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On continuity of measurable group representations and homomorphisms

by

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Abstract. Let G be a locally compact group, and let U be its unitary representation on a Hilbert space H. Endow the space $\mathcal{L}(H)$ of bounded linear operators on H with the weak operator topology. We prove that if U is a measurable map from G to $\mathcal{L}(H)$ then it is continuous. This result was known before for separable H. We also prove that the following statement is consistent with ZFC: every measurable homomorphism from a locally compact group into any topological group is continuous.

Let G be a locally compact group. We consider its *unitary representations*, that is, homomorphisms U from G into the group $\mathcal{U}(H)$ of unitary operators on a Hilbert space H. One gets a rich representation theory if the representations considered are *weakly continuous*, i.e. such that for every $x, y \in H$ the coefficient $f(t) = \langle U(t)x, y\rangle$ is a continuous function on G. This requirement is equivalent to *strong continuity*, i.e. continuity of the function $F(t) = \|U(t)x\|$ for every $x \in H$. Representations satisfying any of these conditions will be further called *continuous*.

In certain cases it happens that every representation is automatically continuous, as, notably, every finite-dimensional unitary representation of a connected semisimple Lie group. This theorem was proved for compact groups by van der Waerden [27] and in the general case by A. I. Shtern [26]. But in general it is easy to construct discontinuous representations, so for automatic continuity, one has to assume some sort of measurability at least. A commonly used notion is as follows. Say that a representation U of a locally compact group G on a Hilbert space H is *weakly measurable* if every coefficient $f(t) = \langle U(t)x, y\rangle$ is a measurable function on G. Every weakly measurable unitary representation must be continuous if it acts on a separable Hilbert space [14, Theorem V.7.3].

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However, in general this does not imply continuity: if G is non-discrete, then the regular representation of G on the space $\ell_2(G)$ of countably summable sequences on G is weakly measurable but discontinuous. In this paper we prove that the separability restriction can be removed if we use a slightly stronger notion of measurability. Let $\mathcal{L}(H)$ be the space of bounded linear operators on the Hilbert space H, endowed with the weak operator topology (it is generated by the functions f_{xy} for all $x, y \in H$, where $f_{xy}(A) = \langle Ax, y\rangle$, $A \in \mathcal{L}(H)$). Say that U is *weakly operator measurable* if $U^{-1}(V)$ is measurable for every open set $V \subset \mathcal{L}(H)$. Now we can formulate the main result of this paper (Theorem 1.5): every weakly operator measurable unitary representation of a locally compact group is continuous.

The proof is based on a generalization of the so-called Four Poles Theorem: if $\mathcal{A}$ is a point-finite family of null sets with nonnull union in a Polish space, then there is a subfamily in $\mathcal{A}$ with a non-measurable union (this was proved initially by L. Bukovský [5] and then in a much simpler way by J. Brzuchowski, J. Cichoń, E. Grzegorek and C. Ryll-Nardzewski [4]). In Lemma 1.4, we prove the same result for subsets of any locally compact group, with the restriction that the cardinality of $\mathcal{A}$ is not more than continuum.

The second part of the paper deals with automatic continuity of more general group homomorphisms. This question is most actively studied for homomorphisms between Polish groups; see a recent review of C. Rosendal [25]. The notion of Haar measurability of $f : G \to H$ is here replaced by *universal measurability*: the inverse image of every open set is measurable with respect to every Radon measure on G. It is known that every universally measurable homomorphism from a locally compact or abelian Polish group into a Polish group, or from a Polish group to a metric group, is continuous. There are also generalizations to other subclasses of Polish groups by S. Solecki and Rosendal. We omit results on other types of measurability (in the sense of Souslin, Christensen etc.).

If G is not supposed to be Polish, the results are fewer. The most general statement is probably the theorem of A. Kleppner [22]: every measurable homomorphism between two locally compact groups is continuous. It has been generalized to some special classes of groups by J. Brzdęk [3]. If one makes no assumptions on the image group, it seems inevitable to impose additional set-theoretic axioms instead. The only result in this direction known to the author belongs to J. P. R. Christensen [6]: under Luzin's hypothesis, every Baire, in particular, every Borel measurable homomorphism from a Polish group to any topological group is continuous. Our Theorem 2.5 is proved under Martin's axiom (MA): every measurable homomorphism from a locally compact group to any topological group is continuous.

