of $\hat{\otimes}_{n,s,\pi} X$, the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a basic sequence, and hence a basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$. Next we show that Q_s is bounded with $\|Q_s\| \le 1$.

Define $s : \otimes_n X \to \otimes_{n,s} X$ by $s(v) = \sum_{k=1}^m x_{1,k} \otimes_s \cdots \otimes_s x_{n,k}$ for every $v = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k} \in \otimes_n X$. Then s is a bounded linear projection and so can be extended to $\hat{\otimes}_{n,\pi} X$ with values in $\hat{\otimes}_{n,s,\pi} X$ (see [7]). Take any $u = \sum_{i_1 \ge \cdots \ge i_n} b_{i_1,\ldots,i_n} e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} \in \hat{\otimes}_{n,s,\pi} X$. For every $p, q \in \mathbb{N}$ with $p < q$, let

$$u_{p,q} = \sum_{i_1 \ge \cdots \ge i_n,\, i_1, \ldots, i_n = p}^{q} b_{i_1,\ldots,i_n} e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}.$$

Then for every $\varepsilon > 0$ there exist $\lambda_k \in \mathbb{R}$ and $x_k = \sum_{i=1}^\infty a_{i,k} e_i \in X$, $k = 1, \ldots, m$, such that

$$u_{p,q} = \sum_{k=1}^{m} \lambda_k x_k \otimes_s \cdots \otimes_s x_k \quad \text{and} \quad \sum_{k=1}^{m} |\lambda_k| \cdot \|x_k\|^n \le \|u_{p,q}\|_{s,\pi} + \varepsilon.$$

Note that $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n,\, i_1 \ge \cdots \ge i_n\}$ is a basis of $\hat{\otimes}_{n,s,\pi} X$ and

$$\begin{aligned} u_{p,q} = s(u_{p,q}) &= \sum_{k=1}^{m} \lambda_k \sum_{i_1, \ldots, i_n} a_{i_1,k} \cdots a_{i_n,k} s(e_{i_1} \otimes \cdots \otimes e_{i_n}) \\ &= \sum_{i_1 \ge \cdots \ge i_n} \xi_{i_1,\ldots,i_n} \Big(\sum_{k=1}^{m} \lambda_k a_{i_1,k} \cdots a_{i_n,k}\Big) e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}, \end{aligned}$$

where $\xi_{i_1,\dots,i_n}$ are positive integers obtained by adding equal terms. In particular, $\xi_{i,\dots,i} = 1$ for $i \in \mathbb{N}$. Thus

$$b_{i,\dots,i} = \sum_{k=1}^{m} \lambda_k a_{i,k}^n, \qquad p \le i \le q.$$

By Rademacher averaging,

$$\begin{aligned}
&\Big\| \sum_{i=p}^{q} b_{i,\dots,i} e_i \otimes_s \cdots \otimes_s e_i \Big\|_{s,\pi} \\
&\quad = \Big\| \sum_{k=1}^{m} \sum_{i=p}^{q} \lambda_k a_{i,k}^n e_i \otimes_s \cdots \otimes_s e_i \Big\|_{s,\pi} \\
&\quad \le \sum_{k=1}^{m} |\lambda_k| \cdot \Big\| \sum_{i=p}^{q} (a_{i,k} e_i) \otimes_s \cdots \otimes_s (a_{i,k} e_i) \Big\|_{s,\pi} \\
&\quad = \sum_{k=1}^{m} |\lambda_k| \Big\| \int_0^1 \Big(\sum_{i=p}^{q} a_{i,k} r_i(t) e_i \Big) \otimes_s \cdots \otimes_s \Big(\sum_{i=p}^{q} a_{i,k} r_i(t) e_i \Big)\, dt \Big\|_{s,\pi} \\
&\quad \le \sum_{k=1}^{m} |\lambda_k| \cdot \int_0^1 \Big\| \sum_{i=p}^{q} a_{i,k} r_i(t) e_i \Big\|^n dt \le \sum_{k=1}^{m} |\lambda_k| \cdot \|x_k\|^n \le \|u_{p,q}\|_{s,\pi} + \varepsilon,
\end{aligned}$$

and hence, for every $p, q \in \mathbb{N}$ with $p < q$,

$$\Big\| \sum_{i=p}^{q} b_{i,\dots,i} e_i \otimes_s \cdots \otimes_s e_i \Big\|_{s,\pi} \le \Big\| \sum_{i_1 \ge \cdots \ge i_n,\, i_1,\dots,i_n = p}^{q} b_{i_1,\dots,i_n} e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} \Big\|_{s,\pi}.$$

It follows that Q_s is well defined and bounded with $\|Q_s\| \le 1$. ∎

REMARK 3.3. Note that for every $u \in \otimes_{n,s} X$ we have $\|u\|_\pi \le \|u\|_{s,\pi} \le (n^n/n!)\|u\|_\pi$ (see [7]). Thus $\Delta(\hat{\otimes}_{n,s,\pi} X)$ is isomorphic to $\Delta(\hat{\otimes}_{n,\pi} X)$, and hence $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is an unconditional basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$. In Section 4 we will show that $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is also a 1-unconditional basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$.

