STUDIA MATHEMATICA 226 (1) (2015)

Linear combinations of generators in multiplicatively invariant spaces

by

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Abstract. Multiplicatively invariant (MI) spaces are closed subspaces of $L^2(\Omega, \mathcal{H})$ that are invariant under multiplication by (some) functions in $L^\infty(\Omega)$; they were first introduced by Bownik and Ross (2014). In this paper we work with MI spaces that are finitely generated. We prove that almost every set of functions constructed by taking linear combinations of the generators of a finitely generated MI space is a new set of generators for the same space, and we give necessary and sufficient conditions on the linear combinations to preserve frame properties. We then apply our results on MI spaces to systems of translates in the context of locally compact abelian groups and we extend some results previously proven for systems of integer translates in $L^2(\mathbb{R}^d)$.

1. Introduction. Given a vector valued space $L^2(\Omega, \mathcal{H})$ where Ω is a σ-finite measure space and $\mathcal{H}$ is a separable Hilbert space, and given a determining set D for $L^1(\Omega)$ (see Section 3.1 for a precise definition), a *multiplicatively invariant* (MI) space is a closed subspace of $L^2(\Omega, \mathcal{H})$ that is invariant under multiplication by functions in D. A particular case of MI spaces are the well-known *doubly invariant spaces* introduced by Helson [16] and Srinivasan [25]. MI spaces as presented here were introduced in [7] where they were also characterized in terms of range functions. The reason why MI spaces appear on the scene is that they are strongly connected to *shift invariant* (SI) spaces. In the classical euclidean case, a SI space is a closed subspace in $L^2(\mathbb{R}^d)$ that is invariant under translations by integers. This type of spaces are typically considered in sampling theory [1, 26, 27, 28] and they also play a fundamental role in approximation theory as well as in frame and wavelet theory [14, 18, 22]. Shift invariant spaces have proven to be very useful models in many problems in signal and image processing. Due to their importance in theory and applications, their structure has been deeply analyzed during the last twenty five years [5, 11, 12, 16, 23].

2010 *Mathematics Subject Classification*: Primary 42C40, 43A70; Secondary 42C15, 22B99.
Key words and phrases: shift invariant space, fibers, range functions, multiplicatively invariant spaces, frame, Gramian, LCA groups.

DOI: 10.4064/sm226-1-1

Every SI space can be generated by a set Φ of functions in $L^2(\mathbb{R}^d)$ in the sense that it is the closure of the space spanned by the integer translations of the functions in Φ. When Φ is a finite set, we say that the SI space is *finitely generated*. Concerning finitely generated SI spaces, a particular problem of interest for us is the following: Suppose that $\Phi = \{\phi_1, \dots, \phi_m\}$ generates the SI space V, that is, $V = \overline{\operatorname{span}}\{T_k\phi_j : k \in \mathbb{Z},\, j = 1, \dots, m\}$. For $\ell \leq m$, let $\Psi = \{\psi_1, \dots, \psi_\ell\}$ be a set of functions constructed by taking linear combinations of the functions in Φ, i.e. $\psi_i = \sum_{j=1}^{m} a_{ij}\phi_j$ for $1 \leq i \leq \ell$. The question is: which linear combinations produce new sets of generators for V? and if in addition we know that $\{T_k\phi_j\}_{k\in\mathbb{Z},\, j=1,\dots,m}$ is a frame for V, when is $\{T_k\psi_i\}_{k\in\mathbb{Z},\, i=1,\dots,\ell}$ also a frame for V? These two questions were completely answered in [6] and [8]. The problem of plain generators was addressed in [6] where the authors proved that almost every set of functions obtained by taking linear combinations of a given set of generators of V generates V as well. Regarding the second question, in [8], the authors exactly characterized those linear combinations that transfer the frame property from $\{T_k\phi_j\}_{k\in\mathbb{Z},\, j=1,\dots,m}$ to $\{T_k\psi_i\}_{k\in\mathbb{Z},\, i=1,\dots,\ell}$.

In the present work we study the questions formulated above but for MI spaces. In our main result we show that almost every linear combination of generators of a MI space produces a new set of generators for the same space. We also characterize those linear combinations that preserve uniform frames (see Definition 3.5). Our results are thus in the spirit of those in [6, 8]. As a first step, we work with finite-dimensional subspaces. We prove that given a finite set $\mathcal{V}$ of vectors in a Hilbert space, almost every finite set of vectors constructed by taking linear combinations of the vectors in $\mathcal{V}$ spans the same subspace that $\mathcal{V}$ spans. This result will be the core of what we then prove for MI spaces, and we also believe it is of interest by itself.

As a consequence, we obtain similar results to [6, 8] but for SI spaces considered in more general contexts than $L^2(\mathbb{R}^d)$. The theory of shift invariant spaces has been extended to the setting of locally compact abelian (LCA) groups, mainly in two different directions. First, in [7, 9, 19] SI spaces are subspaces of $L^2(G)$ where G is an LCA group and the translations are taken along a subgroup H of G such that G/H is compact. The case when H is discrete was addressed in [9, 19], and in the recent paper [7] the authors worked with the non-discrete case. Second, one can consider SI spaces in $L^2(\mathcal{X})$ where $\mathcal{X}$ is a measure space and the translations are defined by the action of a discrete LCA group on $\mathcal{X}$, [4]. In both cases, SI spaces were characterized in terms of range functions, using fiberizations techniques and obtaining results that extend those proven in [5] for SI spaces in $L^2(\mathbb{R}^d)$. This last fact is what connects SI spaces with MI spaces. Thus our results in MI spaces allow us to provide a unified treatment to the problem of when linear combinations of generators in a system of translates preserve generators and frame generators in both contexts described above.

The paper is organized as follows. In Section 2 we show that generators of finite-dimensional subspaces are generally preserved by taking linear combinations. Section 3 is devoted to MI spaces. We first summarize in Section 3.1 the basic properties of MI spaces. We prove in Section 3.2 that almost every linear combination of generators of a MI space yields a new set of generators for the same space (Theorem 3.3). In Section 3.3 we address the problem of preserving uniform frames. Finally, in Section 4 we apply the result obtained for MI spaces to systems of translates.

Notation and definitions. Here we set the notation we will use in the next sections, and we recall the definition of frames and some basic results of linear algebra that will be important in what follows.

DEFINITION 1.1. Let $\mathcal{H}$ be a separable Hilbert space and $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$. The sequence $\{f_k\}_{k\in\mathbb{Z}}$ is said to be a *frame* for $\mathcal{H}$ if there exist $0 < \alpha \le \beta$ such that

$$\alpha\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le \beta\|f\|^2$$

for all $f \in \mathcal{H}$. The constants α and β are called *frame bounds*.

For a set $\mathcal{X} = \{x_1, \dots, x_n\} \subseteq \mathcal{H}$ we denote by $S(\mathcal{X})$ the subspace spanned by $\mathcal{X}$, i.e. $S(\mathcal{X}) = \operatorname{span}\{x_1, \dots, x_n\}$. The *Gramian* associated to $\mathcal{X}$ is the matrix $G_{\mathcal{X}}$ in $\mathbb{C}^{n\times n}$ whose entries are $(G_{\mathcal{X}})_{ij} = \langle x_i, x_j\rangle$. The Gramian is a positive-semidefinite matrix satisfying $G^*_{\mathcal{X}} = G_{\mathcal{X}}$.

