STUDIA MATHEMATICA 221 (2) (2014)

Universal stability of Banach spaces for ε-isometries

by

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Abstract. Let X, Y be real Banach spaces and $\varepsilon > 0$. A standard ε-isometry $f : X \to Y$ is said to be (α,γ)-stable (with respect to $T : L(f) \equiv \overline{\operatorname{span}}\, f(X) \to X$ for some $\alpha,\gamma > 0$) if T is a linear operator with $\|T\| \le \alpha$ such that $Tf - \mathrm{Id}$ is uniformly bounded by $\gamma\varepsilon$ on X. The pair (X,Y) is said to be stable if every standard ε-isometry $f : X \to Y$ is (α,γ)-stable for some $\alpha,\gamma > 0$. The space X [Y] is said to be universally left [right]-stable if (X,Y) is always stable for every Y [X]. In this paper, we show that universally right-stable spaces are just Hilbert spaces; every injective space is universally left-stable; a Banach space X isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is isomorphic to ℓ_∞; and a separable space X has the property that (X,Y) is left-stable for every separable Y if and only if X is isomorphic to c_0.

1. Introduction. The study of properties of isometries between Banach spaces and of their generalizations has continued for 80 years since Mazur and Ulam's 1932 celebrated result [18]: Every surjective isometry between two Banach spaces X and Y is necessarily affine. The simple example of $f : \mathbb{R} \to \ell_\infty^2$ defined by $f(t) = (t, \sin t)$ shows that the surjectivity assumption cannot be omitted. In 1968, Figiel [10] showed the following remarkable result: For every standard isometry $f : X \to Y$ there is a linear operator $T : L(f) \to X$ with $\|T\| = 1$ such that $Tf = \mathrm{Id}$ on X, where $L(f)$ is the closure of $\operatorname{span} f(X)$ in Y (see also [3] and [8]). In 2003, Godefroy and Kalton [12] studied the relationship between isometries and linear isometries and resolved a long-standing problem: Does the existence of an isometry $f : X \to Y$ imply the existence of a linear isometry $U : X \to Y$?

Definition 1.1. Let X, Y be Banach spaces, $\varepsilon \ge 0$, and let $f : X \to Y$ be a mapping.

(1) f is said to be an *ε-isometry* if

2010 *Mathematics Subject Classification*: Primary 46B04, 46B20, 47A58; Secondary 26E25, 46A20, 46A24.
Key words and phrases: ε-isometry, stability, injective space, Banach space.

DOI: 10.4064/sm221-2-3

$$\big|\|f(x)-f(y)\|-\|x-y\|\big|\le\varepsilon\quad \text{forall } x,y\in X. \tag{1.1}$$

In particular, a 0-isometry f is simply called an *isometry*.

(2) We say an ε-isometry f is *standard* if $f(0)=0$.

(3) A standard ε-isometry is (α,γ)-*stable* if there exist $\alpha,\gamma>0$ and a bounded linear operator $T:L(f)\to X$ with $\|T\|\le\alpha$ such that

$$\|Tf(x)-x\|\le\gamma\varepsilon\quad\text{for all } x\in X. \tag{1.2}$$

In this case, we also simply say f is stable, if no confusion arises.

(4) The pair (X,Y) is said to be *stable* if every standard ε-isometry $f:X\to Y$ is (α,γ)-stable for some $\alpha,\gamma>0$.

(5) The pair (X,Y) is called (α,γ)-*stable* for some $\alpha,\gamma>0$ if every standard ε-isometry $f:X\to Y$ is (α,γ)-stable.

In 1945, Hyers and Ulam [15] (see also [19]) asked whether for every pair (X,Y) of Banach spaces there is $\gamma>0$ such that for every standard surjective ε-isometry $f:X\to Y$ there exists a surjective linear isometry $U:X\to Y$ with

$$\|f(x)-Ux\|\le\gamma\varepsilon\quad\text{for all } x\in X. \tag{1.3}$$

After many efforts of a number of mathematicians (see, for instance, [11], [14], [15]), Omladič and Šemrl [19] finally achieved the sharp estimate $\gamma=2$ in (1.3).