By a *Banach lattice with a Schauder basis* we mean a Banach lattice in which the unit vectors form a basis and the order is defined coordinatewise. It follows that such a Schauder basis is 1-unconditional. Conversely, every Banach space with a 1-unconditional basis is a Banach lattice with the order defined coordinatewise. In what follows, E is a Banach lattice with a basis $\{e_i : i \in \mathbb{N}\}$. As a special case of [4, Lemma 22], the set $\{e_i \otimes e_j : (i,j) \in \mathbb{N}^2\}$ with any order is a (1-unconditional) basis of $\hat{\otimes}_{2,|\pi|} E$. The following lemma can be proved in a similar way.

Lemma 3.4. *The tensor basis* $\{e_{i_1}\otimes\cdots\otimes e_{i_n} : (i_1,\dots,i_n)\in\mathbb{N}^n\}$ *with any order is a* (1-*unconditional*) *basis of* $\hat{\otimes}_{n,|\pi|} E$, *and the tensor basis* $\{e_{i_1}\otimes_s\cdots\otimes_s e_{i_n} : (i_1,\dots,i_n)\in\mathbb{N}^n,\ i_1\ge\cdots\ge i_n\}$ *with any order is a* (1-*unconditional*) *basis of* $\hat{\otimes}_{n,s,|\pi|} E$.

Let $\Delta(\hat{\otimes}_{n,|\pi|} E)$ (resp. $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$) denote the *main diagonal space* of $\hat{\otimes}_{n,|\pi|} E$ (resp. $\hat{\otimes}_{n,s,|\pi|} E$), that is, the closed subspace spanned in $\hat{\otimes}_{n,|\pi|} E$ (resp. in $\hat{\otimes}_{n,s,|\pi|} E$) by the *tensor diagonal* $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$. It follows from the above lemma that $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a (1-unconditional) basis of both $\Delta(\hat{\otimes}_{n,|\pi|} E)$ and $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$.

Theorem 3.5. *The three main diagonal spaces* $\Delta(\hat{\otimes}_{n,\pi} E)$, $\Delta(\hat{\otimes}_{n,|\pi|} E)$, *and* $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$ *are pairwise isometrically isomorphic.*

Proof. First we show that $\Delta(\hat{\otimes}_{n,\pi} E)$ is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,|\pi|} E)$. Since $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a basis of both $\Delta(\hat{\otimes}_{n,\pi} E)$ and $\Delta(\hat{\otimes}_{n,|\pi|} E)$, it suffices to show that $\|u\|_\pi = \|u\|_{|\pi|}$ for every $u = \sum_{i=1}^t a_i e_i\otimes\cdots\otimes e_i$.

Let $Q : \hat{\otimes}_{n,\pi} E \to \Delta(\hat{\otimes}_{n,\pi} E)$ be the projection defined in Lemma 3.1, and for every $t\in\mathbb{N}$, define $Q_t : \Delta(\hat{\otimes}_{n,\pi} E)\to\Delta(\hat{\otimes}_{n,\pi} E)$ by

$$Q_t\Big(\sum_{i=1}^{\infty} a_i e_i\otimes\cdots\otimes e_i\Big) = \sum_{i=1}^{t} a_i e_i\otimes\cdots\otimes e_i.$$

Then Q_t is a bounded projection with $\|Q_t\|\le 1$.

On the one hand, for every $\varepsilon>0$, u has a representation

$$u = \sum_{k=1}^m x_{1,k}\otimes\cdots\otimes x_{n,k}\in\otimes_n E$$

such that

$$\sum_{k=1}^m \|x_{1,k}\|\cdots\|x_{n,k}\| \le \|u\|_\pi + \varepsilon.$$

Since $|u| \le \sum_{k=1}^m |x_{1,k}|\otimes\cdots\otimes|x_{n,k}|$, it follows that

$$\|u\|_{|\pi|} \le \sum_{k=1}^m \|x_{1,k}\|\cdots\|x_{n,k}\| \le \|u\|_\pi + \varepsilon,$$

which implies that $\|u\|_{|\pi|}\le\|u\|_\pi$.