Denote by $K_{\mathcal{X}} : \mathbb{C}^n \to \mathcal{H}$ the *synthesis operator* associated to $\mathcal{X}$ given by $K_{\mathcal{X}}c = \sum_{j=1}^n c_j x_j$, and by $K^*_{\mathcal{X}} : \mathcal{H} \to \mathbb{C}^n$ its adjoint, the *analysis operator*, given by $K^*_{\mathcal{X}}h = \{\langle h, x_j\rangle\}_{j=1}^n$. Note that the matrix of the operator $K^*_{\mathcal{X}}K_{\mathcal{X}}$ in the canonical basis on $\mathbb{C}^n$ is $G^t_{\mathcal{X}}$, the transpose of $G_{\mathcal{X}}$. It follows that

$$\begin{aligned}\operatorname{rk}(G_{\mathcal{X}}) = \operatorname{rk}(G^t_{\mathcal{X}}) &= \dim(\operatorname{Im}(K^*_{\mathcal{X}}K_{\mathcal{X}})) = \dim(\operatorname{Im}(K_{\mathcal{X}}K^*_{\mathcal{X}})) \\ &= \dim(\operatorname{Im}(K_{\mathcal{X}})) = \dim(S(\mathcal{X})).\end{aligned} \tag{1.1}$$

The set $\mathcal{X}$ is always a frame for $S(\mathcal{X})$ and its frame bounds are related to the Gramian in the following way: $0 < \alpha \le \beta$ are frame bounds of $\mathcal{X}$ if and only if $\Sigma(G_{\mathcal{X}}) \subseteq \{0\} \cup [\alpha, \beta]$, where $\Sigma(G_{\mathcal{X}})$ is the set of eigenvalues of $G_{\mathcal{X}}$.

If $E \subseteq \mathbb{C}^d$ we indicate by $|E|$ its Lebesgue measure.

2. Linear combinations preserving generators of subspaces. In this section, we are interested in studying which linear combinations of generators of a finite-dimensional subspace still generate the same subspace. Let us explain the problem in detail. Let $\mathcal{H}$ be a separable Hilbert space and consider a finite set $\mathcal{V} = \{v_1, \dots, v_m\}$ of elements in $\mathcal{H}$. Denote by V

the vector whose entries are the elements of $\mathcal{V}$, i.e. $V = (v_1, \dots, v_m)$. For any ℓ such that $r \leq \ell \leq m$, where $r = \dim(S(\mathcal{V}))$, let $\mathcal{W} = \{w_1, \dots, w_\ell\}$ be constructed by taking linear combinations of elements of $\mathcal{V}$. That is, for $i = 1, \dots, \ell$, $w_i = \sum_{j=1}^m a_{ij} v_j$ for some complex scalars a_{ij}. Collecting the coefficients of the linear combinations in a matrix $A = \{a_{ij}\}_{i,j} \in \mathbb{C}^{\ell\times m}$, we can write in matrix notation

$$W = AV^t, \tag{2.1}$$

where $W = (w_1, \dots, w_\ell)$. Therefore, the question is which matrices A transfer the property of being a set of generators for $S(\mathcal{V})$ from $\mathcal{V}$ to $\mathcal{W}$. We shall answer this by showing that for almost every matrix $A \in \mathbb{C}^{\ell\times m}$, the set $\mathcal{W}$ spans $S(\mathcal{V})$.

Theorem 2.1. *Let $\mathcal{V} = \{v_1, \dots, v_m\} \subseteq \mathcal{H}$ and let $r = \dim(S(\mathcal{V}))$. For any ℓ such that $r \leq \ell \leq m$, consider the set of matrices $\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} : S(\mathcal{V}) = S(\mathcal{W})\}$ where $\mathcal{W}$ is obtained from $\mathcal{V}$ by the relationship $W = AV^t$. Then $\mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ has zero Lebesgue measure.*

In order to prove Theorem 2.1 we first give a description of the set $\mathcal{R}$ in terms of the Gramians associated to $\mathcal{V}$ and $\mathcal{W}$. The connection between $G_{\mathcal{V}}$ and $G_{\mathcal{W}}$ is provided in the lemma below (see also [8, Proposition 2.5]).

Lemma 2.2. *Let $\mathcal{V} = \{v_1, \dots, v_m\} \subseteq \mathcal{H}$. If $\mathcal{W} = \{w_1, \dots, w_\ell\}$ is constructed from $\mathcal{V}$ by taking linear combinations of its elements as in* (2.1)*, then the Gramians associated to $\mathcal{V}$ and $\mathcal{W}$ satisfy $G_{\mathcal{W}} = AG_{\mathcal{V}}A^*$.*

Proof. Since $W = AV^t$, we obtain

$$(G_{\mathcal{W}})_{ij} = \Big\langle \sum_{k=1}^m a_{ik} v_k, \sum_{r=1}^m a_{jr} v_r \Big\rangle = \sum_{r,k=1}^m a_{ik}\overline{a_{jr}} \underbrace{\langle v_k, v_r\rangle}_{(G_{\mathcal{V}})_{kr}} = (AG_{\mathcal{V}}A^*)_{ij}. \qquad \blacksquare$$

For $\mathcal{V}$ and $\mathcal{W}$ as in Theorem 2.1, i.e. linked by (2.1), we have $S(\mathcal{W}) \subseteq S(\mathcal{V})$. Hence, $S(\mathcal{W}) = S(\mathcal{V})$ if and only if $\dim(S(\mathcal{W})) = \dim(S(\mathcal{V}))$. Now, by (1.1) and Lemma 2.2, $\dim(S(\mathcal{W})) = \dim(S(\mathcal{V}))$ if and only if $\mathrm{rk}(AG_{\mathcal{V}}A^*) = \mathrm{rk}(G_{\mathcal{V}})$. As a consequence, the set $\mathcal{R}$ in Theorem 2.1 can be described as the set of matrices preserving the rank of $G_{\mathcal{V}}$ under the action $AG_{\mathcal{V}}A^*$:

$$\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} : \mathrm{rk}(AG_{\mathcal{V}}A^*) = \mathrm{rk}(G_{\mathcal{V}})\}. \tag{2.2}$$

Having this description, the proof of Theorem 2.1 follows from the next rank-preserving result:

Proposition 2.3. *Let G be a positive-semidefinite matrix in $\mathbb{C}^{m\times m}$ such that $G = G^*$ and let $r = \mathrm{rk}(G)$. For any $r \leq \ell \leq m$ define $\mathcal{R}(G) = \{A \in \mathbb{C}^{\ell\times m} : \mathrm{rk}(G) = \mathrm{rk}(AGA^*)\}$. Then $\mathcal{N}(G) := \mathbb{C}^{\ell\times m} \setminus \mathcal{R}(G)$ has zero Lebesgue measure.*

Proof. Since G is a self-adjoint positive-semidefinite matrix, there exists a unitary matrix $U \in \mathbb{C}^{m\times m}$ and scalars $\lambda_1 \geq \cdots \geq \lambda_r > 0$ such that $U^*GU = D$ where D is the diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0) \in \mathbb{C}^{m\times m}$. In particular, $r = \mathrm{rk}(G) = \mathrm{rk}(D)$.

Note that for any $A \in \mathbb{C}^{\ell\times m}$, A preserves the rank of G under the action AGA^* if and only if AU preserves the rank of D under the action $AUD(AU)^*$. Therefore,

$$\begin{aligned}\mathcal{N}(G)U &= \{AU : A \in \mathbb{C}^{\ell\times m},\ \mathrm{rk}(G) \neq \mathrm{rk}(AGA^*)\}\\ &= \{B \in \mathbb{C}^{\ell\times m} : \mathrm{rk}(D) \neq \mathrm{rk}(BDB^*)\} = \mathcal{N}(D).\end{aligned}$$

Since U is a unitary matrix, the mapping $A \mapsto AU$ from $\mathbb{C}^{\ell\times m}$ into itself preserves Lebesgue measure, implying $|\mathcal{N}(G)| = |\mathcal{N}(D)|$. Thus, it is enough to show that $|\mathcal{N}(D)| = 0$.