The study of non-surjective ε-isometries has also attracted mathematicians' attention (see, for instance, [2], [4], [5], [7], [19], [20], [21] and [23]). Qian [20] proposed the following problem in 1995.

Problem 1.2. *Establish whether for every pair (X,Y) of Banach spaces there exists $\gamma>0$ such that every standard ε-isometry $f:X\to Y$ is (α,γ)-stable for some $\alpha>0$.*

He showed that the answer is affirmative if both X and Y are L_p spaces. Šemrl and Väisälä [21] proved that $\gamma=2$ is sharp in (1.2) if both X and Y are L_p spaces for $1<p<\infty$. However, Qian [20] presented a counterexample showing that if a separable Banach space Y contains an uncomplemented closed subspace X then for every $\varepsilon>0$ there is a standard ε-isometry $f:X\to Y$ which is not stable. Cheng, Dong and Zhang [4] showed the following weak stability version.

Theorem 1.3 (Cheng–Dong–Zhang). *Let X and Y be Banach spaces, and let $f:X\to Y$ be a standard ε-isometry for some $\varepsilon\ge0$. Then for every $x^*\in X^*$, there exists $\phi\in Y^*$ with $\|\phi\|=\|x^*\|\equiv r$ such that*

$$|\langle\phi,f(x)\rangle-\langle x^*,x\rangle|\le4\varepsilon r\quad\textit{for all } x\in X. \tag{1.4}$$

Concerning the stability of ε-isometries of Banach spaces, the following two questions are very natural.

PROBLEM 1.4. *Is there a characterization for the class of Banach spaces X such that for any Banach space Y, the pair (X, Y) is stable (resp. (α, γ)-stable)?*

Every space X in this class is said to be *universally left-stable* (resp. *universally (α, γ)-left-stable*).

PROBLEM 1.5. *Can we characterize the class of Banach spaces Y such that for any Banach space X, the pair (X, Y) is stable (resp. (α, γ)-stable)?*

Every space Y in this class is called *universally right-stable* (resp. *universally (α, γ)-right-stable*).

In this paper, we study universal stability and universal right-stability of Banach spaces. With the help of Qian's counterexample and Theorem 1.3, incorporating Lindenstrauss–Tzafriri's characterization of Hilbert spaces [17], we show that the universally stable spaces are precisely the spaces of finite dimension; and up to isomorphism, a universally right-stable space is just a Hilbert space. By using Theorem 1.3 we then prove that every injective space is universally left-stable; and a Banach space X which is isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is isomorphic to ℓ_∞. Finally, applying Zippin's theorem [25] we verify that a separable space X has the property that (X, Y) is stable for every separable Y if and only if X is isomorphic to c_0.

All symbols and notations in this paper are standard. We use X to denote a real Banach space and X^* its dual. B_X and S_X denote the closed unit ball and the unit sphere of X, respectively. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$, $\overline{A}$ stands for the closure of A, and $\operatorname{card}(A)$ for its cardinality.

2. Universally (right-) stable spaces for ε-isometries. In this section, we establish some properties of the class of universally left (right)-stable spaces for ε-isometries.

Recall that a Banach space X $[Y]$ is universally left [right]-stable if for every Banach space Y $[X]$ and for every standard ε-isometry $f : X \to Y$, there exist $\alpha, \gamma > 0$ and a bounded operator $T : L(f) \to X$ with $\|T\| \le \alpha$ such that

$$\|Tf(x) - x\| \le \gamma\varepsilon \quad \text{for all } x \in X. \tag{2.1}$$

A *universally stable space* is a Banach space which is both universally left-stable and universally right-stable. We will show that inequality (2.1) holds for every Banach space X if and only if Y is, up to linear isomorphism, a Hilbert space; and universally stable spaces are just finite-dimensional spaces.

The following lemma follows from Qian's counterexample [20].