On the other hand, for every $\varepsilon>0$ there exists $v=\sum_{k=1}^m y_{1,k}\otimes\cdots\otimes y_{n,k}\in \otimes_n E\subseteq\hat{\otimes}_{n,|\pi|} E$ with $y_{j,k}\in E^+$ for $1\le j\le n$ and $1\le k\le m$ such that $|u|\le v$ and

$$\sum_{k=1}^m \|y_{1,k}\|\cdots\|y_{n,k}\| \le \|u\|_{|\pi|} + \varepsilon.$$

Write $y_{j,k} = \sum_{i=1}^{\infty} b_{i,j,k} e_i$ for $1 \le j \le n$ and $1 \le k \le m$. Then

$$v = \sum_{k=1}^{m} \sum_{i_1,\ldots,i_n} b_{i_1,1,k} \cdots b_{i_n,n,k} e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

Note that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of the Banach lattice $\hat{\otimes}_{n,|\pi|} E$ and the original order on $\hat{\otimes}_{n,|\pi|} E$ coincides with the coordinatewise order. Thus

$$|u| = \sum_{i=1}^{t} |a_i| e_i \otimes \cdots \otimes e_i \le \sum_{k=1}^{m} \sum_{i=1}^{t} b_{i,1,k} \cdots b_{i,n,k} e_i \otimes \cdots \otimes e_i.$$

Since $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{n,\pi} E)$, it follows that $\Delta(\hat{\otimes}_{n,\pi} E)$ is a Banach lattice with the order defined coordinatewise. That is, $\|\cdot\|_\pi$ is a lattice norm on $\Delta(\hat{\otimes}_{n,\pi} E)$. Thus

$$\begin{aligned}\|u\|_\pi &\le \Big\| \sum_{k=1}^{m} \sum_{i=1}^{t} b_{i,1,k} \cdots b_{i,n,k} e_i \otimes \cdots \otimes e_i \Big\|_\pi = \|Q_t \circ Q(v)\|_\pi \le \|v\|_\pi \\ &\le \sum_{k=1}^{m} \|y_{1,k}\| \cdots \|y_{n,k}\| \le \|u\|_{|\pi|} + \varepsilon,\end{aligned}$$

which implies that $\|u\|_\pi \le \|u\|_{|\pi|}$.

Next we show that $\Delta(\hat{\otimes}_{n,|\pi|} E)$ is isometrically isomorphic to $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$. Since $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a basis of both $\Delta(\hat{\otimes}_{n,|\pi|} E)$ and $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$, it suffices to show that $\|u\|_{|\pi|} = \|u\|_{s,|\pi|}$ for every $u = \sum_{i=1}^{t} a_i e_i \otimes \cdots \otimes e_i$. It follows from the definitions that $\|u\|_{|\pi|} \le \|u\|_{s,|\pi|}$. Thus we only need to show that $\|u\|_{s,|\pi|} \le \|u\|_{|\pi|}$.

Since E is a Banach lattice, for every $x_1, \ldots, x_n \in E$, one can define (coordinatewise) the expression $x_1^{1/n} \cdots x_n^{1/n}$ to be an element of E (see [15, Section 1.d]). It follows from [15, Proposition 1.d.2] that

$$\| |x_1|^{1/n} \cdots |x_n|^{1/n} \| \le \|x_1\|^{1/n} \cdots \|x_n\|^{1/n}. \tag{3.1}$$

Define $T : E \times \cdots \times E \to \Delta(\hat{\otimes}_{n,s,|\pi|} E)$ by

$$T(x_1, \ldots, x_n) = \sum_{i=1}^{\infty} a_{1,i} \cdots a_{n,i} e_i \otimes_s \cdots \otimes_s e_i$$

for every $x_k = \sum_{i=1}^{\infty} a_{k,i} e_i \in E$ for $1 \le k \le n$. Now for any $p, q \in \mathbb{N}$ with

$p < q$, by (3.1) and Lemma 3.2 we have

$$\begin{aligned}
&\Big\| \sum_{i=p}^{q} |a_{1,i} \cdots a_{n,i}| e_i \otimes_s \cdots \otimes_s e_i \Big\|_{s,|\pi|} \\
&\quad = \Big\| \sum_{i=p}^{q} (|a_{1,i} \cdots a_{n,i}|^{1/n} e_i) \otimes_s \cdots \otimes_s (|a_{1,i} \cdots a_{n,i}|^{1/n} e_i) \Big\|_{s,|\pi|} \\
&\quad = \Big\| Q_s \Big(\sum_{i_1,\ldots,i_n=p}^{q} (|a_{1,i_1} \cdots a_{n,i_1}|^{1/n} e_{i_1}) \otimes_s \cdots \otimes_s (|a_{1,i_n} \cdots a_{n,i_n}|^{1/n} e_{i_n}) \Big) \Big\|_{s,|\pi|} \\
&\quad \le \Big\| \sum_{i_1,\ldots,i_n=p}^{q} (|a_{1,i_1} \cdots a_{n,i_1}|^{1/n} e_{i_1}) \otimes_s \cdots \otimes_s (|a_{1,i_n} \cdots a_{n,i_n}|^{1/n} e_{i_n}) \Big\|_{s,|\pi|} \\
&\quad = \Big\| \Big(\sum_{i=p}^{q} |a_{1,i} \cdots a_{n,i}|^{1/n} e_i \Big) \otimes_s \cdots \otimes_s \Big(\sum_{i=p}^{q} |a_{1,i} \cdots a_{n,i}|^{1/n} e_i \Big) \Big\|_{s,|\pi|} \\
&\quad = \Big\| \sum_{i=p}^{q} |a_{1,i} \cdots a_{n,i}|^{1/n} e_i \Big\|^n = \Big\| \Big(\sum_{i=p}^{q} |a_{1,i}| e_i \Big)^{1/n} \cdots \Big(\sum_{i=p}^{q} |a_{n,i}| e_i \Big)^{1/n} \Big\|^n \\
&\quad \le \Big\| \sum_{i=p}^{q} a_{1,i} e_i \Big\| \cdots \Big\| \sum_{i=p}^{q} a_{n,i} e_i \Big\|,
\end{aligned}$$