Let $B \in \mathbb{C}^{\ell\times m}$ be written by column-blocks as $B = (B_1|B_2)$ where the columns of B_1 are the first r columns of B and the columns of B_2 are the last $m - r$ columns of B. Then

$$\mathrm{rk}(BDB^*) = \mathrm{rk}(BD^{1/2}(BD^{1/2})^*) = \mathrm{rk}(BD^{1/2}) = \mathrm{rk}(B_1).$$

Thus,

$$\mathcal{N}(D) = \{B = (B_1|B_2) \in \mathbb{C}^{\ell\times m} : B_1 \in \mathbb{C}^{\ell\times r},\ B_2 \in \mathbb{C}^{\ell\times(m-r)},\ \mathrm{rk}(B_1) < r\}.$$

Since the set of matrices in $\mathbb{C}^{\ell\times r}$ which are not full rank has zero Lebesgue measure, the result follows. ■

3. Linear combinations of generators of MI spaces. In the previous section we showed that almost all linear combinations of generators of a finite-dimensional subspace produce a new set of vectors spanning the same subspace. We now want to study a similar problem but in the context of multiplicatively invariant (MI) spaces in $L^2(\Omega, \mathcal{H})$. The concept of MI spaces was recently introduced in [7] in the general setting of $L^2(\Omega, \mathcal{H})$ as a generalization of the very well-known doubly invariant spaces proposed by Helson [16] and Srinivasan [25] for $\Omega = \mathbb{T}$. We shall prove that a result analogous to Theorem 2.1 can be obtained for MI spaces in $L^2(\Omega, \mathcal{H})$. The main difference here lies in the meaning of the word "generator" which, for MI spaces, differs from the notion of generator for a subspace. To properly state the result we shall prove in this case, we first summarize the basic properties of MI spaces.

3.1. Multiplicatively invariant spaces in $L^2(\Omega, \mathcal{H})$. The material we collect here is a summary of the content of [7, Section 2]. See [7] for details and proofs.

Let (Ω, μ) be a σ-finite measure space and let $\mathcal{H}$ be a separable Hilbert space. The vector valued space $L^2(\Omega, \mathcal{H})$ is the space of measurable functions

$\Phi : \Omega \to \mathcal{H}$ such that $\|\Phi\|^2 = \int_\Omega \|\Phi(\omega)\|^2_{\mathcal{H}}\, d\mu(\omega) < \infty$. The inner product in $L^2(\Omega, \mathcal{H})$ is given by $\langle \Phi, \Psi\rangle = \int_\Omega \langle \Phi(\omega), \Psi(\omega)\rangle_{\mathcal{H}}\, d\mu(\omega)$.

To define MI spaces in $L^2(\Omega, \mathcal{H})$ we need the concept of determining set for $L^1(\Omega)$. A set $D \subseteq L^\infty(\Omega)$ is said to be a *determining set* for $L^1(\Omega)$ if for every $f \in L^1(\Omega)$ such that $\int_\Omega f(\omega)g(\omega)\, d\mu(\omega) = 0$ for all $g \in D$, one has $f = 0$. In the setting of Helson [16], a determining set is the set of exponentials with integer parameter, $D = \{e^{2\pi ik\,\cdot}\}_{k\in\mathbb{Z}} \subseteq L^\infty(\mathbb{T})$.

Definition 3.1. A closed subspace $M \subseteq L^2(\Omega, \mathcal{H})$ is *multiplicatively invariant* with respect to the determining set D for $L^1(\Omega)$ (MI space for short) if

$$\Phi \in M \;\Rightarrow\; g\Phi \in M \text{ for any } g \in D.$$

For an at most countable (meaning finite or countable) subset $\boldsymbol{\Phi} \subseteq L^2(\Omega, \mathcal{H})$ define $M_D(\boldsymbol{\Phi}) = \overline{\operatorname{span}}\{g\Phi\colon \Phi \in \boldsymbol{\Phi},\ g \in D\}$. The subspace $M_D(\boldsymbol{\Phi})$ is called the multiplicatively invariant space generated by $\boldsymbol{\Phi}$, and we say that $\boldsymbol{\Phi}$ is a set of generators for $M_D(\boldsymbol{\Phi})$. When $\boldsymbol{\Phi}$ is finite, $M = M_D(\boldsymbol{\Phi})$ is said to be *finitely generated* by $\boldsymbol{\Phi}$. In that case, we define the *length* of M as

$$\ell(M) = \min\{n \in \mathbb{N}\colon \exists \Phi_1, \ldots, \Phi_n \in M \text{ with } M = M_D(\Phi_1, \ldots, \Phi_n)\}.$$

One of the most important properties of MI spaces is their characterization in terms of measurable range functions. A *range function* is a mapping $J : \Omega \to \{\text{closed subspaces of } \mathcal{H}\}$ equipped with the orthogonal projections $P_J(\omega)$ of $\mathcal{H}$ onto $J(\omega)$. A range function is said to be *measurable* if for any $a, b \in \mathcal{H}$, $\omega \mapsto \langle P_J(\omega)a, b\rangle$ is measurable as a function from Ω to $\mathbb{C}$.

Theorem 3.2 ([7, Theorem 2.4]). *Suppose that $L^2(\Omega)$ is separable, so that $L^2(\Omega, \mathcal{H})$ is also separable. Let M be a closed subspace of $L^2(\Omega, \mathcal{H})$ and D a determining set for $L^1(\Omega)$. Then M is an MI space with respect to D if and only if there exists a measurable range function J such that*

$$M = \{\Phi \in L^2(\Omega, \mathcal{H}) : \Phi(\omega) \in J(\omega) \textit{ for a.e. } \omega \in \Omega\}.$$

Identifying range functions that are equal almost everywhere, the correspondence between MI spaces and measurable range functions is one-to-one and onto.

Moreover, when $M = M_D(\boldsymbol{\Phi})$ for some at most countable set $\boldsymbol{\Phi} \subseteq L^2(\Omega, \mathcal{H})$, the range function associated to M is

$$J(\omega) = \overline{\operatorname{span}}\{\Phi(\omega) : \Phi \in \boldsymbol{\Phi}\} \quad \textit{for a.e. } \omega \in \Omega.$$

3.2. Linear combinations of MI-generators. We can now properly state what we want to prove. Fix a determining set $D \subseteq L^\infty(\Omega)$ for $L^1(\Omega)$. Suppose that M is a finitely generated MI space with respect to D. That is, $M = M_D(\boldsymbol{\Phi})$ where $\boldsymbol{\Phi} = \{\Phi_1, \ldots, \Phi_m\} \subseteq L^2(\Omega, \mathcal{H})$. For a number ℓ such that $\ell(M) \le \ell \le m$ we construct a new set of functions of M, $\boldsymbol{\Psi} = \{\Psi_1, \ldots, \Psi_\ell\}$ say, by taking linear combinations of $\{\Phi_1, \ldots, \Phi_m\}$ as we did for the case of generators for a finite-dimensional Hilbert space in Section 2. More precisely, for each $1 \leq i \leq \ell$, $\Psi_i = \sum_{j=1}^{m} a_{ij}\Phi_j$, and collecting the coefficients in a matrix $A \in \mathbb{C}^{\ell\times m}$, we write $\mathit{\Psi} = A\mathit{\Phi}^t$ where $\mathit{\Phi} = (\Phi_1,\dots,\Phi_m)$ and $\mathit{\Psi} = (\Psi_1,\dots,\Psi_\ell)$. The question is now which matrices $A \in \mathbb{C}^{\ell\times m}$ transfer the property of being a generator set for M from $\boldsymbol{\Phi}$ to $\boldsymbol{\Psi}$.

Theorem 3.3. *Let M be a finitely generated MI space and*

$$\boldsymbol{\Phi} = \{\Phi_1,\dots,\Phi_m\} \subseteq L^2(\Omega,\mathcal{H})$$

be such that $M = M_D(\boldsymbol{\Phi})$ where $\ell(M) \leq m$. For $\ell(M) \leq \ell \leq m$, set $\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} : M = M_D(\boldsymbol{\Psi}), \mathit{\Psi} = A\mathit{\Phi}^t\}$. Then $\mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ has zero Lebesgue measure.

Observe that this result is analogous to the one we proved for the case of generators for subspaces, Theorem 2.1. As mentioned before, the generator set $\boldsymbol{\Phi}$ generates $M_D(\boldsymbol{\Phi})$ as a MI space. This fact changes the nature of the problem, and so the proof of Theorem 3.3 requires more subtle techniques than those used to prove Theorem 2.1.

For the proof of the above theorem we need the following known result.