Theorem 2.5 is reduced to the following question. Let G be a locally compact group; call a set $A \subset G$ *extra-measurable* if SA is measurable for any $S \subset G$. An obvious example of an extra-measurable set is any open set. Existence of discontinuous measurable homomorphisms implies existence of null extra-measurable sets; but under MA, as Theorem 2.4 shows, the latter do not exist, so every measurable homomorphism is continuous. In the commutative case, the question of automatic continuity is even equivalent to the existence of a certain sequence of null extra-measurable sets (Proposition 2.2).

1. Continuity of unitary representations

Definitions and notations. On a locally compact group G, we fix a left Haar measure μ and the corresponding outer measure μ^*. A map $f : G \to Y$, where Y is a topological space, is called *measurable* if $f^{-1}(Y)$ is Haar measurable for every open set $U \subset Y$. For a set A, $|A|$ denotes its cardinality.

There are two approaches to the construction of Haar measures. One, used by E. Hewitt and K. A. Ross [16], yields an outer regular measure: for every measurable set E, one has $\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\}$. Another one, taken by D. H. Fremlin [13], leads to an inner regular measure: $\mu(E) = \sup\{\mu(F) : F \subset E,\ F \text{ compact}\}$.

In the σ-finite case, in particular, on a σ-compact group, both constructions give the same resulting measure, which is both inner and outer regular. If G is not σ-compact, the approach of [16] gives rise to the following pathological sets. A set $A \subset G$ is called *locally null* [16, 11.26] if $\mu(A \cap K) = 0$ for every compact set $K \subset G$. Equivalently, A does not contain any set of positive finite measure. Of course, if A is null then A is locally null. Every locally null set A is measurable, and either $\mu(A) = 0$ or $\mu(A) = \infty$. In Fremlin's treatment, there are no locally null nonnull sets.

The results of this paper, in particular Theorem 1.5, are valid for both definitions of the Haar measure.

It is known [14, IV.2.16 and V.7.2] that every unitary representation of a locally compact group may be decomposed into a direct sum $U = U_1 \oplus U_2$, where U_1 is continuous and every coefficient of U_2 is (locally) almost everywhere zero. We will say that U_2 is *singular*. If U acts on a separable space then $U_2 = 0$ [14, Theorem V.7.3].

Let U act on a Hilbert space H. Endow the space $\mathcal{L}(H)$ of bounded linear operators on H with the weak operator topology (generated by the functions f_{xy} for all $x, y \in H$, where $f_{xy}(A) = \langle Ax, y\rangle$, $A \in \mathcal{L}(H)$). If U is a measurable map from G to $\mathcal{L}(H)$, we will say that U is *weakly operator measurable*.

The following lemma is known [13, 443P].

LEMMA 1.1. *Let G be a σ-compact locally compact group, K its compact normal subgroup, and let $\pi : G \to G/K$ be the quotient map. If $A \subset G$ is such that $A = AK$ then A is measurable* (*resp. null*) *in G if and only if $\pi(A)$ is measurable* (*resp. null*) *in G/K.*

The following two facts will be used in further proofs several times, so we prefer to state them separately.

REMARK 1.2 (Pro-Lie and Polish groups). Recall that a topological group is called *pro-Lie* if it is an inverse (projective) limit of (finite-dimensional) Lie groups (see [18]). It is known that in every locally compact group there is an open pro-Lie subgroup [18, p. 165]. If G is a locally compact group and $G = \varprojlim_{i\in I} G_i$, where every G_i is a Lie group, then these groups can be chosen as $G_i = G/K_i$, where every K_i is a compact normal subgroup of G, and the order on I is just inclusion of K_i. Every σ-compact Lie group is Polish (it is first countable, hence metrizable [17, Theorem A4.16], and further apply [2, Chapter IX, §6, Corollary of Proposition 2]). If all G_i are σ-compact and I is countable, G is Polish too ([2, §6, Proposition 1a,b]).