which implies that T is well defined and that $\|T\| \le 1$. It is clear that T is a positive n-linear operator. Note that every Banach lattice with a (1-unconditional) basis is Dedekind complete. By [3, Proposition 3.3] there exists a positive linear operator $T^{\otimes} : \hat{\otimes}_{n,|\pi|} E \to \Delta(\hat{\otimes}_{n,s,|\pi|} E)$ such that $\|T^{\otimes}\| = \|T\| \le 1$ and $T^{\otimes}(x_1 \otimes \cdots \otimes x_n) = T(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in E$. Since

$$T^{\otimes}(u) = \sum_{i=1}^{t} a_i T(e_i, \ldots, e_i) = \sum_{i=1}^{t} a_i e_i \otimes \cdots \otimes e_i = u,$$

it follows that $\|u\|_{s,|\pi|} = \|T^{\otimes}(u)\|_{s,|\pi|} \le \|u\|_{|\pi|}$. ■

4. Relations to concavification. In this section we assume that E is a Banach lattice with a (1-unconditional) basis $\{e_i : i \in \mathbb{N}\}$. For every $a \in \mathbb{R}$ and every $\alpha > 0$ we define $a^{\alpha} = \operatorname{sign}(a) \cdot |a|^{\alpha}$. For every $x = \sum_{i=1}^{\infty} a_i e_i$ we define x^{α} coordinatewise, that is,

$$x^{\alpha} = \sum_{i=1}^{\infty} a_i^{\alpha} e_i.$$

It follows that if $x = \sum_{i=1}^{\infty} a_i e_i \in E$ (that is, $\sum_{i=1}^{\infty} a_i e_i$ converges in E) then $x^n = \sum_{i=1}^{\infty} a_i^n e_i \in E$. Let $E_{(n)}$ denote the n-concavification of E (see [15, Section 1.d]). It follows that

$$E_{(n)} = \Big\{x^n = \sum_{i=1}^{\infty} a_i^n e_i : \ x = \sum_{i=1}^{\infty} a_i e_i \in E\Big\}.$$

In other words,

$$E_{(n)} = \Big\{x = \sum_{i=1}^{\infty} a_i e_i : \ x^{1/n} = \sum_{i=1}^{\infty} a_i^{1/n} e_i \in E\Big\}.$$

For every $x \in E_{(n)}$ we define

$$\|x\|_{E_{(n)}} = \inf\Big\{\sum_{k=1}^{m} \|x_k^{1/n}\|_E^n : x_k \in E_{(n)}^+,\ 1 \le k \le m,\ |x| \le \sum_{k=1}^{m} x_k\Big\}.$$

Then $\|\cdot\|_{E_{(n)}}$ is a lattice norm on $E_{(n)}$, which may not be complete (see [4]). Let $E_{[n]}$ denote the completion of $E_{(n)}$ with respect to $\|\cdot\|_{E_{(n)}}$. Then $E_{[n]}$ is a Banach lattice. Note that $E_{(n)}$, being a vector lattice, satisfies the Riesz Decomposition Property (see [1, Theorem 1.13]). Thus the lattice norm $\|\cdot\|_{E_{(n)}}$ on $E_{(n)}$ has the following equivalent form:

$$\|x\|_{E_{(n)}} = \inf\Big\{\sum_{k=1}^{m} \|x_k^{1/n}\|_E^n : x_k \in E_{(n)}^+,\ 1 \le k \le m,\ |x| = \sum_{k=1}^{m} x_k\Big\}, \quad x \in E_{(n)}.$$

REMARK 4.1. If $E = \ell_p$ for $1 \le p < \infty$ then $E_{(n)} = \ell_{p/n}$ as vector spaces. In the case that $p \ge n$, E is n-convex, and hence $E_{[n]} = E_{(n)} = \ell_{p/n}$ as Banach spaces. Thus the norm $\|\cdot\|_{E_{(n)}}$ is the $\ell_{p/n}$-norm on $E_{(n)}$. For $p \le n$, E satisfies the lower n-estimate. It follows from [4, Proposition 21] that $E_{[n]} = \ell_1$ as Banach spaces. Thus the norm $\|\cdot\|_{E_{(n)}}$ is the ℓ_1-norm on $E_{(n)}$.