Lemma 3.4. *Let (X,μ) and (Y,ν) be measure spaces and $F \subseteq X\times Y$ be $\mu\times\nu$-measurable. Then $(\mu\times\nu)(F) = 0$ if and only if $\nu(F_x) = 0$ for μ-a.e. $x \in X$ if and only if $\mu(F_y) = 0$ for ν-a.e. $y \in Y$, where $F_x = \{y \in Y : (x,y) \in F\}$ and $F_y = \{x \in X : (x,y) \in F\}$.*

Proof of Theorem 3.3. Along this proof the relationship between $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ will always be $\mathit{\Psi} = A\mathit{\Phi}^t$ for some matrix $A \in \mathbb{C}^{\ell\times m}$, so we will not repeat this again. For each $\omega \in \Omega$, let $\boldsymbol{\Phi}(\omega) = \{\Phi_1(\omega),\dots,\Phi_m(\omega)\}$ and $\boldsymbol{\Psi}(\omega) = \{\Psi_1(\omega),\dots,\Psi_\ell(\omega)\}$. We denote by $J_{\boldsymbol{\Phi}}$ and $J_{\boldsymbol{\Psi}}$ the measurable range functions associated to $M_D(\boldsymbol{\Phi})$ and $M_D(\boldsymbol{\Psi})$ respectively. Note that since $\mathit{\Psi} = A\mathit{\Phi}^t$, we have $\mathit{\Psi}(\omega) = A\mathit{\Phi}(\omega)^t$, where $\mathit{\Phi}(\omega) = (\Phi_1(\omega),\dots,\Phi_m(\omega))$ and $\mathit{\Psi}(\omega) = (\Psi_1(\omega),\dots,\Psi_\ell(\omega))$. We now proceed as in [6]. By Theorem 3.2 and and the reasoning we used to obtain (2.2), we deduce that

$$\begin{aligned}
\mathcal{R} &= \{A \in \mathbb{C}^{\ell\times m} : M_D(\boldsymbol{\Phi}) = M_D(\boldsymbol{\Psi})\} \\
&= \{A \in \mathbb{C}^{\ell\times m} : J_{\boldsymbol{\Phi}}(\omega) = J_{\boldsymbol{\Psi}}(\omega) \text{ for a.e. } \omega\in\Omega\} \\
&= \{A \in \mathbb{C}^{\ell\times m} : S(\boldsymbol{\Phi}(\omega)) = S(\boldsymbol{\Psi}(\omega)) \text{ for a.e. } \omega\in\Omega\} \\
&= \{A \in \mathbb{C}^{\ell\times m} : \operatorname{rk}(G_{\boldsymbol{\Phi}(\omega)}) = \operatorname{rk}(AG_{\boldsymbol{\Phi}(\omega)}A^*) \text{ for a.e. } \omega\in\Omega\},
\end{aligned}$$

where in the last equality, $G_{\boldsymbol{\Phi}(\omega)}$ is the Gramian associated to $\boldsymbol{\Phi}(\omega)$. Since $\operatorname{rk}(G_{\boldsymbol{\Phi}(\omega)}) \geq \operatorname{rk}(AG_{\boldsymbol{\Phi}(\omega)}A^*)$ for a.e. $\omega\in\Omega$, we want to prove that the set

$$\begin{aligned}
\{A \in \mathbb{C}^{\ell\times m} : \operatorname{rk}(G_{\boldsymbol{\Phi}(\omega)}) > \operatorname{rk}(AG_{\boldsymbol{\Phi}(\omega)}A^*) \\
\text{for } \omega \text{ in a set of positive measure}\}
\end{aligned} \tag{3.1}$$

has zero Lebesgue measure.

Let $\mathcal{F} = \{(\omega, A) \in \Omega \times \mathbb{C}^{\ell\times m} : \mathrm{rk}(G_{\boldsymbol{\Phi}(\omega)}) > \mathrm{rk}(AG_{\boldsymbol{\Phi}(\omega)}A^*)\}$. Since each Φ_i is a measurable function, so are the entries of $G_{\boldsymbol{\Phi}(\omega)}$. On the other hand, the rank of any matrix is the largest of the absolute values of its minors. Thus, since the determinant is a polynomial in the entries of the matrix, it follows that the rank of a matrix with measurable entries is a measurable function. Now, since $\mathcal{F} = f^{-1}((0,\infty))$ where f is the measurable function $f(\omega, A) = \mathrm{rk}(G_{\boldsymbol{\Phi}(\omega)}) - \mathrm{rk}(AG_{\boldsymbol{\Phi}(\omega)}A^*)$, it turns out that $\mathcal{F}$ is a measurable subset of $\Omega \times \mathbb{C}^{\ell\times m}$.

The sections of $\mathcal{F}$ are denoted by $\mathcal{F}_\omega$ and $\mathcal{F}_A$. By Proposition 2.3 we know that $|\mathcal{F}_\omega| = 0$ for a.e. $\omega \in \Omega$ and hence, by Lemma 3.4, $\mu(\mathcal{F}_A) = 0$ for a.e. $A \in \mathbb{C}^{\ell\times m}$. Note that the set given in (3.1) is exactly $\{A \in \mathbb{C}^{\ell\times m} : \mu(\mathcal{F}_A) > 0\}$. Therefore, it has zero Lebesgue measure. ■

3.3. Linear combinations preserving uniform frames. As mentioned in the introduction, we want to give a unified treatment for the problem of when linear combinations preserve generators and frame generators in systems of translates, where the "systems of translates" are considered in different contexts. That is why we work at the level of vector valued functions. To address the frame case, we need to introduce the following definition which, at this point, may seem a bit artificial. However, we shall see that it makes sense in each of the different contexts we want to consider.

Definition 3.5. Let $\boldsymbol{\Phi} \subseteq L^2(\Omega, \mathcal{H})$ be an at most countable set and let J be the measurable range function defined as $J(\omega) = \overline{\mathrm{span}}\{\Phi(\omega) : \Phi \in \boldsymbol{\Phi}\}$ for a.e. $\omega \in \Omega$. We say that $\boldsymbol{\Phi}$ is a *uniform frame* for J if there exist constants $0 < \alpha \leq \beta$ such that, for a.e. $\omega \in \Omega$, the set $\{\Phi(\omega) : \Phi \in \boldsymbol{\Phi}\}$ is a frame for $J(\omega)$ with frame bounds α and β.

Fix a determining set $D \subseteq L^\infty(\Omega)$ for $L^1(\Omega)$ and suppose that $\boldsymbol{\Phi}$ is a finite set of functions in $L^2(\Omega, \mathcal{H})$ that is a uniform frame for J where J is the measurable range function associated to $M = M_D(\boldsymbol{\Phi})$. Then Theorem 3.3 tells us that almost every linear combination of the functions in $\boldsymbol{\Phi}$ produces a new set of generators $\boldsymbol{\Psi}$ of M. In particular, this says that for a.e. $\omega \in \Omega$, $\boldsymbol{\Psi}(\omega)$ is a new set of generators for $J(\omega)$. Thus, we are interested in knowing which linear combinations also preserve uniform frames. That is, if $\varPsi = A\varPhi^t$, what property must A have for $\boldsymbol{\Psi}$ to be a uniform frame for J? We shall answer this question by completely characterizing matrices A that preserve uniform frames in terms of angles between subspaces. To this end, we first recall the notion of Friedrichs angle [13, 15, 20].

Let $S, T \neq \{0\}$ be subspaces of $\mathbb{C}^n$. The *Friedrichs angle* between S and T is the angle in $[0, \pi/2]$ whose cosine is

$$\boldsymbol{\mathcal{G}}[S,T] = \sup\{|\langle x, y\rangle| : x \in S \cap (S\cap T)^\perp,\ y \in T \cap (S\cap T)^\perp,\ \|x\| = \|y\| = 1\}.$$

We define $\boldsymbol{\mathcal{G}}[S,T]=0$ if either $S=\{0\}$, $T=\{0\}$, $S\subseteq T$ or $T\subseteq S$. As usual, the sine of the Friedrichs angle is defined as $\boldsymbol{\mathcal{F}}[S,T]=\sqrt{1-\boldsymbol{\mathcal{G}}[S,T]^2}$.