REMARK 1.3 (Baire sets in direct products). Baire sets ([15, §51]) are the elements of the σ-algebra generated by all compact G_δ-sets. In the case of a σ-compact locally compact group (the only case we will need), this is also the σ-algebra generated by all zero sets of continuous functions (in [16, 11.1], this latter property is taken as a definition). Consider the direct product of a family of locally compact groups, $\mathcal{G} = \prod_{j\in J} G_j$. This is a topological group, which is not necessarily locally compact. Let $G \subset \mathcal{G}$ be a closed σ-compact subgroup. For any $I \subset J$ let $\pi_I : \mathcal{G} \to \prod_{j\in I} G_j$ be the natural projection. We say that a set $X \subset G$ *depends on the coordinates* $I \subset J$ if $X = G \cap \pi_I^{-1}(\pi_I X)$. If $F = \bigcap U_n$ is a compact G_δ set in G, then for every n the open set U_n can be chosen as a finite union of basic neighbourhoods in $\mathcal{G}$, each depending on a finite number of coordinates. It follows that every such F, and as a consequence every Baire set, depends on a countable set of coordinates.

LEMMA 1.4. *Let $\mathcal{A} = \{A_s : s \in S\}$ be a point-finite family of null subsets of a σ-compact locally compact group G. If $|\mathcal{A}| \le \mathfrak{c}$ and $\bigcup \mathcal{A}$ is nonnull, then there is $\mathcal{B} \subset \mathcal{A}$ such that $\bigcup \mathcal{B}$ is nonmeasurable.*

Proof. Let $H \subset G$ be an open pro-Lie subgroup, which we can assume to be compactly generated. Since G is a countable union of H-cosets, there is $t_0 \in G$ such that $(\bigcup \mathcal{A}) \cap t_0 H$ is nonnull. Define $A'_s = (t_0^{-1} A_s) \cap H$ for all s; then the family $\mathcal{A}' = \{A'_s : s \in S\}$ satisfies all the conditions of the lemma and is contained in H. Moreover, if a union $\bigcup\{A'_s : s \in T\}$ is nonmeasurable, then so is $\bigcup\{A_s : s \in T\}$. Thus, we can assume that $G = H$, i.e. $G = \varprojlim_{i\in I} G_i$ is σ-compact and pro-Lie. In this case every $G_i = G/K_i$ is a σ-compact Lie group, hence Polish. We assume that every G_i is nontrivial, otherwise $G = \{1\}$ and the family $\mathcal{A}$ would not exist.

We can assume that $S \subset \mathbb{R}$. Let $\mathbb{Q} = \{q_m : m \in \mathbb{N}\}$ be an enumeration of the rationals, and let $W_{mn} = \bigcup\{A_s : |s - q_m| < 1/n\}$. For every $s \in S$, choose sequences $n_k^{(s)}, m_k^{(s)}$ so that $|s - q_{m_k^{(s)}}| < 1/n_k^{(s)}$ and $n_k^{(s)} \to \infty$ while $n_{k+1}^{(s)} > n_k^{(s)}$ for all k. Then $A_s = \bigcap_k W_{n_k^{(s)} m_k^{(s)}}$ for every s. Indeed, every point $x \in A_s$ is contained in this intersection; if $x \notin A_s$ then $x \in A_{t_i}$ for an at most finite set of points $t_i \neq s$; and every t_i can be separated from s by some interval $|q_{m_k^{(s)}} - s| < 1/n_k^{(s)}$, so that $x \notin W_{n_k^{(s)} m_k^{(s)}}$.