THEOREM 4.2. *Let E be a Banach lattice with a basis. Then $E_{[n]}$ is isometrically lattice isomorphic to $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$.*

Proof. Define $\phi : E_{(n)} \to \Delta(\hat{\otimes}_{n,s,|\pi|} E)$ by

$$\phi(z^n) = \phi\Big(\sum_{i=1}^{\infty} a_i^n e_i\Big) = \sum_{i=1}^{\infty} a_i^n e_i \otimes_s \cdots \otimes_s e_i$$

for every $z^n \in E_{(n)}$, where $z = \sum_{i=1}^{\infty} a_i e_i \in E$. Now for every $p, q \in \mathbb{N}$ with

$p < q$, we have

$$\Big\|\sum_{i=p}^{q} a_i^n e_i \otimes_s \cdots \otimes_s e_i\Big\|_{s,|\pi|} = \Big\|\sum_{i=p}^{q} |a_i|^n e_i \otimes_s \cdots \otimes_s e_i\Big\|_{s,|\pi|}$$
$$\leq \Big\|\sum_{i_1,\ldots,i_n=p}^{q} |a_{i_1}|e_{i_1} \otimes_s \cdots \otimes_s |a_{i_n}|e_{i_n}\Big\|_{s,|\pi|}$$
$$= \Big\|\Big(\sum_{i=p}^{q} |a_i|e_i\Big) \otimes_s \cdots \otimes_s \Big(\sum_{i=p}^{q} |a_i|e_i\Big)\Big\|_{s,|\pi|}$$
$$= \Big\|\sum_{i=p}^{q} a_i e_i\Big\|_E^n.$$

It follows that ϕ is well defined and

$$\|\phi(z^n)\|_{s,|\pi|} \leq \|z\|_E^n, \qquad \forall z = \sum_{i=1}^{\infty} a_i e_i \in E. \tag{4.1}$$

It is easy to see that ϕ is a vector lattice homomorphism. Next we show that it is an isometry.

Take any $x = \sum_{i=1}^{\infty} a_i^n e_i \in E_{(n)}$. For every $\varepsilon > 0$, choose $x_k = \sum_{i=1}^{\infty} a_{i,k}^n e_i \in E_{(n)}^+$ for $1 \leq k \leq m$ such that $|x| \leq \sum_{k=1}^{m} x_k$ and

$$\sum_{k=1}^{m} \|x_k^{1/n}\|_E^n \leq \|x\|_{E_{(n)}} + \varepsilon.$$

Then, by (4.1),

$$\|\phi(x)\|_{s,|\pi|} = \|\phi(|x|)\|_{s,|\pi|} \leq \Big\|\phi\Big(\sum_{k=1}^{m} x_k\Big)\Big\|_{s,|\pi|}$$
$$\leq \sum_{k=1}^{m} \|\phi(x_k)\|_{s,|\pi|} \leq \sum_{k=1}^{m} \|x_k^{1/n}\|_E^n \leq \|x\|_{E_{(n)}} + \varepsilon,$$

which implies that $\|\phi(x)\|_{s,|\pi|} \leq \|x\|_{E_{(n)}}$.

For the reverse inequality, for every $\varepsilon > 0$ choose $\lambda_k \in \mathbb{R}^+$ and $y_k = \sum_{i=1}^{\infty} b_{k,i} e_i \in E^+$ for $1 \leq k \leq m$ such that $|\phi(x)| \leq \sum_{k=1}^{m} \lambda_k y_k \otimes_s \cdots \otimes_s y_k$ and

$$\sum_{k=1}^{m} \lambda_k \|y_k\|^n \leq \|\phi(x)\|_{s,|\pi|} + \varepsilon.$$

Then

$$\sum_{i=1}^{\infty} |a_i^n| e_i \otimes_s \cdots \otimes_s e_i = \phi(|x|) = |\phi(x)|$$
$$\leq \sum_{k=1}^{m} \lambda_k y_k \otimes_s \cdots \otimes_s y_k$$
$$= \sum_{k=1}^{m} \lambda_k \sum_{i_1,\ldots,i_n} b_{k,i_1} e_{i_1} \otimes_s \cdots \otimes_s b_{k,i_n} e_{i_n}.$$

Note that the order on $\hat{\otimes}_{n,s,|\pi|} E$ is coordinatewise. Thus

$$\phi(|x|) = \sum_{i=1}^{\infty} |a_i^n| e_i \otimes_s \cdots \otimes_s e_i \leq \sum_{k=1}^{m} \lambda_k \sum_{i=1}^{\infty} b_{k,i}^n e_i \otimes_s \cdots \otimes_s e_i = \phi\Big(\sum_{k=1}^{m} \lambda_k y_k^n\Big).$$

Since ϕ is bipositive, we have $|x| \leq \sum_{k=1}^{m} \lambda_k y_k^n$, and hence

$$\|x\|_{E_{(n)}} \leq \sum_{k=1}^{m} \lambda_k \|y_k\|^n \leq \|\phi(x)\|_{s,|\pi|} + \varepsilon,$$

which implies that $\|x\|_{E_{(n)}} \leq \|\phi(x)\|_{s,|\pi|}$.