We can now state a characterization of matrices that preserve uniform frames.

THEOREM 3.6. *Let $\boldsymbol{\Phi}=\{\Phi_1,\dots,\Phi_m\}\subseteq L^2(\Omega,\mathcal{H})$ be a uniform frame for J where J is the measurable range function associated to $M=M_D(\boldsymbol{\Phi})$ and suppose that $\ell(M)\leq \ell\leq m$. Let $A\in\mathbb{C}^{\ell\times m}$ be a matrix and consider $\boldsymbol{\Psi}=\{\Psi_1,\dots,\Psi_\ell\}$ where $\Psi=A\Phi^t$. Then $\boldsymbol{\Psi}$ is a uniform frame for J if and only if A satisfies the following two conditions:*

(1) *$A\in\mathcal{R}$ where $\mathcal{R}$ is as in Theorem* 3.3.
(2) *There exists $\delta>0$ such that $\boldsymbol{\mathcal{F}}[\mathrm{Ker}(A),\mathrm{Im}(G_{\boldsymbol{\Phi}(\omega)})]\geq\delta$ for a.e. $\omega\in\Omega$.*

The proof is based on the fact that $\boldsymbol{\Phi}$ is a uniform frame with frame bounds α and β for J if and only if $\Sigma(G_{\boldsymbol{\Phi}(\omega)})\subseteq[\alpha,\beta]\cup\{0\}$ for a.e. $\omega\in\Omega$. Therefore, the task is to prove that conditions (1) and (2) of Theorem 3.6 guarantee that the positive eigenvalues of $AG_{\boldsymbol{\Phi}(\omega)}A^*$ are uniformly bounded. This can be done using [8, Proposition 3.3], which is an adaptation of a result on singular values of a composition of operators of Antezana et al. [2]. Having these results at hand, the complete proof of Theorem 3.6 is an immediate adaptation of [8, proof of Theorem 4.4]. For the convenience of the reader we provide it here.

Proof of Theorem 3.6. For a positive-semidefinite matrix G such that $G=G^*$ we denote by $\lambda_-(G)$ its smallest non-zero eigenvalue. For any matrix B we denote by $\sigma(B)$ the smallest non-zero singular value of B.

Let $0<\alpha\leq\beta$ be the frame bounds of $\boldsymbol{\Phi}$. Then since $\Sigma(G_{\boldsymbol{\Phi}(\omega)})\subseteq[\alpha,\beta]\cup\{0\}$ for a.e. $\omega\in\Omega$, we have $\alpha\leq\lambda_-(G_{\boldsymbol{\Phi}(\omega)})$ and $\|G_{\boldsymbol{\Phi}(\omega)}\|\leq\beta$ for a.e. $\omega\in\Omega$.

Suppose first that $\boldsymbol{\Psi}$ is a uniform frame for J with frame bounds $0<\alpha'\leq\beta'$. In particular, since the correspondence between MI spaces and range functions is one-to-one and onto, $\boldsymbol{\Psi}$ is a generator set for $M=M_D(\boldsymbol{\Phi})$ and hence $A\in\mathcal{R}$. Thus, we are under the hypotheses of [8, Proposition 3.3] and so

$$\begin{aligned}\lambda_-(G_{\boldsymbol{\Psi}(\omega)})=\lambda_-(AG_{\boldsymbol{\Phi}(\omega)}A^*)&\leq\|A\|^2\|G_{\boldsymbol{\Phi}(\omega)}\|\,\boldsymbol{\mathcal{F}}[\mathrm{Ker}(A),\mathrm{Im}(G_{\boldsymbol{\Phi}(\omega)})]\\&\leq\|A\|^2\beta\,\boldsymbol{\mathcal{F}}[\mathrm{Ker}(A),\mathrm{Im}(G_{\boldsymbol{\Phi}(\omega)})].\end{aligned}$$

Thus, $\alpha'\leq\|A\|^2\beta\,\boldsymbol{\mathcal{F}}[\mathrm{Ker}(A),\mathrm{Im}(G_{\boldsymbol{\Phi}(\omega)})]$ and condition (2) follows with $\delta=\alpha'/(\|A\|^2\beta)$.

Suppose now that (1) and (2) are satisfied for some matrix A. Then we again apply [8, Proposition 3.3] to get

$$\lambda_-(AG_{\boldsymbol{\Phi}(\omega)}A^*) \geq \sigma(A)^2\lambda_-(G_{\boldsymbol{\Phi}(\omega)})\,\boldsymbol{\mathcal{F}}[\mathrm{Ker}(A), \mathrm{Im}(G_{\boldsymbol{\Phi}(\omega)})]^2 \geq \sigma(A)^2\alpha\delta^2 \tag{3.2}$$

for a.e. $\omega \in \Omega$.

Note that $\|G_{\boldsymbol{\Psi}(\omega)}\| = \|AG_{\boldsymbol{\Phi}(\omega)}A^*\| \leq \|A\|^2\|G_{\boldsymbol{\Phi}(\omega)}\| \leq \|A\|^2\beta$ and hence the eigenvalues of $G_{\boldsymbol{\Psi}(\omega)}$ are bounded above by $\|A\|^2\beta$. Combining this fact with (3.2) we obtain $\Sigma(G_{\boldsymbol{\Psi}(\omega)}) \subseteq [\sigma(A)^2\alpha\delta^2, \|A\|^2\beta] \cup \{0\}$ for a.e. $\omega \in \Omega$ and so $\boldsymbol{\Psi}$ is a uniform frame for J. ∎

When the new set of generators has exactly $\ell(M)$ elements, the following theorem can be shown. For its proof see [8, Theorem 4.7].

THEOREM 3.7. *Let $\boldsymbol{\Phi} = \{\Phi_1, \ldots, \Phi_m\} \subseteq L^2(\Omega, \mathcal{H})$ be a uniform frame for J where J is the measurable range function associated to $M = M_D(\boldsymbol{\Phi})$, and let $\ell(M) = \ell \leq m$. Let $A \in \mathbb{C}^{\ell\times m}$ and $\boldsymbol{\Psi} = \{\Psi_1, \ldots, \Psi_\ell\}$ where $\Psi = A\Phi^t$. Then $\boldsymbol{\Psi}$ is a uniform frame for J if and only if AA^* is invertible and*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|(I_m - A^*(AA^*)^{-1}A)G_{\boldsymbol{\Phi}(\omega)}G^{\dagger}_{\boldsymbol{\Phi}(\omega)}\| < 1.$$

Here, I_m is the identity in $\mathbb{C}^{m\times m}$ and $G^{\dagger}_{\boldsymbol{\Phi}(\omega)}$ is the Moore–Penrose pseudo-inverse of $G_{\boldsymbol{\Phi}(\omega)}$.

REMARK 3.8. It might be the case that condition (2) in Theorem 3.6 is not satisfied for any matrix A. An example is given in [8, Example 4.12] for a system of translates in $\mathbb{R}^d$ but it can be easily adapted to the setting of MI spaces. Indeed, in $L^2((-1/2, 1/2]^2, \ell^2(\mathbb{Z}^2))$ consider the vector valued functions Φ_1 and Φ_2 given by $\Phi_1(\omega_1, \omega_2) = -\sin(2\pi\omega_1)e_0$ and $\Phi_1(\omega_1, \omega_2) = e^{2\pi i\omega_2}\cos(2\pi\omega_1)e_0$ where e_0 is the sequence in $\ell^2(\mathbb{Z}^2)$ that takes the value 1 at $(0, 0)$ and 0 otherwise. As a determining set take $D = \{e^{2\pi i\langle(k,j),\cdot\rangle}\}_{(k,j)\in\mathbb{Z}^2}$. Then for $M_D(\Phi_2, \Phi_2)$ there is no matrix satisfying condition (2) in Theorem 3.6. See [8, Example 4.12] for details.