If one of the sets W_{mn} is nonmeasurable, the lemma is proved. Suppose that every W_{mn} is measurable. By [16, 19.30b], there exists a Baire set $B_{mn} \subset W_{mn}$ such that $N_{mn} = W_{mn} \setminus B_{mn}$ is null. Further, for every n, m there is a null Baire set $N'_{mn} \supset N_{mn}$. Let $N = \bigcup_{m,n} N'_{mn}$. Then N is a Baire set, so $W_{mn} \setminus N = B_{mn} \setminus N$ is Baire for all m, n. Let $W_{mn} \setminus N$ depend on the countable set of coordinates J_{mn}. Then every $A_s \setminus N = \bigcap_k (W_{n_k^{(s)} m_k^{(s)}} \setminus N)$ depends on the coordinates $J = \bigcup_{m,n} J_{mn}$, and the set J is countable.

Extending J if necessary, we can assume that the family $\{K_j : j \in J\}$ is closed under finite intersections. Denote $K = \bigcap_{j\in J} K_j$. Then $G/K = \varprojlim_{j\in J} G/K_j$ is a Polish group. Let $\pi : G \to G/K$ be the quotient map, and put $A'_s = \pi(A_s \setminus N)$. Then, since $A_s \setminus N = (A_s \setminus N)K$, the family $\mathcal{A}' = \{A'_s : s \in S\}$ is point-finite, and by Lemma 1.1 we see that A'_s is null for all s, while $\bigcup \mathcal{A}' = \pi(\bigcup \mathcal{A})$ is nonnull. By the Four Poles Theorem for the Polish case [4] we get $\mathcal{B}' \subset \mathcal{A}'$ such that $\bigcup \mathcal{B}'$ is nonmeasurable. Put $\mathcal{B} = \{A_s : A'_s \in \mathcal{B}'\}$. Then $\bigcup \mathcal{B} \setminus N = \pi^{-1}(\cup \mathcal{B}')$ is nonmeasurable, so $\mathcal{B}$ is as desired. ∎

A simple example shows that in the Hewitt & Ross approach, this theorem is not true for a non-σ-compact group. Let $\mathbb{R}_d$ be the real line with the discrete topology, and consider the direct product $\mathbb{R}_d \times \mathbb{R}$. Then $X = \mathbb{R}_d \times \{0\}$ is measurable of infinite measure (this is an example of a locally null, nonnull set [16, 11.33]). Every uncountable subset of X is also measurable with infinite measure, and every countable subset is null. Thus, if we put $A_t = \{(t, 0)\}$ and $\mathcal{A} = \{A_t : t \in \mathbb{R}_d\}$, then every A_t is null, $\bigcup \mathcal{A}$ is nonnull, but every subfamily of $\mathcal{A}$ has a measurable union. In Fremlin's approach, this example does not appear.

Theorem 1.5. *Let G be a locally compact group. Then every weakly operator measurable unitary representation of G is continuous.*

Proof. Let $U : G \to \mathcal{L}(H)$ be a representation acting on a Hilbert space H. Clearly U is continuous if and only if its restriction to any open subgroup is continuous. In G, there is an open compactly generated pro-Lie subgroup (e.g., the intersection of an open pro-Lie subgroup and the subgroup generated by a pre-compact neighbourhood of identity). So we can assume that $G = \varprojlim_{i\in I} G/K_i$ is itself compactly generated and pro-Lie; in particular, G is σ-compact.

Take any $x \in H$ with $\|x\| = 1$. Put $f(t) = \langle U(t)x, x\rangle$ and $S = \{t \in G : f(t) \neq 0\}$. We can assume that U is singular; then S is null. Suppose towards a contradiction that $U \not\equiv 0$; then $e \in S$.

Let us show that S has a null projection onto a Polish quotient of G. By [16, 19.30b], there exists a null Baire set $B \supset S$. Every Baire set (Remark 1.3) depends on a countable number of coordinates. Let $J \subset I$ be a countable set such that $B = \pi_J^{-1}\pi_J B$. By extending J if necessary, we can assume that the family $\{K_j : j \in J\}$ is closed under finite intersections. Define $K = \bigcap_{j\in J} K_j$. Then $G/K = \varprojlim_{j\in J} G/K_j$ is a Polish group. Let $\pi : G \to G/K$ be the quotient map and let $S' = \pi(S)$; then $S' \subset \pi(B)$ is null.