Lemma 2.1. *Let X be a closed subspace of a Banach space Y. If* $\mathrm{card}(X) = \mathrm{card}(Y)$, *then for every $\varepsilon > 0$ there is a standard ε-isometry $f : X \to Y$ such that*

(1) $L(f) \equiv \overline{\mathrm{span}}\, f(X) = Y$;
(2) *X is complemented whenever f is stable.*

Theorem 2.2. *Let Y be a Banach space. Then the following statements are equivalent:*

(i) *Y is universally right-stable;*
(ii) *Y is isomorphic to a Hilbert space;*
(iii) *Y is universally $(\alpha, 4)$-right-stable for some $\alpha > 0$.*

Proof. (i)⇒(ii). By definition, every closed subspace of Y is again universally right-stable. Fix any closed separable subspace Z of Y. By Lemma 2.1, universal right-stability of Z entails that every closed subspace of Z is complemented in Z. According to Lindenstrauss–Tzafriri's theorem [17]: "a Banach space such that every closed subspace is complemented is isomorphic to a Hilbert space", Z is isomorphic to a (separable) Hilbert space. Hence, Y itself is isomorphic to a Hilbert space.

(ii)⇒(iii). Suppose that Y is isomorphic to a Hilbert space H. Let $\alpha = \mathrm{dist}(Y, H)$, the Banach–Mazur distance between Y and H. Then every closed subspace of Y is α-complemented in Y. Given $\varepsilon > 0$ and any standard ε-isometry $f : X \to Y$, according to Theorem 4.8 of [4], inequality (2.1) holds for some $T : L(f) \to X$ with $\|T\| \le \alpha$ and with $\gamma = 4$, i.e., Y is universally $(\alpha, 4)$-right-stable.

(iii)⇒(i). This is trivial. ∎

Theorem 2.3. *A normed space X is universally stable if and only if it is finite-dimensional.*

Proof. Sufficiency. Since every finite-dimensional normed space is isomorphic to a Euclidean space, Theorem 2.2 entails that it is universally right-stable. Moreover, Theorem 3.4 of [4] says that n-dimensional spaces are universally left-stable with $\gamma = 4n$.

Necessity. Suppose, to the contrary, that X is infinite-dimensional. Since X is also universally right-stable, according to Theorem 2.2 we have just proven, it is isomorphic to a Hilbert space. Since every closed subspace of a universally right-stable space is again universally right-stable, we can assume that X is separable. Thus, X is isometric to a subspace of ℓ_∞. Since ℓ_∞ is prime [16] (i.e. every complemented infinite-dimensional subspace is isomorphic to it), X is uncomplemented in ℓ_∞. Note $\mathrm{card}(X) = \mathrm{card}(\ell_\infty)$. By Lemma 2.1, there is an unstable standard ε-isometry $f : X \to \ell_\infty$ for every $\varepsilon > 0$, which contradicts the universal stability of X. ∎

3. Universally left-stable spaces. In this section, we consider properties of universally left-stable spaces. We shall show that (1) every injective Banach space is universally left-stable; (2) a Banach space isomorphic to a subspace of ℓ_∞ is universally left-stable if and only if it is isomorphic to ℓ_∞, and (3) for a separable Banach space X, (X, Y) is stable for every separable Banach space Y if and only if X is a separably injective Banach space.

A Banach space X is said to be λ-*injective* (or simply injective) if it has the following extension property: Every bounded linear operator T from a closed subspace of a Banach space into X can be extended to a bounded operator on the whole space with norm at most $\lambda\|T\|$ (see, for instance, [1]). Goodner [13] introduced a family of Banach spaces coinciding with the family of injective spaces: for any $\lambda \geq 1$, a Banach space X is a P_λ-*space* if, whenever X is isometrically embedded in another Banach space, there is a projection onto the image of X with norm not larger than λ. The following result is due to Day [6] (see also Wolfe [24] and Fabian et al. [9, p. 242]).

Proposition 3.1. *A Banach space X is λ-injective if and only if it is a P_λ-space.*

Remark 3.2. For any set Γ, that $\ell_\infty(\Gamma)$ is 1-injective follows from the Hahn–Banach theorem.