4. Application to systems of translates. In this section we show how the previous results can be applied to systems of translates. As we will see, there exists a connection between systems of translates and vector valued functions which of course depends on the context where the systems of translates are considered. The link is what we call *fiberization isometry.*

4.1. Systems of translates on LCA groups. Here we work with systems of translates in the context of locally compact abelian groups. Given a second countable LCA group G written additively, we consider translates of functions in $L^2(G)$ along a subgroup $H \subseteq G$ such that G/H is compact. A closed subspace $V \subseteq L^2(G)$ is said to be *H-invariant* (or invariant under translations in H) if for every $f \in V$, $T_hf \in V$ for all $h \in H$ where T_h denotes translation by h, i.e. $T_hf(x) = f(x-h)$. Subspaces that are H-invariant were characterized using range functions and fiberization techniques in [9, 19] when H is discrete. Recently in [7], a similar characterization was obtained only assuming that G/H is compact (i.e. H not necessarily discrete). An important point in getting these characterizations is to see the space $L^2(G)$ as a vector valued space of the form $L^2(\Omega, \mathcal{H})$ for some particular choices of Ω and $\mathcal{H}$. Now we briefly describe how to do this.

Let $\widehat{G}$ be the dual group of G, that is, the set of continuous characters on G. For $x \in G$ and $\gamma \in \widehat{G}$ we use the notation (x, γ) for the complex value that γ takes at x. For any subgroup $H \subseteq G$, the *annihilator* of H is the subgroup $H^* = \{\gamma \in \widehat{G} : (h, \gamma) = 1, \forall h \in H\}$ of $\widehat{G}$. Let us assume from now on that H is a *co-compact* subgroup of G, that is, G/H is compact. Then, by the duality theorem [24, Lemma 2.1.3], H^* is discrete. Now fix a measurable section $\Omega \subseteq \widehat{G}$ of the quotient $\widehat{G}/H^*$ whose existence is a consequence of [21, Lemma 1.1]. When the Haar measures of the groups involved here are appropriately chosen, the following result shows that $L^2(G)$ is isometrically isomorphic to the vector valued space $L^2(\Omega, \ell^2(H^*))$. For its proof see [9, Proposition 3.3] and [7, Proposition 3.7].

Proposition 4.1. *The fiberization mapping $\mathcal{T} : L^2(G) \to L^2(\Omega, \ell^2(H^*))$ defined by*

$$\mathcal{T}f(\omega) = \{\widehat{f}(\omega + \delta)\}_{\delta \in H^*}$$

is an isometric isomorphism and it satisfies $\mathcal{T}T_h f(\omega) = (-h, w)\mathcal{T}f(\omega)$ for all $f \in L^2(G)$ and all $h \in H$. Here, $\widehat{f}$ denotes the Fourier transform of f.

The fiberization isometry of Proposition 4.1 allows us to see $L^2(G)$ as the vector valued space $L^2(\Omega, \ell^2(H^*))$. Under this isometry, H-invariant spaces of $L^2(G)$ exactly correspond to MI spaces in $L^2(\Omega, \ell^2(H^*))$. Let us explain this correspondence in detail. The determining set behind the notion of MI spaces in $L^2(\Omega, \ell^2(H^*))$ is $D = \{(h, \cdot)\chi_\Omega(\cdot)\}_{h \in H}$ [7, Corollary 3.6]. Thus, by Proposition 4.1, $V \subseteq L^2(G)$ is an H-invariant space if and only if $M = \mathcal{T}V$ is a MI space with respect to D. Therefore, we can also identify H-invariant spaces with measurable range functions, as shown in [9, Theorem 3.10] and [7, Theorem 3.8]:

Theorem 4.2. *Let $V \subseteq L^2(G)$ be a closed subspace and $\mathcal{T}$ the mapping defined in Proposition 4.1. Then V is H-invariant if and only if there exists a measurable range function J such that*

$$V = \{f \in L^2(G) : \mathcal{T}f(\omega) \in J(\omega) \text{ for a.e. } \omega \in \Omega\}.$$

Once one identifies range functions that are equal almost everywhere, the correspondence between measurable range functions and H-invariant spaces is one-to-one and onto. When $V = \overline{\operatorname{span}}\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ for an at most countable set $\mathcal{A} \subseteq L^2(G)$, the measurable range function associated to

V is given by

$$J(\omega) = \overline{\operatorname{span}}\{\mathcal{T}\varphi(\omega) : \varphi \in \mathcal{A}\} \quad \textit{for a.e. } \omega \in \Omega.$$

Frames of translates can also be characterized using range functions and the fiberization isometry. Indeed, when $\mathcal{A} \subseteq L^2(G)$ is a countable set, frames of the form $\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ for $V = \overline{\operatorname{span}}\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ correspond to uniform frames for J where J is the measurable range function associated with V. For H discrete, this fact was proven in [9, Theorem 4.1]. When H is not discrete but co-compact, the set $\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ is not indexed by a discrete set and one needs to work with the notion of continuous frame (see [7, Definition 5.1] for details). The characterization of continuous frames in terms of range functions was given in [7, Theorem 5.1]. In the next theorem, we state a characterization of frames of translates using range functions without distinguishing between the discrete and the continuous cases. The reader should keep in mind that when H is not discrete, the word "frame" refers to the notion of continuous frame as in [7, Definition 5.1].

Theorem 4.3. *Let $\mathcal{A} \subseteq L^2(G)$ be a countable set, let J be the measurable range function associated to $V = \overline{\operatorname{span}}\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ and let $\mathcal{T}$ be the mapping of Proposition* 4.1*. Then the following conditions are equivalent:*

(1) *$\{T_h\varphi : h \in H, \varphi \in \mathcal{A}\}$ is a frame for V with frame bounds $0 < \alpha \leq \beta$.*
(2) *$\{\mathcal{T}\varphi : \varphi \in \mathcal{A}\}$ is a uniform frame for J with frame bounds $0 < \alpha \leq \beta$. That is, for a.e. $\omega \in \Omega$, $\{\mathcal{T}\varphi(\omega) : \varphi \in \mathcal{A}\}$ is a frame for $J(\omega)$ with* (*uniform*) *frame bounds $0 < \alpha \leq \beta$.*

We already have all the ingredients we need to see how the results of Section 3 can be applied to this setting. Fix $\{\phi_1, \ldots, \phi_m\} \subseteq L^2(G)$ and consider the H-invariant space generated by $\{\phi_1, \ldots, \phi_m\}$, $V = \overline{\operatorname{span}}\{T_h\phi_j : h \in H, 1 \leq j \leq m\}$. By taking linear combinations of $\{\phi_1, \ldots, \phi_m\}$ we want to construct new sets of generators for V. As we did before for subspaces and for MI spaces, we consider sets $\{\psi_1, \ldots, \psi_\ell\}$ of functions in $L^2(G)$, where for every $1 \leq j \leq m$, $\psi_j = \sum_{i=1}^m a_{ij}\phi_j$ and ℓ is a number between the length of the MI space $\mathcal{T}V$ and m. Collecting the coefficients of the linear combinations in a matrix $A \in \mathbb{C}^{\ell\times m}$ and letting $\Phi = (\phi_1, \ldots, \phi_m)$ and $\Psi = (\psi_1, \ldots, \psi_\ell)$ we can write $\Psi = A\Phi^t$. In the next theorem we prove that for almost every matrix $A \in \mathbb{C}^{\ell\times m}$, the functions $\{\psi_1, \ldots, \psi_\ell\}$ generate V. This result extends [6, Theorem 1] to the context of LCA groups, and moreover, since H is allowed to be non-discrete, it is new even in the case when $G = \mathbb{R}^d$.

Theorem 4.4. *Given $\{\phi_1, \ldots, \phi_m\} \subseteq L^2(G)$ let $V = \overline{\operatorname{span}}\{T_h\phi_j : h \in H, 1 \leq j \leq m\}$ and let $\ell(M)$ be the length of $M = \mathcal{T}V$ where $\mathcal{T}$ is the fiberization isometry of Proposition* 4.1*. For $\ell(M) \leq \ell \leq m$, let $\mathcal{R}$ be the set of matrices $A = \{a_{ij}\}_{ij} \in \mathbb{C}^{\ell\times m}$ such that the linear combinations $\psi_j = \sum_{i=1}^m a_{ij}\phi_j$*

generate V, i.e. $V = \overline{\operatorname{span}}\{T_h\psi_i : h \in H, 1 \le i \le \ell\}$. Then $\mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ has Lebesgue zero measure.