Choose an enumeration (probably with repetitions) $\{P_\alpha : \alpha < \mathfrak{c}\}$ of perfect nonnull sets in G/K. It is known that there are at most continuum many such sets. By induction, we will choose points t_α, $\alpha < \mathfrak{n}$, with some ordinal $\mathfrak{n} \le \mathfrak{c}$ so that $\bigcup\{t_\alpha S' : \alpha < \mathfrak{n}\}$ is nonnull (in G/K) and $t_\alpha \notin \bigcup\{t_\beta S' : \beta < \alpha\}$ for every $\alpha > 0$. Set $t_0 = e$. For every α, let $T_\alpha = \{t_\beta : \beta < \alpha\}$. If $T_\alpha S'$ is nonnull, stop the procedure. Otherwise $P_\alpha \setminus T_\alpha S' \neq \emptyset$, and choose t_α as any point of this set. Let $\mathfrak{n}$ be the ordinal on which we stopped the induction, or $\mathfrak{n} = \mathfrak{c}$ if it was not stopped. Set $T = T_\mathfrak{n}$. If $\mathfrak{n} < \mathfrak{c}$ then as assumed $\mu^*(TS') > 0$; if $\mathfrak{n} = \mathfrak{c}$ then TS' intersects every nonnull perfect set in G/K, so it is of full measure. In either case TS' is nonnull.

For every $\alpha < \mathfrak{n}$, choose any $z_\alpha \in \pi^{-1}(t_\alpha)$ and set $Z = \{z_\alpha : \alpha < \mathfrak{n}\}$. It follows that $ZSK = \pi^{-1}(\pi(ZS)) = \pi^{-1}(TS')$ is nonnull in G. Recall that K is a normal subgroup, so $ZSK = ZKS$. Define now

$$S_n = \{t \in G : |f(t)| = |\langle U(t)x, x\rangle| > 1/n\}. \tag{1.1}$$

Then $S = \bigcup_n S_n$ and $ZKS = \bigcup_n ZKS_n$. It follows that ZKS_N is nonnull for some $N \in \mathbb{N}$.

We claim that the family $\mathcal{A} = \{z_\alpha KS_N : \alpha < \mathfrak{n}\}$ is point-finite. To prove this, we first show that $U(z_\alpha k_2)x \perp U(z_\beta k_1)x$ for any $\alpha \neq \beta$ and any $k_1, k_2 \in K$. Suppose that $\alpha > \beta$. Then $(z_\beta k_1)^{-1} z_\alpha k_2 \notin S$, because otherwise we would have $z_\alpha \in z_\beta k_1 S k_2^{-1} \subset z_\beta KSK = z_\beta SK^2 = z_\beta SK$ and hence $t_\alpha = \pi(z_\alpha) \in \pi(z_\beta SK) = t_\beta S'$, which is impossible by the choice of t_α. This gives us

$$0 = f((z_\beta k_1)^{-1} z_\alpha k_2) = \langle U((z_\beta k_1)^{-1} z_\alpha k_2)x, x\rangle = \langle U(z_\alpha k_2)x, U(z_\beta k_1)x\rangle,$$

that is, $U(z_\alpha k_2)x \perp U(z_\beta k_1)x$.

Next, if $t \in z_\alpha KS_N$ then there is $k_\alpha \in K$ such that $(z_a k_a)^{-1} t \in S_N$, i.e. $|\langle U(t)x, U(z_\alpha k_\alpha)x\rangle| > 1/N$. As we have shown above, $U(z_\alpha k_\alpha)x$ are orthogonal for different α; since U is unitary, they have norm 1. By Bessel's inequality we have, for any $t \in G$,

$$1 = \|x\|^2 = \|U(t)x\|^2 \geq \sum_{\alpha:\, t \in z_\alpha KS_N} |\langle U(t)x, U(z_\alpha k_\alpha)x\rangle|^2$$
$$> N^{-2} |\{\alpha : t \in z_\alpha KS_N\}|.$$