In conclusion, we have shown that ϕ is an isometry, and hence ϕ can be extended isometrically from $E_{(n)}$ to its completion $E_{[n]}$, still denoted by ϕ.

We can now easily show that ϕ is onto. Indeed, take any $\sum_{i=1}^{\infty} a_i e_i \otimes_s \cdots \otimes_s e_i \in \Delta(\hat{\otimes}_{n,s,|\pi|} E)$. Define $x_m = \sum_{i=1}^{m} a_i^{1/n} e_i \in E$ for each $m \in \mathbb{N}$. Then $x_m^n = \sum_{i=1}^{m} a_i e_i \in E_{(n)}$ and $\phi(x_m^n) = \sum_{i=1}^{m} a_i e_i \otimes_s \cdots \otimes_s e_i$, which in turn converges to $\sum_{i=1}^{\infty} a_i e_i \otimes_s \cdots \otimes_s e_i$ in $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$. ∎

In particular, if E is n-convex then $E_{(n)}$ is a Banach lattice (see [15, Section 1.d]), and hence $E_{(n)} = E_{[n]}$. This yields the following.

Corollary 4.3. *If E is n-convex then $\Delta(\hat{\otimes}_{n,s,|\pi|} E) = E_{(n)}$ lattice isometrically.*

The following special case of our results is Proposition 21 of [4].

Proposition 4.4. *If E satisfies the lower n-estimate with constant M then $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$ is lattice isomorphic (and isometric if $M = 1$) to ℓ_1.*

We now arrive at the main result of this paper.

Theorem 4.5. *All four main diagonal spaces $\Delta(\hat{\otimes}_{n,\pi} E)$, $\Delta(\hat{\otimes}_{n,s,\pi} E)$, $\Delta(\hat{\otimes}_{n,|\pi|} E)$, and $\Delta(\hat{\otimes}_{n,s,|\pi|} E)$ are pairwise isometrically isomorphic.*

Proof. It follows from Remark 3.3 and Theorems 3.5 and 4.2 that $\Delta(\hat{\otimes}_{n,s,\pi} E)$ is isomorphic to $E_{[n]}$ via the mapping $\phi : E_{[n]} \to \Delta(\hat{\otimes}_{n,s,\pi} E)$

defined by

$$\phi(z^n) = \sum_{i=1}^{\infty} a_i^n e_i \otimes_s \cdots \otimes_s e_i$$

for every $z^n = \sum_{i=1}^{\infty} a_i^n e_i \in E_{(n)}$ where $z = \sum_{i=1}^{\infty} a_i e_i \in E$, with

$$\|x\|_{E_{(n)}} \le \|\phi(x)\|_{\Delta(\hat{\otimes}_{n,s,\pi} E)} \le \frac{n^n}{n!}\|x\|_{E_{(n)}}, \qquad x \in E_{(n)}. \tag{4.2}$$

Since $E_{[n]}$ is a vector lattice, there is an order (which is the coordinatewise order) in $\Delta(\hat{\otimes}_{n,s,\pi} E)$ induced by $E_{[n]}$ such that $\Delta(\hat{\otimes}_{n,s,\pi} E)$ is a vector lattice. Next we show that the norm on $\Delta(\hat{\otimes}_{n,s,\pi} E)$ is a lattice norm.

Let $Q_s : \hat{\otimes}_{n,s,\pi} E \to \Delta(\hat{\otimes}_{n,s,\pi} E)$ be the projection defined in Lemma 3.2. Take any $x = \sum_{i=1}^{\infty} a_i^n e_i \in E_{(n)}^+$. For every $\varepsilon > 0$, choose $x_k = \sum_{i=1}^{\infty} a_{i,k}^n e_i \in E_{(n)}^+$ for $1 \le k \le m$ such that $x = \sum_{k=1}^{m} x_k$ and