Proof. Let $A \in \mathbb{C}^{\ell\times m}$ and $\Psi = \{\psi_1, \dots, \psi_\ell\}$ where $\Psi = A\Phi^t$. Then $\{\psi_1, \dots, \psi_\ell\}$ is a set of generators for V if and only if $\{\mathcal{T}\psi_1, \dots, \mathcal{T}\psi_\ell\}$ generates M as a MI space with respect to $D = \{(h, \cdot)\chi_\Omega(\cdot)\}_{h\in H}$. Denoting $\boldsymbol{\Phi} = \{\mathcal{T}\phi_1, \dots, \mathcal{T}\phi_m\}$, $\boldsymbol{\Psi} = \{\mathcal{T}\psi_1, \dots, \mathcal{T}\psi_\ell\}$, $\mathit{\Psi} = (\mathcal{T}\psi_1, \dots, \mathcal{T}\psi_\ell)$ and $\mathit{\Phi} = (\mathcal{T}\phi_1, \dots, \mathcal{T}\phi_m)$, we see that $\mathcal{R} = \{A \in \mathbb{C}^{\ell\times m} : M = M_D(\boldsymbol{\Psi}), \mathit{\Psi} = A\mathit{\Phi}^t\}$. Thus, by Theorem 3.3, $\mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ has Lebesgue zero measure. ∎

The next theorem is an extension of [8, Theorem 4.4] to LCA groups.

THEOREM 4.5. *Let $\{\phi_1, \dots, \phi_m\} \subseteq L^2(G)$ be such that $\{T_h\phi_j : h \in H, 1 \le j \le m\}$ is a frame for $V = \overline{\operatorname{span}}\{T_h\varphi_j : h \in H, 1 \le j \le m\}$ and suppose that $\ell(M) \le \ell \le m$, where $M = \mathcal{T}V$ and $\mathcal{T}$ is as in Proposition* 4.1. *Let $A \in \mathbb{C}^{\ell\times m}$ and consider $\Psi = \{\psi_1, \dots, \psi_\ell\}$ where $\Psi = A\Phi^t$. Then $\{T_h\psi_i : h \in H, 1 \le i \le \ell\}$ is a frame for V if and only if A satisfies the following two conditions:*

1. *(1) $A \in \mathcal{R}$, where $\mathcal{R}$ is as in Theorem* 4.4.
2. *(2) There is $\delta > 0$ such that $\boldsymbol{\mathcal{F}}[\operatorname{Ker}(A), \operatorname{Im}(G_{\boldsymbol{\Phi}(\omega)})] \ge \delta$ for a.e. $\omega \in \Omega$, where $G_{\boldsymbol{\Phi}(\omega)}$ is the Gramian associated to $\{\mathcal{T}\phi_1(\omega), \dots, \mathcal{T}\phi_m(\omega)\}$.*

REMARK 4.6. Given a co-compact subgroup $\Delta \subseteq \widehat{G}$ of the dual group of G and $\mathcal{A} \subseteq L^2(G)$, consider the system $\{M_\delta\phi : \phi \in \mathcal{A}, \delta \in \Delta\}$ where M_δ is the modulation operator given by $M_\delta\phi(x) = (x, \delta)\phi(x)$. Since under the Fourier transform, modulations become translations, all the results we have proven for systems of translates can be reformulated for systems of modulations. Furthermore, one may also consider systems of time-frequency translates $\{M_\delta T_h\phi : \phi \in \mathcal{A}, \delta \in \Delta, h \in H\}$ where $H \subseteq G$ and $\Delta \subseteq \widehat{G}$ are discrete subgroups and $\mathcal{A} \subseteq L^2(G)$. The closure of the span of a system of time-frequency translates is called a *shift-modulation invariant space* or Gabor space. Using fiberization techniques and range functions, a characterization of such spaces was given in [10]. Therefore, this setting is one more example where the results of Section 3 can be applied.

4.2. Discrete LCA groups acting on σ-finite measure spaces. We are now interested in systems of functions constructed by the action of a discrete LCA group Γ on $L^2(\mathcal{X})$ where $(\mathcal{X}, \mu)$ is a σ-finite measure space. We will work with *quasi-Γ-invariant* actions. This notion was introduced in [17] and then extended to the non-abelian case in [3]. Fix a discrete countable LCA group Γ. Let $(\mathcal{X}, \mu)$ be a σ-finite measure space and $\sigma : \Gamma \times \mathcal{X} \to \mathcal{X}$ a measurable action satisfying the following conditions:

(i) for each $\gamma \in \Gamma$ the map $\sigma_\gamma : \mathcal{X} \to \mathcal{X}$ given by $\sigma_\gamma(x) := \sigma(\gamma, x)$ is μ-measurable;

(ii) $\sigma_\gamma(\sigma_{\gamma'}(x)) = \sigma_{\gamma+\gamma'}(x)$ for all $\gamma, \gamma' \in \Gamma$ and for all $x \in \mathcal{X}$;
(iii) $\sigma_e(x) = x$ for all $x \in \mathcal{X}$, where e is the identity of Γ.

The action σ is said to be *quasi-Γ-invariant* if there exists a measurable function $J_\sigma : \Gamma \times \mathcal{X} \to \mathbb{R}^+$, called the *Jacobian* of σ, such that $d\mu(\sigma_\gamma(x)) = J_\sigma(\gamma, x) d\mu$. To each quasi-$\Gamma$-invariant action σ we can associate a unitary representation T_σ of Γ on $L^2(\mathcal{X})$ given by $T_\sigma(\gamma) f(x) = J_\sigma(-\gamma, x)^{1/2} f(\sigma_{-\gamma}(x))$.

Given a quasi-Γ-invariant action σ, we say that a closed subspace V of $L^2(\mathcal{X})$ is *Γ-invariant* if

$$f \in V \;\Rightarrow\; T_\sigma(\gamma) f \in V \text{ for any } \gamma \in \Gamma.$$

When $L^2(\mathcal{X})$ is separable, each Γ-invariant spaces is of the form $V = \overline{\operatorname{span}}\{T_\sigma(\gamma)\varphi : \gamma \in \Gamma,\, \varphi \in \mathcal{A}\}$ for some at most countable set $\mathcal{A} \subseteq L^2(\mathcal{X})$.

In order to obtain the analogues of Theorems 4.4 and 4.5 for systems of the form $\{T_\sigma(\gamma)\phi_j\}_{j=1}^m$ using the machinery of MI spaces of Section 3, we first need to establish a connection between $L^2(\mathcal{X})$ and a vector valued space of the type $L^2(\Omega, \mathcal{H})$. We can do this by assuming that the quasi-Γ-invariant action σ has the *tiling property*: there exists a measurable subset $C \subseteq \mathcal{X}$ such that $\mu(\mathcal{X} \setminus \bigcup_{\gamma\in\Gamma} \sigma_\gamma(C)) = 0$ and $\mu(\sigma_\gamma(C) \cap \sigma_{\gamma'}(C)) = 0$ whenever $\gamma \neq \gamma'$. In this case it can be shown (see [3] and [4]) that there exists an isometric isomorphism between $L^2(\mathcal{X})$ and $L^2(\widehat{\Gamma}, L^2(C))$.

Proposition 4.7 ([4, Proposition 3.3]). *The mapping $\mathcal{T}_\sigma : L^2(\mathcal{X}) \to L^2(\widehat{\Gamma}, L^2(C))$ defined by*

$$\mathcal{T}_\sigma[\psi](\alpha)(x) := \sum_{\gamma\in\Gamma} [(T_\sigma(\gamma)\psi)(x)](-\gamma, \alpha)$$

is an isometric isomorphism, and $\mathcal{T}_\sigma[T_\sigma(\gamma)\psi](\alpha) = (\gamma, \alpha)\mathcal{T}_\sigma[\psi]$.