So $\mathcal{A}$ is point-finite. It has cardinality $|\mathcal{A}| = |Z| = \mathfrak{n} \leq \mathfrak{c}$ and a nonnull union $\bigcup \mathcal{A} = ZKS_N$. Every $z_\alpha KS_N \subset z_\alpha KS = z_\alpha \pi^{-1}(S')$ is a null set. Thus, we can apply Lemma 1.4 to get $\mathcal{B} \subset \mathcal{A}$ such that $\bigcup \mathcal{B}$ is nonmeasurable. Now recall that by formula (1.1), S_N is the inverse image of an open set in $\mathcal{L}(H)$. The same is true for every translate of S_N and for unions of such translates, in particular for every $z_\alpha KS_N$ and for $\bigcup \mathcal{B}$. Being the inverse image of an open set, $\bigcup \mathcal{B}$ must be measurable. This contradiction proves the theorem. ∎

2. Continuity of group homomorphisms. The content of this section is valid for both treatments of the Haar measure. For the inner regular variant adopted in [13], it suffices to ignore the bracketed "locally" everywhere.

Let G be a locally compact group. Call a set $A \subset G$ *extra-measurable* if SA is measurable for every set $S \subset G$. Every open set is extra-measurable, while a one-point set is not, unless the group is discrete. As shown below (Theorem 2.1), existence of discontinuous measurable homomorphisms implies existence of [locally] null (definition below) extra-measurable sets. It is consistent with ZFC (Theorem 2.4) that a nonempty [locally] null set cannot be extra-measurable, so it is consistent that every measurable homomorphism from a locally compact group to any topological group is continuous. It is an open question whether this statement is true in ZFC without any additional axioms. Already in the basic case of the real line the answer is unknown, but for commutative groups one can make the question more precise (Proposition 2.2).

THEOREM 2.1. *Let G be a locally compact group. If there exists a homomorphism $\varphi : G \to H$ to a topological group H which is measurable but discontinuous, then there is a family $\mathcal{A}$ of nonempty* [*locally*] *null extra-measurable sets such that for every $A \in \mathcal{A}$:*

$$(2.1) \qquad A^{-1} = A; \quad \exists B \in \mathcal{A} : B^2 \subset A; \quad \forall x \in G\ \exists C \in \mathcal{A} : x^{-1}Cx \subset A.$$

Proof. Suppose that such a φ exists. Let U be an open neighbourhood of identity in H and let $A = \varphi^{-1}(U)$. Then for any $S \subset G$ we have $SA = \varphi^{-1}(\varphi(S)U)$. Indeed, the inclusion $\varphi(SA) \subset \varphi(S)\varphi(A) = \varphi(S)U$ is obvious. For the opposite inclusion, take $z \in \varphi^{-1}(\varphi(S)U)$ and choose $s \in S$, $a \in A$

such that $\varphi(z) = \varphi(s)\varphi(a) = \varphi(sa)$. Let $(sa)^{-1}z = t$; then $t \in \ker\varphi$. Since $\varphi(at) = \varphi(a) \in U$, we have $at \in A$ and $z = sat \in SA$.

Now SA is the inverse image of an open set $\varphi(S)U$, so it must be measurable. Thus, A is extra-measurable.

Suppose that $\varphi^{-1}(U)$ is not [locally] null for every U. Take an open neighbourhood of identity V such that $V^{-1}V \subset U$. Then $B = \varphi^{-1}(V)$ is by assumption also non-[locally] null. It then contains a set C with $0 < \mu(C) < \infty$, so $C^{-1}C$ contains a neighbourhood of identity $W \subset G$ [16, 20.17]. Then $\varphi(W) \subset \varphi(C^{-1}C) \subset V^{-1}V \subset U$, so $\varphi^{-1}(U) \supset W$. Since U was arbitrary, this implies that φ is continuous.