$$\sum_{k=1}^{m} \|x_k^{1/n}\|_E^n \le \|x\|_{E_{(n)}} + \varepsilon.$$

Thus

$$\begin{aligned}
\|\phi(x)\|_{s,\pi} &\le \sum_{k=1}^{m} \|\phi(x_k)\|_{s,\pi} = \sum_{k=1}^{m} \Big\| \sum_{i=1}^{\infty} a_{i,k}^n e_i \otimes_s \cdots \otimes_s e_i \Big\|_{s,\pi} \\
&= \sum_{k=1}^{m} \Big\| Q_s\Big(\sum_{i_1,\ldots,i_n} (a_{i_1,k} e_{i_1}) \otimes_s \cdots \otimes_s (a_{i_n,k} e_{i_n}) \Big) \Big\|_{s,\pi} \\
&\le \sum_{k=1}^{m} \|Q_s\| \, \Big\| \sum_{i_1,\ldots,i_n} (a_{i_1,k} e_{i_1}) \otimes_s \cdots \otimes_s (a_{i_n,k} e_{i_n}) \Big\|_{s,\pi} \\
&\le \sum_{k=1}^{m} \Big\| \Big(\sum_{i=1}^{\infty} a_{i,k} e_i \Big) \otimes_s \cdots \otimes_s \Big(\sum_{i=1}^{\infty} a_{i,k} e_i \Big) \Big\|_{s,\pi} \\
&= \sum_{k=1}^{m} \Big\| \sum_{i=1}^{\infty} a_{i,k} e_i \Big\|_E^n = \sum_{k=1}^{m} \|x_k^{1/n}\|_E^n \le \|x\|_{E_{(n)}} + \varepsilon,
\end{aligned}$$

which implies that $\|\phi(x)\|_{s,\pi} \le \|x\|_{E_{(n)}}$, and hence $\|\phi(x)\|_{s,\pi} = \|x\|_{E_{(n)}}$ by (4.2). Since $E_{(n)}$ is dense in $E_{[n]}$, it follows that $\|\phi(x)\|_{s,\pi} = \|x\|_{E_{(n)}}$ for every $x \in E_{[n]}^+$. Note that $\|\cdot\|_{E_{(n)}}$ is a lattice norm on $E_{[n]}$. Thus $\|\cdot\|_{s,\pi}$ is also a lattice norm on $\Delta(\hat{\otimes}_{n,s,\pi} E)$. Hence for every $x \in E_{[n]}$, we have $\|\phi(x)\|_{s,\pi} = \big\||\phi(x)|\big\|_{s,\pi} = \|\phi(|x|)\|_{s,\pi} = \big\||x|\big\|_{E_{(n)}} = \|x\|_{E_{(n)}}$, which implies that $\Delta(\hat{\otimes}_{n,s,\pi} E)$ is isometrically isomorphic to $E_{[n]}$. The proof is complete by Theorems 3.5 and 4.2. ■

COROLLARY 4.6. *The tensor diagonal* $\{e_i \otimes \cdots \otimes e_i : \in \mathbb{N}\}$ *is a* 1-*unconditional basis of* $\Delta(\hat{\otimes}_{n,s,\pi} E)$.

5. Applications to polynomials. In this section we assume that Y is a Banach space and E is a Banach lattice with a (1-unconditional) basis $\{e_i : i \in \mathbb{N}\}$. Recall that an n-linear operator $T : E \times \cdots \times E \to Y$ is called *orthosymmetric* if $T(x_1, \ldots, x_n) = 0$ whenever $x_1, \ldots, x_n \in E$ with $x_i \perp x_j$ for some $i \neq j$ and $i, j = 1, \ldots, n$. Also recall that an n-homogeneous polynomial $P : E \to Y$ is called *orthogonally additive* if $P(x + y) = P(x) + P(y)$ whenever $x, y \in E$ with $x \perp y$. Let $\mathcal{P}_o(^nE; Y)$ denote the space of all continuous n-homogeneous orthogonally additive polynomials from E to Y. In particular, denote $\mathcal{P}_o(^nE; \mathbb{R})$ by $\mathcal{P}_o(^nE)$.

For Banach spaces Z and Y, let $\mathcal{L}(Z; Y)$ denote the space of all continuous linear operators from Z to Y, and let $Z^* := \mathcal{L}(Z; \mathbb{R})$. Theorem 4.2, Proposition 4.4, and [3, Corollary 4.4] have the following consequences (see also [2, Theorem 2.3]).

COROLLARY 5.1. $\mathcal{P}_o(^nE; Y)$ *is isometrically isomorphic to* $\mathcal{L}(E_{[n]}; Y)$. *In particular,* $\mathcal{P}_o(^nE)$ *is isometrically isomorphic to* $(E_{[n]})^* = (E_{(n)}, \|\cdot\|_{E_{(n)}})^*$.

COROLLARY 5.2. *If* E *satisfies the lower* n-*estimate with constant* 1 *then* $\mathcal{P}_o(^nE; Y)$ *is isometrically isomorphic to* $\mathcal{L}(\ell_1; Y)$. *In particular,* $\mathcal{P}_o(^nE)$ *is isometrically isomorphic to* ℓ_∞.