Theorem 3.3. *Every λ-injective space is universally $(\lambda, 4\lambda)$-left-stable.*

Proof. Let X be a λ-injective Banach space. We can assume that X is a closed complemented subspace of $\ell_\infty(\Gamma)$; otherwise, we can identify X with its canonical embedding $J(X)$ as a subspace of $\ell_\infty(\Gamma)$, where Γ denotes the closed ball B_{X^*} of X^*. Hence, it is λ-complemented in $\ell_\infty(\Gamma)$. Let $P : \ell_\infty(\Gamma) \to X$ be a projection such that $\|P\| \leq \lambda$. Given any $\beta \in \Gamma$, let $\delta_\beta \in \ell_\infty(\Gamma)^*$ be defined for $x = (x(\gamma))_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ by $\delta_\beta(x) = x(\beta)$. Assume that $f : X \to Y$ is a standard ε-isometry. For every $x^* \in X^*$, by Theorem 1.3, there is $\phi \in Y^*$ with $\|\phi\| = \|x^*\|$ such that

$$|\langle \phi, f(x)\rangle - \langle x^*, x\rangle| \leq 4\varepsilon\|x^*\| \quad \text{for all } x \in X. \tag{3.1}$$

In particular, letting $x^* = \delta_\gamma$ in (3.1) for every fixed $\gamma \in \Gamma$, we obtain a linear functional $\phi_\gamma \in Y^*$ satisfying (3.1) with $\|\phi_\gamma\| = \|\delta_\gamma\|_X \leq 1$. Therefore, $(\phi_\gamma(y))_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ for every $y \in Y$.

Let $T(y) = P(\phi_\gamma(y))_{\gamma\in\Gamma}$ for all $y \in Y$, and note that $P|_X = I_X$, the identity from X to itself. Then $\|T\| \leq \|P\| \leq \lambda$ and for all $x \in X$,

$$\begin{aligned}\|Tf(x) - x\| &= \|P(\phi_\gamma(f(x)))_{\gamma\in\Gamma} - (\delta_\gamma(x))_{\gamma\in\Gamma}\| \\ &= \|P(\phi_\gamma(f(x)))_{\gamma\in\Gamma} - P((\delta_\gamma(x))_{\gamma\in\Gamma})\| \\ &\leq \|P\| \cdot \|(\phi_\gamma(f(x)))_{\gamma\in\Gamma} - (\delta_\gamma(x))_{\gamma\in\Gamma}\|_\infty \leq 4\lambda\varepsilon. \ \blacksquare\end{aligned}$$

Theorem 3.4. *Let X be a Banach space isomorphic to an infinite-dimensional subspace of ℓ_∞. Then the following statements are equivalent:*

(i) *X is universally left-stable;*
(ii) *X is isomorphic to ℓ_∞;*
(iii) *X is universally $(\lambda, 4\lambda)$-left-stable, where $\lambda = \mathrm{dist}(X, \ell_\infty)$.*

Proof. (i)⇒(ii). Since $\dim X = \infty$ and since it is isomorphic to a subspace of ℓ_∞, we have

$$\mathrm{card}(X) \geq \aleph = \aleph^{\mathbb{N}} = \mathrm{card}(\mathbb{R}^{\mathbb{N}}) = \mathrm{card}(\ell_\infty) \geq \mathrm{card}(X). \tag{3.2}$$

Assume that X is universally left-stable. We can put an equivalent norm $|||\cdot|||$ on ℓ_∞ such that X is isometric to a closed subspace of $(\ell_\infty, |||\cdot|||)$. Indeed, let $T : X \to \ell_\infty$ be a linear embedding and let $|\cdot|$ on $Z \equiv T(X)$ be defined by $|z| = \|x\|$ for all $z = Tx \in Z$. Then, we choose a sufficiently large $\lambda > 0$ and define $|||\cdot|||$ on ℓ_∞ by $|||u||| = \inf\{|v| + \lambda\|u - v\| : v \in Z\}$. Clearly, the norm $|||\cdot|||$ has the property we desired. Applying Lemma 2.1, we observe that X is complemented in $(\ell_\infty, |||\cdot|||)$, hence in ℓ_∞. By Lindenstrauss' theorem [16], X is isomorphic to ℓ_∞.