Thus, under the assumptions of the theorem there is U such that $\varphi^{-1}(U)$ is [locally] null. Let $\mathcal{V}$ be a base of neighbourhoods of the identity in H such that $V \subset U$ and $V = V^{-1}$ for every $V \in \mathcal{V}$. Denote $\mathcal{A} = \{\varphi^{-1}(V) : V \in \mathcal{V}\}$; then this family has the properties (2.1). ∎

The conditions (2.1) guarantee that if we take $\mathcal{A}$ as a base of neighbourhoods of the identity in G, this turns G into a topological group [2, IV, §2]. However, it does not follow immediately that the identity map on G is measurable, i.e. in general we do not get a converse of this theorem. Equivalence holds in the commutative case:

Proposition 2.2. *Let G be a commutative locally compact group. The following are equivalent:*

(i) *There is a homomorphism $\varphi : G \to H$ to a topological group H which is measurable but discontinuous.*
(ii) *There is a sequence of [locally] null extra-measurable sets A_n such that, for every n, $A_n^{-1} = A_n$ and $A_{n+1}^2 \subset A_n$.*

Proof. (i)⇒(ii): Proved in Theorem 2.1.

(ii)⇒(i): Take the sets A_n as a base of neighbourhoods of the identity in G; this turns G into a topological group which we can denote H. (Note that H is metrizable if $\bigcap A_n = \emptyset$.) The identity map $\varphi : G \to H$ is obviously discontinuous. At the same time, for every open set $U \subset H$ we have $U = \bigcup T_n A_n$ for some sets T_n; all $T_n A_n$ and hence $U = \varphi^{-1}(U)$ are measurable, so φ is a measurable map and (i) holds. ∎

Existence of sets as in Proposition 2.2(ii) is an open question even on the real line. Known results on automatic continuity mostly concern Polish groups; here they do not give a ready answer, since the group H obtained in the proof may not be complete (i.e. not Polish).

Let $\operatorname{add}(\mathcal{N})$ be the minimal cardinality of a family $\mathcal{J}$ of null sets on the real line G such that $\bigcup \mathcal{J}$ is nonnull. This is called the *additivity* of the ideal $\mathcal{N}$ of Lebesgue null sets in G. It is known that additivity of the ideal of Haar null sets is the same for every nondiscrete locally compact Polish group [13, 522Va]. It is consistent with ZFC that $\mathrm{add}(\mathcal{N}) < \mathfrak{c}$, but it follows from Martin's axiom (MA) that $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$ (see [12]). This is, in fact, the assumption that we use in our proof. It is known that Martin's axiom follows from the continuum hypothesis, but is also consistent with its negation. For further discussion of Martin's axiom, we refer to Fremlin's monograph [12].

A. Kharazishvili has indicated in private correspondence that the following statement holds for a commutative Polish group:

LEMMA 2.3. (MA) *Let G be a locally compact Polish group, and let $A \subset G$ be a nonempty set of measure zero. Then there is a set $S \subset G$ such that SA is nonmeasurable.*

Proof. If G is countable, then by local compactness it has all points isolated, and the measure of every point is positive. Then the set A in the assumption cannot exist. Thus, G is uncountable without isolated points. Note that G is σ-compact [11, Theorems 3.3.1, 3.8.1, 3.8.C(b)].

We will construct S so that both SA and $G \setminus SA$ intersect every perfect set of positive measure. Then SA must be nonmeasurable, since the inner measure of SA and $G \setminus SA$ is zero.

By translating A, and then S, if necessary, one can assume that $e \in A$. Note that A^{-1} also has measure zero—this follows, e.g., from [16, 20.2] or [13, 442K].

Since G is separable and uncountable, there are exactly continuum many closed sets in it. Let $\{P_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all perfect sets of positive measure. By induction, we will choose $s_\xi, d_\xi \in P_\xi$ so that the condition $d_\xi \in P_\xi \setminus s_\eta A$ holds for every ξ, η. Then $S = \{s_\xi : \xi < \mathfrak{c}\}$ will be as needed, since $s_\xi \in S \cap P_\xi \subset SA \cap P_\xi$ and $d_\xi \in P_\xi \setminus SA$, so both $P_\xi \cap (SA)$ and $P_\xi \cap (G \setminus SA)$ are nonempty.