REMARK 5.3. We cover the well known results of Sundaresan [19] (see also [2]). If $E = \ell_p$ for $1 \leq p < \infty$ then by Remark 4.1, $E_{[n]} = \ell_{p/n}$ if $p \geq n$, and $(E_{(n)}, \|\cdot\|_{E_{(n)}}) = (\ell_{p/n}, \|\cdot\|_{\ell_1})$ if $p \leq n$. It follows from Corollary 5.1 that $\mathcal{P}_o(^n\ell_p) = \ell_{p/(p-n)}$ if $p > n$, and $\mathcal{P}_o(^n\ell_p) = \ell_\infty$ if $p \leq n$.

For an n-homogeneous polynomial $P : Z \to Y$, let $T_P : Z \times \cdots \times Z \to Y$ denote the symmetric n-linear operator associated to P. The Polarization Inequality states that $\|P\| \leq \|T_P\| \leq (n^n/n!)\|P\|$ (see [7]). It is known that if Z is a Hilbert space then for every n-homogeneous polynomial $P : Z \to Y$, we have $\|T_P\| = \|P\|$ (see [6, Proposition 1.44] and [7]). Now define the n-homogeneous polynomial $P(x) = a_1 \cdots a_n$ for every $x = (a_i)_i \in \ell_1$. Then $P : \ell_1 \to \mathbb{R}$ and $\|T_P\| = (n^n/n!)\|P\|$ (see [6, Example 1.39] and [7]). Next we will show that every continuous n-homogeneous orthogonally additive polynomial $P : E \to Y$ satisfies $\|T_P\| = \|P\|$.

THEOREM 5.4. *For every* $P \in \mathcal{P}_o(^nE; Y)$, $\|T_P\| = \|P\|$.

Proof. Take any $P \in \mathcal{P}_o(^nE; Y)$. It follows from [3, Lemma 4.1] that its associated symmetric n-linear operator T_P is orthosymmetric. By linearization of P there exists $\tilde{P} \in \mathcal{L}(\hat{\otimes}_{n,s,\pi} E; Y)$ such that $\|\tilde{P}\| = \|P\|$ and $\tilde{P}(x \otimes \cdots \otimes x) = P(x)$ for every $x \in E$. By linearization of T_P there exists $\tilde{T}_P \in \mathcal{L}(\hat{\otimes}_{n,\pi} E; Y)$ such that $\|\tilde{T}_P\| = \|T_P\|$ and $\tilde{T}_P(x_1 \otimes \cdots \otimes x_n) = T_P(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in E$. For every $\varepsilon > 0$ there exists $u \in \hat{\otimes}_{n,\pi} E$ such that $\|u\|_\pi \le 1$ and $\|\tilde{T}_P\| \le \|\tilde{T}_P(u)\| + \varepsilon$. Note that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ with the order defined in [11] is a basis of $\hat{\otimes}_{n,\pi} E$. We write

$$u = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}.$$

Let $Q : \hat{\otimes}_{n,\pi} E \to \Delta(\hat{\otimes}_{n,\pi} E)$ be the projection defined in Lemma 3.1. Then

$$\Big\| \sum_{i=1}^{\infty} a_{i,\ldots,i} e_i \otimes \cdots \otimes e_i \Big\|_\pi = \|Q(u)\|_\pi \le \|u\|_\pi \le 1.$$

It follows from Theorem 4.5 that

$$\begin{aligned}
\|T_P\| = \|\tilde{T}_P\| \le \|\tilde{T}_P(u)\| + \varepsilon &= \Big\| \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} T_P(e_{i_1}, \ldots, e_{i_n}) \Big\| + \varepsilon \\
&= \Big\| \sum_{i=1}^{\infty} a_{i,\ldots,i} T_P(e_i, \ldots, e_i) \Big\| + \varepsilon = \Big\| \sum_{i=1}^{\infty} a_{i,\ldots,i} P(e_i) \Big\| + \varepsilon \\
&= \Big\| \tilde{P}\Big(\sum_{i=1}^{\infty} a_{i,\ldots,i} e_i \otimes \cdots \otimes e_i \Big) \Big\| + \varepsilon \\
&\le \|\tilde{P}\| \Big\| \sum_{i=1}^{\infty} a_{i,\ldots,i} e_i \otimes \cdots \otimes e_i \Big\|_{s,\pi} + \varepsilon \\
&= \|\tilde{P}\| \Big\| \sum_{i=1}^{\infty} a_{i,\ldots,i} e_i \otimes \cdots \otimes e_i \Big\|_{\pi} + \varepsilon \le \|\tilde{P}\| + \varepsilon = \|P\| + \varepsilon,
\end{aligned}$$

which implies that $\|T_P\| \le \|P\|$, and hence $\|T_P\| = \|P\|$. ■

Acknowledgments. The authors are grateful to the referee for suggestions to revise the paper.

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Received October 21, 2012
Revised version February 4, 2014 (7672)