(ii)⇒(iii). Suppose that X is isomorphic to ℓ_∞. Since ℓ_∞ is 1-injective (Remark 3.2), X is necessarily λ-injective ($\lambda = \mathrm{dist}(X, \ell_\infty)$). By Theorem 3.3, X is universally $(\lambda, 4\lambda)$-left-stable.

(iii)⇒(i). This is trivial. ■

A separable Banach space X is said to be *separably injective* if it has the following extension property: Every bounded linear operator from a closed subspace of a separable Banach space into X can be extended to a bounded operator on the whole space. In 1941, Sobczyk [22] showed that c_0 is separably injective, and Zippin ([25], 1977) further proved that c_0 is, up to isomorphism, the only separable separably injective space.

With the aid of Zippin's theorem, we can prove the following theorem, which says that c_0 is (up to isomorphism) the only space satisfying inequality (2.1) for every separable Y.

THEOREM 3.5. *Let X be a separable Banach space. Then the following statements are equivalent:*

(i) *(X, Y) is stable for every separable Banach space Y;*
(ii) *X is isomorphic to c_0;*
(iii) *(X, Y) is $(2\alpha, 8\alpha)$-stable for every separable Banach space Y, where $\alpha = \|T\|\,\|T^{-1}\|$ for any isomorphism $T : X \to c_0$.*

Proof. (i)⇒(ii). Suppose that X is not isomorphic to c_0. Then by Zippin's theorem, X is not separably injective. Therefore, there exists a separable Banach space Y which contains X as an uncomplemented subspace. Clearly, $\mathrm{card}(X) = \mathrm{card}(Y)$. By Lemma 2.1 again, for every $\varepsilon > 0$, there is a standard ε-isometry $f : X \to Y$ which is not stable.

(ii)⇒(iii). Let X be a Banach space isomorphic to c_0 and $T : X \to c_0$ be an isomorphism. Assume that $(e_n)_{n=1}^{\infty}$ is the canonical basis of c_0 with the standard biorthogonal functionals $(e_n^*)_{n=1}^{\infty} \subset \ell_1$. Let $(x_n) \subset X$ satisfy $Tx_n = e_n$ for all $n \in \mathbb{N}$, and let $T^* : \ell_1 \to X^*$ be the conjugate operator of T. Then

$$Tx = \sum (T^* e_n^*)(x) e_n \quad \text{and} \quad x = \sum (T^* e_n^*)(x) T^{-1} e_n, \qquad \text{for all } x \in X.$$

Let $\alpha = \|T\| \cdot \|T^{-1}\|$, $x_n^* = T^* e_n^* \in \|T\| B_{X^*}$ for all $n \in \mathbb{N}$, and note $x_n = T^{-1} e_n \in X$. By Theorem 1.3 there exists $\phi_n \in \|T\| B_{Y^*}$ with $\|\phi_n\| = \|x_n^*\|$ such that

$$|\langle \phi_n, f(x)\rangle - \langle x_n^*, x\rangle| \le 4\varepsilon \|T\| \quad \text{for all } x \in X. \tag{3.3}$$

Since $e_n^* \to 0$ in the w^*-topology of $\ell_1 = c_0^*$, we have $x_n^* = T^* e_n^* \to 0$ in the w^*-topology of X^*. Let

$$K = \{\psi \in \|T\| B(Y^*) : |\langle \psi, f(x)\rangle| \le 4\varepsilon \|T\| \text{ for all } x \in X\}. \tag{3.4}$$