Just as for ordinary translates of the previous section, the isomorphism $\mathcal{T}_\sigma$ of Proposition 4.7 connects Γ-invariant spaces in $L^2(\mathcal{X})$ with MI spaces in $L^2(\widehat{\Gamma}, L^2(C))$. Here the determining set D is the set of characters of $\widehat{\Gamma}$. More precisely, for every $\gamma \in \Gamma$, let $X_\gamma : \widehat{\Gamma} \to \mathbb{C}$ be the homomorphism defined as $X_\gamma(\alpha) = (\gamma, \alpha)$. Then, by Pontryagin duality [24, Theorem 1.7.2], $\{X_\gamma\}_{\gamma\in\Gamma}$ is the set of characters of $\widehat{\Gamma}$ and thus, by uniqueness of the Fourier transform, $D = \{X_\gamma\}_{\gamma\in\Gamma}$ is a determining set for $L^1(\widehat{\Gamma})$. Therefore, one can characterize Γ-invariant spaces using range functions, obtaining a similar result to Theorem 4.2. Furthermore, one can also prove characterizations of frames of the form $\{T_\sigma(\gamma)\varphi : \gamma \in \Gamma,\, \varphi \in \mathcal{A}\}$ for $V = \overline{\operatorname{span}}\{T_\sigma(\gamma)\varphi : \gamma \in \Gamma,\, \varphi \in \mathcal{A}\}$ in the spirit of Theorem 4.3. We do not include here the complete statements of these results because we think they should be clear for the reader (see [4, Theorems 4.3 and 5.1] for details and proofs).

In a similar way to Theorem 4.4, the following result can be shown:

THEOREM 4.8. *Given* $\{\phi_1, \ldots, \phi_m\} \subseteq L^2(\mathcal{X})$ *let* $V = \overline{\operatorname{span}}\{T_\sigma(\gamma)\phi_j : \gamma \in \Gamma,\ 1 \leq j \leq m\}$ *and let* $\ell(M)$ *be the length of* $M = \mathcal{T}_\sigma[V]$ *where* $\mathcal{T}_\sigma$ *is as in Proposition* 4.7. *For* $\ell(M) \leq \ell \leq m$, *let* $\mathcal{R}$ *be the set of matrices* $A = \{a_{ij}\}_{ij} \in \mathbb{C}^{\ell\times m}$ *such that the linear combinations* $\psi_j = \sum_{i=1}^m a_{ij}\phi_j$ *generate* V, *i.e.* $V = \overline{\operatorname{span}}\{T_\sigma(\gamma)\psi_i : \gamma \in \Gamma,\ 1 \leq i \leq \ell\}$. *Then* $\mathbb{C}^{\ell\times m} \setminus \mathcal{R}$ *has Lebesgue zero measure.*

If in addition $\{T_\sigma(\gamma)\phi_j : \gamma \in \Gamma,\ 1 \leq j \leq m\}$ *is a frame for* V, *then* $\{T_\sigma(\gamma)\psi_i : \gamma \in \Gamma,\ 1 \leq i \leq \ell\}$ *is also a frame for* V *if and only if* $A \in \mathcal{R}$ *and there exists* $\delta > 0$ *such that* $\boldsymbol{\mathcal{F}}\,[\operatorname{Ker}(A), \operatorname{Im}(G_{\boldsymbol{\Phi}(\alpha)})] \geq \delta$ *for a.e.* $\alpha \in \widehat{\Gamma}$, *where* $G_{\boldsymbol{\Phi}(\alpha)}$ *is the Gramian associated to* $\{\mathcal{T}_\sigma[\phi_1](\alpha), \ldots, \mathcal{T}_\sigma[\phi_m](\alpha)\}$.

Acknowledgements. This project was carried out when the author was supported by a fellowship for postdoctoral researchers from the Alexander von Humboldt Foundation. She wants to thank Gitta Kutyniok for supporting her stay in Technische Universität Berlin. Also the author wants to thank the referee for his/her comments which helped to improve the manuscript and to Carlos Cabrelli for carefully reading the paper.

References

[1] A. Aldroubi and K. Gröchenig, *Nonuniform sampling and reconstruction in shift-invariant spaces*, SIAM Rev. 43 (2001), 585–620.

[2] J. Antezana, G. Corach, M. Ruiz, and D. Stojanoff, *Nullspaces and frames*, J. Math. Anal. Appl. 309 (2005), 709–723.

[3] D. Barbieri, E. Hernández, and J. Parcet, *Riesz and frame systems generated by unitary actions of discrete groups*, Appl. Comput. Harmon. Anal. (2014, online).

[4] D. Barbieri, E. Hernández, and V. Paternostro, *The Zak Transform and the structure of spaces invariant under the action of an LCA group*, preprint, 2014.

[5] M. Bownik, *The structure of shift-invariant subspaces of* $L^2(\mathbb{R}^n)$, J. Funct. Anal. 177 (2000), 282–309.

[6] M. Bownik and N. Kaiblinger, *Minimal generator sets for finitely generated shift-invariant subspaces of* $L^2(\mathbb{R}^n)$, J. Math. Anal. Appl. 313 (2006), 342–352.

[7] M. Bownik and K. A. Ross, *The structure of translation-invariant spaces on locally compact abelian groups*, preprint, 2014.

[8] C. Cabrelli, C. Mosquera, and V. Paternostro, *Linear combination of frame generators in systems of translates*, J. Math. Anal. Appl. 413 (2014), 776–788.

[9] C. Cabrelli and V. Paternostro, *Shift-invariant spaces on LCA groups*, J. Funct. Anal. 258 (2010), 2034–2059.

[10] C. Cabrelli and V. Paternostro, *Shift-modulation invariant spaces on LCA groups*, Studia Math. 211 (2012), 1–19.

[11] C. de Boor, R. A. DeVore, and A. Ron, *Approximation from shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (1994), 787–806.

[12] C. de Boor, R. A. DeVore, and A. Ron, *The structure of finitely generated shift-invariant spaces in* $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994), 37–78.

[13] F. Deutsch, *The angle between subspaces of a Hilbert space*, in: Approximation Theory, Wavelets and Applications (Maratea, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 454, Kluwer, Dordrecht, 1995, 107–130.

[14] K. Gröchenig, *Localization of frames, Banach frames, and the invertibility of the frame operator*, J. Fourier Anal. Appl. 10 (2004), 105–132.

[15] V. P. Havin and B. Jöricke, *The uncertainty principle in harmonic analysis*, in: Commutative Harmonic Analysis, III, Encyclopaedia Math. Sci. 72, Springer, Berlin, 1995, 177–259, 261–266.

[16] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[17] E. Hernández, H. Šikić, G. Weiss, and E. Wilson, *Cyclic subspaces for unitary representations of LCA groups; generalized Zak transform*, Colloq. Math. 118 (2010), 313–332.

[18] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, FL, 1996.

[19] R. A. Kamyabi Gol and R. Raisi Tousi, *A range function approach to shift-invariant spaces on locally compact abelian groups*, Int. J. Wavelets Multiresolut. Inf. Process. 8 (2010), 49–59.

[20] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York, 1976.

[21] G. W. Mackey, *Induced representations of locally compact groups. I*, Ann. of Math. 55 (1952), 101–139.

[22] S. G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc. 315 (1989), 69–87.

[23] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Canad. J. Math. 47 (1995), 1051–1094.

[24] W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

[25] T. P. Srinivasan, *Doubly invariant subspaces*, Pacific J. Math. 14 (1964), 701–707.

[26] Q. Sun, *Local reconstruction for sampling in shift-invariant spaces*, Adv. Comput. Math. 32 (2010), 335–352.

[27] W. Sun, *Sampling theorems for multivariate shift invariant subspaces*, Sampl. Theory Signal Image Process. 4 (2005), 73–98.

[28] P. Zhao, C. Zhao, and P. G. Casazza, *Perturbation of regular sampling in shift-invariant spaces for frames*, IEEE Trans. Inform. Theory 52 (2006), 4643–4648.

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Received April 4, 2014
Revised version February 12, 2015 (7942)