Suppose that for all $\eta < \xi$ such s_η, d_η have been chosen, or that $\xi = 0$ (the base of induction). Set $D_\xi = \{d_\eta : \eta < \xi\}$ and note that $|D_\xi| < \mathfrak{c}$. Since P_ξ cannot be covered by less than continuum many translates of A^{-1} (here we use Martin's axiom), we can choose a point $s_\xi \in P_\xi \setminus D_\xi A^{-1} \neq \emptyset$. This implies $(s_\xi A) \cap D_\xi = \emptyset$.

Next, set $S_\xi = \{s_\eta : \eta \leq \xi\}$. Then $|S_\xi| < \mathfrak{c}$ and similarly we can choose $d_\xi \in P_\xi \setminus S_\xi A$. By this choice, we have $d_\xi \notin s_\eta A$ for all $\eta \leq \xi$, and for $\eta > \xi$ we have $d_\xi \notin s_\eta A$ by the choice of s_η. This concludes the proof. ∎

THEOREM 2.4. (MA) *Let G be a locally compact group, and let $A \subset G$ be a nonempty* [*locally*] *null set. Then there is a set $S \subset G$ such that SA is nonmeasurable.*

Proof. Let H be an open pro-Lie subgroup of G. Clearly, H can be chosen σ-compact (e.g., generated by any pre-compact neighbourhood of the identity).

Translating A if necessary, we can assume that $e \in A$. Then $A_1 = A \cap H$ is nonempty and [locally] null with respect to the Haar measure of H, and due to σ-compactness it is just null in H. If we find a set $S \subset H$ such that SA_1 is nonmeasurable in H, then $(SA) \cap H = SA_1$ is nonmeasurable in G, and so SA is nonmeasurable too. We can therefore assume that $G = H$, that is, G is σ-compact and pro-Lie, and A is null.

As in the proof of Theorem 1.5, either G is Polish (and we can apply Lemma 2.3), or we can find a Polish quotient G/K such that $\pi(A)$ is null, where $\pi : G \to G/K$ is the quotient map. By Lemma 2.3, there is a set $S_1 \subset G/K$ such that $S_1\pi(A)$ is nonmeasurable. By Lemma 1.1, $\pi^{-1}(S_1\pi(A)) = \pi^{-1}(S_1)A$ is also nonmeasurable. Thus, we can take $S = \pi^{-1}(S_1)$, and the theorem is proved. ∎

This result together with Theorem 2.1 implies:

Theorem 2.5. (MA) *Every measurable homomorphism from a locally compact group to any topological group is continuous.*

In conclusion, let us review some closely related results. Say that a set S is *small* if the union of every family of translates of S of cardinality less than continuum is null. We use Martin's axiom to guarantee that every null set is small. Without MA, this depends on the set S. As proved by Gruenhage [9], the ternary Cantor set is small, and Darji and Keleti showed that every subset of $\mathbb{R}$ of packing dimension less than 1 is small. On the other hand, Elekes and Tóth [10] and Abért [1] proved the following: it is consistent with ZFC that in every locally compact group there is a nonsmall compact set of measure zero. It is however unknown whether for a nonsmall set the statement of Theorem 2.4 is false.

Finally, we say a few words on results in ZFC concerning nonmeasurable products of sets. One should better say "sums of sets" because there is a tradition to do everything in the commutative case. This restriction is reasonable since the principal difficulties appear already in the case of the real line. The advances most close to our topic are: For every null set S on the real line such that $S+S$ has positive outer measure there is a set $A \subset S$ such that $A+A$ is nonmeasurable (Ciesielski, Fejzić and Freiling [8]). Cichoń, Morayne, Rałowski, Ryll-Nardzewski, and Żeberski [7] proved that there is a subset A of the Cantor set C such that $A+C$ is nonmeasurable, and under additional axioms they proved the same statement for every closed null set P such that $P+P$ has positive measure. There is also a series of results going back to Sierpiński which exhibit null sets A and B such that $A+B$ is nonmeasurable (see, e.g., the monograph [19] and the recent papers of Kharazishvili and Kirtadze [21], [20]), where the task is to make A and B "maximally negligible" (in different senses), and $A+B$ "maximally nonmeasurable".

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