Then K is a nonempty w^*-closed compact subset of Y^*. Since Y is separable, $(\|T\| B_{Y^*}, w^*)$ is metrizable. Let ρ be a metric such that $(\|T\| B_{Y^*}, \rho)$ is isomorphic to $(\|T\| B_{Y^*}, w^*)$. Since $(\|T\| B_{Y^*}, \rho)$ is a compact metric space and since K is a compact subset of it, $(\phi_n) \subset K$ has at least one sequential ρ-cluster point. Since (x_n^*) is a w^*-null sequence in X^*, inequality (3.3) entails that any ρ-cluster point ϕ of (ϕ_n) is in K and $\|\phi\| \le \|T^*\| = \|T\|$. This further implies that $\operatorname{dist}_\rho(\phi_n, K) \to 0$. Consequently, there is a sequence $(\psi_n) \subset K$ such that $\operatorname{dist}_\rho(\phi_n, \psi_n) \to 0$, or equivalently $\phi_n - \psi_n \to 0$ in the w^*-topology of Y^*. Hence, for every $y \in Y$,

$$Uy \equiv \sum_{n=1}^{\infty} \langle \phi_n - \psi_n, y\rangle e_n \in c_0 \tag{3.5}$$

and

$$\|Uy\| \le \Big(\sup_{n\in\mathbb{N}} \|\phi_n - \psi_n\|\Big) \|y\| \le 2\|T\| \, \|y\|, \tag{3.6}$$

that is, $\|U\| \le 2\|T\|$.

Finally, let

$$S(y) = T^{-1}(Uy) = \sum_{n=1}^{\infty} \langle \phi_n - \psi_n, y\rangle x_n \quad \text{for all } y \in Y. \tag{3.7}$$

Then

$$\|S\| = \|T^{-1} U\| \le 2\|T\| \cdot \|T^{-1}\| = 2\alpha$$

and

$$
\begin{aligned}
\|Sf(x)-x\| &= \Big\|\sum_{n=1}^{\infty}\langle\phi_n-\psi_n,f(x)\rangle x_n-\sum_{n=1}^{\infty}\langle x_n^*,x\rangle x_n\Big\| \\
&= \lim_{n\to\infty}\Big\|\sum_{i=1}^{n}\langle\phi_i-\psi_i,f(x)\rangle x_i-\sum_{i=1}^{n}\langle x_i^*,x\rangle x_i\Big\| \\
&= \lim_{n\to\infty}\Big\|\sum_{i=1}^{n}(\langle\phi_i,f(x)\rangle-\langle x_i^*,x\rangle)x_i-\sum_{i=1}^{n}\langle\psi_i,f(x)\rangle x_i\Big\| \\
&\le \limsup_{n\to\infty}\Big\|\sum_{i=1}^{n}(\langle\phi_i,f(x)-\langle x_i^*,x\rangle)x_i\Big\|+\limsup_{n\to\infty}\Big\|\sum_{i=1}^{n}\langle\psi_i,f(x)\rangle x_i\Big\| \\
&= \limsup_{n\to\infty}\Big\|T^{-1}\sum_{i=1}^{n}(\langle\phi_i,f(x)\rangle-\langle x_i^*,x\rangle)e_i\Big\|+\limsup_{n\to\infty}\Big\|T^{-1}\Big(\sum_{i=1}^{n}\langle\psi_i,f(x)\rangle e_i\Big)\Big\| \\
&\le \|T^{-1}\|\limsup_{n\to\infty}\Big(\Big\|\sum_{i=1}^{n}(\langle\phi_i,f(x)\rangle-\langle x_i^*,x\rangle)e_i\Big\|+\Big\|\sum_{i=1}^{n}\langle\psi_i,f(x)\rangle e_i\Big\|\Big) \\
&\le \|T^{-1}\|\Big(\sup_n|\langle\phi_i,f(x)\rangle-\langle x_i^*,x\rangle|+\sup_n|\langle\psi_i,f(x)\rangle|\Big) \\
&\le 8\varepsilon\|T\|\cdot\|T^{-1}\|=8\varepsilon\alpha.
\end{aligned}
$$

Thus, our proof is complete. ∎

Acknowledgements. The authors would like to thank the referee for his (her) insightful and helpful suggestions on this paper.

Research of L. X. Cheng was supported in part by NSFC, grants 11071201 & 11371296.

Research of D. X. Dai was supported in part by a fund of China Scholarship Council and Texas A&M University.

Research of Y. B. Dong was supported in part by NSFC, grant 11201353.

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Received April 16, 2013
Revised version November 16, 2013 (7777)